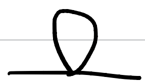
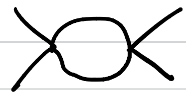
Computation of the one-loop integrals

We would like to compute regularized versions of

$$I = \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2+m^2}$$

$$V = \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2+m^2} \frac{1}{(k-p)^2+m^2}$$

that appear in  and , for

① momentum cut-off

$$\frac{1}{k^2+m^2} = \int_0^\infty d\alpha\, e^{-\alpha(k^2+m^2)}$$

$$\longrightarrow \int_{1/\Lambda^2}^\infty d\alpha\, e^{-\alpha(k^2+m^2)} = \frac{e^{-\frac{k^2+m^2}{\Lambda^2}}}{k^2+m^2}$$

and

③ dimensional regularization : $4 \rightsquigarrow d = 4-\epsilon$.

Exercise : Do the computation.

(Option 1) Explain the steps marked ! in the following.

(Option 2) Do it in your own way.

$$I_{①} = \int \frac{d^4k}{(2\pi)^4} \int_{1/\Lambda^2}^{\infty} d\alpha \; e^{-\alpha(k^2+m^2)}$$

$$\overset{!}{=} \int_{1/\Lambda^2}^{\infty} d\alpha \frac{e^{-\alpha m^2}}{(4\pi)^2 \alpha^2}$$

$$\overset{!}{=} \frac{1}{(4\pi)^2} \left[\Lambda^2 - m^2 \left(\log\left(\frac{\Lambda^2}{m^2}\right) + 1 - \gamma \right) + m^2 O\left(\frac{m^2}{\Lambda^2}\right) \right].$$

$$I_{③} = \mu_{DR}^{4-d} \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2+m^2} = \mu_{DR}^{4-d} \frac{\mathrm{Vol}(S^{d-1})}{2(2\pi)^d} \int_0^{\infty} (k^2)^{d/2-1} dk^2 \frac{1}{k^2+m^2}$$

$$\overset{!}{=} \frac{\mu_{DR}^{4-d} m^{d-2}}{(4\pi)^{d/2}\Gamma(d/2)} B\left(\frac{d}{2}, 1-\frac{d}{2}\right) = \Gamma\left(\frac{d}{2}\right)\Gamma\left(1-\frac{d}{2}\right)$$

$$= m^2 \left(\frac{\mu_{DR}}{m}\right)^{4-d} \frac{1}{(4\pi)^{d/2}} \Gamma\left(1-\frac{d}{2}\right)$$

$d=4-\epsilon$

$$\overset{!}{=} -\frac{m^2}{(4\pi)^2} \left[\frac{2}{\epsilon} + \log\left(\frac{4\pi\mu_{DR}^2}{m^2}\right) + 1 - \gamma + O(\epsilon) \right].$$

$$V_{①} = \int \frac{d^4k}{(2\pi)^4} \int_{1/\Lambda^2}^{\infty} \int_{1/\Lambda^2}^{\infty} d\alpha\, d\beta \; e^{-\alpha(k^2+m^2)-\beta((k-p)^2+m^2)}$$

$$\overset{!}{=} \frac{1}{(4\pi)^2} \int_{1/\Lambda^2}^{\infty} \int_{1/\Lambda^2}^{\infty} \frac{d\alpha\, d\beta}{(\alpha+\beta)^2} \; e^{-\frac{\alpha\beta}{\alpha+\beta}p^2 - (\alpha+\beta)m^2}$$

insert $1 = \int_{2/\Lambda^2}^{\infty} d\lambda\, \delta(\lambda-\alpha-\beta)$ & substitute $\alpha \to \lambda x$, $\beta \to \lambda y$

$$= \frac{1}{(4\pi)^2} \int_{2/\Lambda^2}^{\infty} d\lambda \int_{1/\lambda\Lambda^2}^{\infty} \int_{1/\lambda\Lambda^2}^{\infty} \frac{dx\, dy}{(x+y)^2} \; e^{-\lambda\left(\frac{xy}{x+y}p^2 + (x+y)m^2\right)} \underbrace{\delta(\lambda(1-x-y))}_{\frac{1}{\lambda}\delta(1-x-y)}$$

$$= \frac{1}{(4\pi)^2} \int_{2/\Lambda^2}^{\infty} \frac{d\lambda}{\lambda} \underbrace{\int_{1/\lambda\Lambda^2}^{1-1/\lambda\Lambda^2}}_{= \underbrace{\int_0^1}_{X} - 2\underbrace{\int_0^{1/\lambda\Lambda^2}}_{Y}} dx \; e^{-\lambda(x(1-x)p^2+m^2)}$$

$$= X - 2Y.$$

$$X = \frac{1}{(4\pi)^2} \int_{2/\Lambda^2}^{\infty} \frac{d\lambda}{\lambda} \int_0^1 dx \; e^{-\lambda(x(1-x)p^2+m^2)}$$

$$\overset{!}{=} \frac{1}{(4\pi)^2} \left[\log\left(\frac{\Lambda^2}{2m^2}\right) - \gamma - \int_0^1 dx \log\left(1 + x(1-x)\frac{p^2}{m^2}\right) + O\left(\frac{p^2}{\Lambda^2}, \frac{m^2}{\Lambda^2}\right) \right].$$

