Some math exercises

I. Fieldstrength, Bianchi identity, Yang-Mills equation, etc.

A $\mathfrak{g} = \mathrm{Lie}(G)$-valued function ϵ is called <u>covariant</u> when it transforms as

$$\epsilon \mapsto g^{-1} \epsilon g$$

under gauge transformation by $g \in \mathcal{G}$

① Show that, if ϵ is covariant, its covariant derivative

$$D_\mu \epsilon := \partial_\mu \epsilon + [A_\mu, \epsilon]$$ is also covariant.

(※ This was shown in the class for a general representation R.
The exercise is to do it (again) for $R = \mathfrak{g}$.)

② Show that the field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial A_\mu + [A_\mu, A_\nu]$ is covariant.

In particular, $D_\mu F_{\nu\lambda}$ is also covariant.

③ Show that $D_\mu F_{\nu\lambda} + D_\nu F_{\lambda\mu} + D_\lambda F_{\mu\nu} = 0$ holds.

This is called <u>Bianchi identity</u>.

④ Show that an arbitrary variation δA_μ of A_μ is covariant,

In particular $D_\mu \delta A_\nu$ is also covariant.

⑤ Show that under an arbitrary variation of A_μ, the fieldstrength varies as $\delta F_{\mu\nu} = D_\mu \delta A_\nu - D_\nu \delta A_\mu$.

⑥ For two covariant $\mathfrak{g}$-valued functions ϵ_1 and ϵ_2, $\epsilon_1 \cdot \epsilon_2$ is gauge invariant (by the property of the scalar product "$\cdot$"). Show that

$$\partial_\mu (\epsilon_1 \cdot \epsilon_2) = D_\mu \epsilon_1 \cdot \epsilon_2 + \epsilon_1 \cdot D_\mu \epsilon_2 .$$

⑦ Show that Euler-Lagrange equation of the Yang Mills action is

$$D^\mu F_{\mu\nu} = 0.$$

This is called <u>Yang-Mills equation</u>.

(Hint : use ⑤ & ⑥)

II. Actions of Lie groups.

Suppose a Lie group G acts on a manifold M, and it is a right action: $x(gh) = (xg)h$ for $x \in M$, $g, h \in G$.

Variation of a function f on M by $X \in \mathfrak{g} = \mathrm{Lie}(G)$ is a function $\delta_X f$ on M defined by

$$(\delta_X f)(x) = \frac{d}{dt} f(xe^{tX})\Big|_{t=0}.$$

Show that, for X & $Y \in \mathfrak{g}$,

$$\delta_X \delta_Y f - \delta_Y \delta_X f = \delta_{[X,Y]} f$$

Suppose it is a left action instead, $(gh)x = g(hx)$.

The variation is $(\delta_X f)(x) = \frac{d}{dt} f(e^{tX}x)\Big|_{t=0}$.

Show that, for X & $Y \in \mathfrak{g}$

$$\delta_X \delta_Y f - \delta_Y \delta_X f = -\delta_{[X,Y]} f$$