Examples

Computation of $Z(t_f, q_f ; t_i, q_i)$ & $Z(S^1_T)$

in examples : 1. free particle in a line

2. harmonic oscillator

3. free particle in a circle

3)' " with a theta term

1. Free particle in a line

$$L = \frac{m}{2}\dot{q}^2, \qquad \hat{H} = \frac{1}{2m}\hat{p}^2.$$

Spectrum : $E_p = \frac{1}{2m}p^2$, $\Psi_p(q) = \dfrac{e^{ipq/\hbar}}{\sqrt{2\pi\hbar}}$

Using operator formalism we have

$$Z(t_f, q_f ; t_i, q_i) = \langle q_f | e^{-i\frac{t_f - t_i}{\hbar}\hat{H}} | q_i \rangle$$

$$= \sqrt{\frac{m}{2\pi\hbar\, i(t_f - t_i)}}\; e^{i\frac{m(q_f - q_i)^2}{2\hbar(t_f - t_i)}}$$

Let us try to reproduce this by path-integral:

A configuration $q(t)$ s.t. $q(t_i) = q_i$, $q(t_f) = q_f$ can be written as

$$q(t) = q_i + (q_f - q_i)\frac{t - t_i}{t - t_f} + \sum_{n=1}^{\infty} q_n \sqrt{\frac{2}{t_f - t_i}} \sin\left(\frac{\pi n}{t_f - t_i}(t - t_i)\right)$$

A motivation of "$\sqrt{\frac{2}{t_f - t_i}}$":

$$\varphi_n(t) = \sqrt{\frac{2}{t_f - t_i}} \sin\left(\frac{\pi n}{t_f - t_i}(t - t_i)\right) \quad \text{are orthonormal}$$

with respect to the inner product $(\varphi, \varphi') = \int_{t_i}^{t_f} dt\, \varphi(t)^* \varphi'(t)$:

$$(\varphi_n, \varphi_m) = \delta_{n,m}$$

For this

$$S[q] = \frac{m(q_f - q_i)^2}{2(t_f - t_i)} + \sum_{n=1}^{\infty} \frac{m}{2}\left(\frac{\pi n}{t_f - t_i}\right)^2 q_n^2$$

Let us "define" the path-integral measure by

$$\mathcal{D}q = C \prod_{n=1}^{\infty} dq_n \quad \text{for some constant } C.$$

Then,

$$Z(t_f, q_f; t_i, q_i) = \int \mathcal{D}q\, e^{\frac{i}{\hbar} S[q]}$$

$$= e^{\frac{i}{\hbar}\frac{m(q_f - q_i)^2}{2(t_f - t_i)}} \cdot C \prod_{n=1}^{\infty} \sqrt{\frac{2\pi i \hbar (t_f - t_i)^2}{m(\pi n)^2}}$$

The product $\prod_{n=1}^{\infty} \sqrt{\frac{2\pi i \hbar (t_f - t_i)^2}{m(\pi n)^2}}$ is divergent (or zero),

but let us "tune" the constant C so that

$$C \cdot \prod_{n=1}^{\infty} \sqrt{\frac{2\pi i \hbar (t_f - t_i)^2}{m(\pi n)^2}}$$ is finite (and non-zero).

Several ways:

(I) Choose C_N so that

$$\lim_{N \to \infty} C_N \prod_{n=1}^{N} \sqrt{\frac{2\pi i \hbar (t_f - t_i)^2}{m(\pi n)^2}}$$ is finite.

(II) Zeta function regularization

$$\prod_{n=1}^{\infty} \frac{x}{n} = (2\pi x)^{-\frac{1}{2}}$$

Zeta function regularization

$$\zeta(s) := \sum_{n=1}^{\infty} n^{-s}$$ Riemann's zeta function

- absolutely convergent if $\mathrm{Re}(s) > 1$
- $\exists$ analytic continuation & regular at $s=0$.

$$\zeta(s) = \sum_{n=1}^{\infty} e^{-s\log n}$$

$$\zeta'(s) = \sum_{n=1}^{\infty} (-\log n)\, e^{-s\log n}$$

known: $\zeta(0) = -\frac{1}{2}$, $\zeta'(0) = -\frac{1}{2}\log 2\pi$

$$\prod_{n=1}^{\infty} \frac{X}{n} = \exp\left(\sum_{n=1}^{\infty} \log\left(\frac{X}{n}\right)\right) = \exp\left(\sum_{n=1}^{\infty} (\log X - \log n)\right)$$

$$= \exp\left(\zeta(0)\log X + \zeta'(0)\right)$$

$$= \exp\left(-\frac{1}{2}\log X - \frac{1}{2}\log 2\pi\right) = (2\pi X)^{-\frac{1}{2}}.$$

Back to path-integral.

