# Multiple pairs

Consider the mechanics of $n$-pairs of anticommuting variables

$\psi_1, \overline{\psi}_1, \psi_2, \overline{\psi}_2, \cdots, \psi_n, \overline{\psi}_n$ with Lagrangian

$$L = i \sum_{i=1}^{n} \overline{\psi}_i \dot{\psi}_i - H(\overline{\psi}, \psi)$$

where $H(\overline{\psi}, \psi)$ is a function of $\psi_1, \overline{\psi}_1, \cdots, \psi_n, \overline{\psi}_n$.

- Show that the energy of the system is $H(\overline{\psi}, \psi)$.

- The canonical commutation relation is

$$\{\hat{\psi}_i, \hat{\overline{\psi}}_j\} = \hbar \delta_{i,j}, \quad \{\hat{\psi}_i, \hat{\psi}_j\} = \{\hat{\overline{\psi}}_i, \hat{\overline{\psi}}_j\} = 0,$$

--- the Clifford algebra.

Show this using Ward identity.

- An irreducible representation $\mathcal{H}$ of this algebra is spanned by

$$|0\rangle, \quad \underset{1 \le i \le n}{\hat{\overline{\psi}}_i |0\rangle}, \quad \underset{1 \le i < j \le n}{\hat{\overline{\psi}}_i \hat{\overline{\psi}}_j |0\rangle}, \quad \underset{1 \le i < j < k \le n}{\hat{\overline{\psi}}_i \hat{\overline{\psi}}_j \hat{\overline{\psi}}_k |0\rangle}, \quad \cdots, \quad \hat{\overline{\psi}}_1 \hat{\overline{\psi}}_2 \cdots \hat{\overline{\psi}}_n |0\rangle$$

where $|0\rangle$ is a state annihilated by all $\hat{\psi}_i$'s.

It has dimension $1 + n + \binom{n}{2} + \cdots + \binom{n}{n} = (1+1)^n = 2^n$.

- In what follows, we assume that $H(\bar{\varphi},\varphi)$ is real,

$$H(\bar{\varphi},\varphi)^* = H(\bar{\varphi},\varphi), \quad \text{under} \quad \underline{\varphi_i^* = \bar{\varphi}_i, \ \bar{\varphi}_i^* = \varphi_i}.$$

(For example, $H(\bar{\varphi},\varphi) = \sum_{i=1}^{n} \omega_i \bar{\varphi}_i \varphi_i$ with $\omega_i \in \mathbb{R}$.)

Then, the Lagrangian is also real modulo total derivative

$$L^* = L + \frac{d}{dt}(\ - \ ).$$

Correspondingly, we'd like the representation $\mathcal{H}$ to have an inner product s.t. $\underline{\hat{\varphi}_i^\dagger = \hat{\bar{\varphi}}_i, \ \hat{\bar{\varphi}}_i^\dagger = \hat{\varphi}_i}$. Show that the inner product s.t. $\langle 0|0\rangle = 1$ is positive definite.

- Put $|\varphi\rangle := \prod_{i=1}^{n} \left(1 + \frac{1}{\hbar}\hat{\varphi}_i^\dagger \varphi_i\right)|0\rangle$

$$|\bar{\varphi}\rangle := (\bar{\varphi}_1 - \hat{\varphi}_1^\dagger)(\bar{\varphi}_2 - \hat{\varphi}_2^\dagger)\cdots(\bar{\varphi}_n - \hat{\varphi}_n^\dagger)|0\rangle$$

$$\langle\varphi| := |\bar{\varphi}\rangle^\dagger = \langle 0|(\varphi_n - \hat{\varphi}_n)\cdots(\varphi_2 - \hat{\varphi}_2)(\varphi_1 - \hat{\varphi}_1)$$

$$\langle\bar{\varphi}| := |\varphi\rangle^\dagger = \langle 0|\prod_{i=1}^{n}\left(1 + \frac{1}{\hbar}\bar{\varphi}_i\hat{\varphi}_i\right).$$