$$Y = \frac{1}{(4\pi)^2} \int_{2/\Lambda^2}^{\infty} \frac{d\lambda}{\lambda} \int_0^{1/\lambda\Lambda^2} dx\; e^{-\lambda\left(x(1-x)p^2+m^2\right)}$$

$$\Big[\quad 0 \le \lambda x \le \frac{1}{\Lambda^2} \Rightarrow e^{-\lambda\left(x(1-x)p^2+m^2\right)} = e^{-\lambda m^2}\left(1+O\left(\frac{p^2}{\Lambda^2}\right)\right)$$

$$= \frac{1}{(4\pi)^2} \int_{2/\Lambda^2}^{\infty} \frac{d\lambda}{\lambda} \frac{1}{\lambda\Lambda^2} e^{-\lambda m^2} \left(1+O\left(\frac{p^2}{\Lambda^2}\right)\right)$$

$$= \frac{1}{(4\pi)^2}\left(\frac{1}{2} + O\left(\frac{m^2}{\Lambda^2}, \frac{p^2}{\Lambda^2}\right)\right).$$

$$\therefore\ V_{①} = X - 2Y$$

$$= \frac{1}{(4\pi)^2}\Big(\log\left(\frac{\Lambda^2}{2m^2}\right) - \gamma - 1 - \int_0^1 dx \log\left(1 + x(1-x)\frac{p^2}{m^2}\right) + O\left(\frac{p^2}{\Lambda^2}, \frac{m^2}{\Lambda^2}\right)\Big].$$

$$V_{③} = \mu_{DR}^{4-d} \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2+m^2} \frac{1}{(k-p)^2+m^2}$$

$$\overset{!}{=} \mu_{DR}^{4-d} \int \frac{d^dk'}{(2\pi)^d} \int_0^1 \frac{dx}{\big(k'^2 + \underbrace{x(1-x)p^2+m^2}_{=:\Delta}\big)^2}$$

$$\overset{!}{=} \frac{\mu_{DR}^{4-d}}{(4\pi)^{d/2}\Gamma(d/2)} \int_0^1 dx\, \Delta^{\frac{d}{2}-2}\, B\left(\frac{d}{2}, 2-\frac{d}{2}\right) = \Gamma\left(\frac{d}{2}\right)\Gamma\left(2-\frac{d}{2}\right)$$

$$= \frac{\mu_{DR}^{4-d}}{(4\pi)^{d/2}} \Gamma\left(2-\frac{d}{2}\right) \int_0^1 dx \, \Delta^{\frac{d}{2}-2}$$

$d=4-\epsilon$

$$\overset{!}{=} \frac{1}{(4\pi)^2} \left[\frac{2}{\epsilon} - \gamma - \int_0^1 dx \log\left(\frac{\Delta}{4\pi\mu_{DR}^2}\right) + O(\epsilon) \right]$$

$$= \frac{1}{(4\pi)^2} \left[\frac{2}{\epsilon} + \log\left(\frac{4\pi\mu_{DR}^2}{m^2}\right) - \gamma - \int_0^1 dx \log\left(1 + x(1-x)\frac{p^2}{m^2}\right) + O(\epsilon) \right]$$

You may use

- $$\int_\epsilon^\infty \frac{dt}{t} e^{-t} = -\log\epsilon - \gamma + O(\epsilon)$$

- $$B(p,q) = \int_0^\infty \frac{y^{p-1}dy}{(1+y)^{p+q}} = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

- $$\Gamma(n+1) = n! \quad \text{for} \quad n=0,1,2,\cdots$$

- $$\Gamma(z) = \frac{1}{z} - \gamma + O(z) \quad \text{as} \quad z \to 0$$

- $$\Gamma(-1+z) = -\frac{1}{z} + \gamma - 1 + O(z) \quad \text{as} \quad z \to 0$$

- $$\frac{1}{AB} = \int_0^1 \frac{dx}{(xA+(1-x)B)^2}$$

Determination of the one-loop counter terms

We have determined the one loop counter terms of 4d ϕ^4 theory,

$a_1(\Lambda)$, $b_1(\Lambda)$, $C_1(\Lambda)$ for momentum cutoff

$a_1(\epsilon)$, $b_1(\epsilon)$, $C_1(\epsilon)$ for dimensional regularization

so that on-shell renormalization condition is satisfied.

Do the same for

intermediate renormalization

$$\begin{cases} \Gamma(-p,p)\big|_{p^2=0} = m^2 \\ \frac{d}{dp^2}\Gamma(-p,p)\big|_{p^2=0} = 1 \\ \Gamma(p_1,\cdots,p_4)\big|_{p_i\cdot p_j=0} = \lambda \end{cases}$$

and

"another R.C."

$$\begin{cases} \Gamma(-p,p)\big|_{p^2=\mu^2} = \mu^2+m^2 \\ \frac{d}{dp^2}\Gamma(-p,p)\big|_{p^2=\mu^2} = 1 \\ \Gamma(p_1,\cdots,p_4)\big|_{p_i\cdot p_j=\begin{cases}\mu^2 & i=j \\ -\mu^2/3 & i\neq j\end{cases}} = \lambda \end{cases}$$