Show that they are eigenstates of $\hat{\varphi}_i$ or $\hat{\varphi}_i^\dagger$:

$$\hat{\varphi}_i|\varphi\rangle = \varphi_i|\varphi\rangle, \quad \hat{\varphi}_i^\dagger|\bar{\varphi}\rangle = \bar{\varphi}_i|\bar{\varphi}\rangle,$$

$$\langle\varphi|\hat{\varphi}_i = \langle\varphi|\varphi_i, \quad \langle\bar{\varphi}|\hat{\varphi}_i^\dagger = \langle\bar{\varphi}|\bar{\varphi}_i.$$

- Compute $\langle \bar{\psi} | \psi' \rangle$, $\langle \psi | \bar{\psi}' \rangle$, $\langle \psi | \psi' \rangle$, $\langle \bar{\psi} | \bar{\psi}' \rangle$.
- Show that

$$\int d\psi_1 \cdots d\psi_n \, |\psi\rangle\langle\psi| = \int d\bar{\psi}_n \cdots d\bar{\psi}_1 \, |\bar{\psi}\rangle\langle\bar{\psi}| = id_{\mathcal{H}}$$

$$\mathrm{Tr}_{\mathcal{H}} A = \int \prod_{i=1}^{n} d\bar{\psi}_i d\psi_i \, \langle -\bar{\psi} | A | \psi \rangle \langle \psi | \bar{\psi} \rangle$$

$$\mathrm{Tr}_{\mathcal{H}} (-1)^F A = \int \prod_{i=1}^{n} d\bar{\psi}_i d\psi_i \, \langle \bar{\psi} | A | \psi \rangle \langle \psi | \bar{\psi} \rangle$$

Note: $d\bar{\psi}_n \cdots d\bar{\psi}_1 \cdot d\psi_1 \cdots d\psi_n = \prod_{i=1}^{n} d\bar{\psi}_i d\psi_i$

- We may represent a state $|\Psi\rangle$ by its wave function

$$\Psi(\bar{\psi}) := \langle \bar{\psi} | \Psi \rangle.$$

Find the form of the inner product & time evolution of the wave functions.

- Show that the transition amplitude

$$Z(t_f, \bar{\psi}^f; t_i, \psi^i) = \langle \bar{\psi}^f | e^{-i\frac{t_f - t_i}{\hbar}\hat{H}} | \psi^i \rangle$$

and the partition functions can be expressed by path integral as

$$Z(t_f, \overline{\psi}^f; t_i, \psi^i)$$

$$= \int \mathcal{D}\overline{\psi}\mathcal{D}\psi \; e^{\frac{1}{\hbar}\overline{\psi}\psi|_{t_f} + \frac{i}{\hbar}\int_{t_i}^{t_f} dt \left( i\overline{\psi}\dot{\psi} - H(\overline{\psi},\psi) \right)}$$

$$\overline{\psi}(t_f) = \overline{\psi}^f, \quad \psi(t_i) = \psi^i$$

where $\overline{\psi}\psi := \sum_{i=1}^{n} \overline{\psi}_i \psi_i$,

$$\mathrm{Tr}_{\mathcal{H}}\, e^{-\frac{T}{\hbar}\hat{H}} = \int \mathcal{D}\overline{\psi}\mathcal{D}\psi \; e^{-\frac{1}{\hbar}\int_{\mathbb{R}/T\mathbb{Z}} d\tau \, L_E}$$

$$\overline{\psi}(\tau+T) = -\overline{\psi}(\tau), \quad \psi(\tau+T) = -\psi(\tau),$$

antiperiodic

$$\mathrm{Tr}_{\mathcal{H}}(-1)^F e^{-\frac{T}{\hbar}\hat{H}} = \int \mathcal{D}\overline{\psi}\mathcal{D}\psi \; e^{-\frac{1}{\hbar}\int_{\mathbb{R}/T\mathbb{Z}} d\tau \, L_E}$$

$$\overline{\psi}(\tau+T) = \overline{\psi}(\tau), \quad \psi(\tau+T) = \psi(\tau).$$

periodic