Let us use zeta function regularization $\prod_{n=1}^{\infty} \frac{X}{n} = (2\pi X)^{-\frac{1}{2}}$

Let us define

$$\mathcal{D}q = \prod_{n=1}^{\infty} c\, dq_n \qquad \text{for some constant } c.$$

Then,

$$Z(t_f, q_f : t_i, q_i) = e^{\frac{i}{\hbar}\frac{m(q_f-q_i)^2}{2(t_f-t_i)}} \cdot \prod_{n=1}^{\infty} c\sqrt{\frac{2\pi i\hbar(t_f-t_i)^2}{m(\pi n)^2}}$$

$$= e^{\frac{i}{\hbar}\frac{m(q_f-q_i)^2}{2(t_f-t_i)}} \left(2\pi c\sqrt{\frac{2\pi i\hbar(t_f-t_i)^2}{m\pi^2}}\right)^{-\frac{1}{2}}$$

$$= e^{\frac{i}{\hbar}\frac{m(q_f-q_i)^2}{2(t_f-t_i)}} \left(\frac{m}{8\pi i c^2\hbar(t_f-t_i)^2}\right)^{\frac{1}{4}}$$

If we choose $c^2 = i\pi\hbar/2m$

$$\downarrow$$

$$= e^{\frac{i}{\hbar}\frac{m(q_f-q_i)^2}{2(t_f-t_i)}} \sqrt{\frac{m}{2\pi\hbar i(t_f-t_i)}}$$

match with the operator-result.

More about the definition of $\mathcal{D}q = C \prod_{n=1}^{\infty} dq_n$

For any way to define "$C \prod_{n=1}^{\infty} \sqrt{\dfrac{2\pi i \hbar (t_f - t_i)^2}{m(\pi n)^2}}$"

it is a function of $t_f - t_i$ (and $\hbar$ & m).

Let us denote it by $F(t_f - t_i)$ so that

$$Z(t_f, q_f ; t_i, q_i) = F(t_f - t_i)\, e^{\frac{i m (q_f - q_i)^2}{2\hbar (t_f - t_i)}} .$$

Let us try to fix the function $F(T)$ by the properties

$Z(t_f, q_f ; t_i, q_i)$ is supposed to have :

(a) unitarity of time evolution

$$\Psi(q) \longmapsto \int dq' \, Z(T, q ; 0, q') \Psi(q')$$

(b) composition law :

$$\int dq_2 \, Z(t_3, q_3 ; t_2, q_2)\, Z(t_2, q_2 ; t_1, q_1) = Z(t_3, q_3 ; t_2, q_2).$$

Exercise : Show that

(a) $\Leftrightarrow$ $|F(T)|^2 = \dfrac{m}{2\pi\hbar T}$.

(b) $\Leftrightarrow$ $F(t_3-t_2)F(t_2-t_1) = \sqrt{\dfrac{m(t_3-t_1)}{2\pi i\hbar(t_3-t_2)(t_2-t_1)}}\, F(t_3-t_1)$.

These reqire that

$$F(T) = \sqrt{\frac{m}{2\pi i\hbar T}}\, e^{iAT}$$

for some real constant A that may depend on m & $\hbar$.

This A can be absorbed by redefining Lagrangian

$$L(q,\dot{q}) \to L(q,\dot{q}) - \hbar A.$$

Thus, modulo a possible shift of Lagrangian by a constant, we may set

$$F(T) = \sqrt{\frac{m}{2\pi i\hbar T}}.$$

Partition function.

$$\mathrm{Tr}\left(e^{-\frac{T}{\hbar}\hat{H}}\right) = \int dq \underbrace{\langle q | e^{-\frac{T}{\hbar}\hat{H}} | q \rangle}_{\sqrt{\frac{m}{2\pi\hbar T}}\, e^{-\frac{1}{\hbar}\frac{m(q-q)^2}{2\,T}}}$$

$$= \int dq \sqrt{\frac{m}{2\pi\hbar T}} = \text{divergent !}$$

Let us "regularize" this by pretending that the particle is in a segment of large but finite length L.

Then $\mathrm{Tr}\left(e^{-\frac{T}{\hbar}\hat{H}}\right) = L \cdot \sqrt{\frac{m}{2\pi\hbar T}}$.

Let us try to reproduce this by path-integral:

$$Z(S^1_T) = \int \mathcal{D}q\, e^{-\frac{1}{\hbar}\int_{S^1_T} d\tau\, \frac{m}{2}\left(\frac{dq}{d\tau}\right)^2}$$

$q(\tau)$: periodic in $\tau \to \tau + T$

A periodic configuration $q(\tau) = q(\tau + T)$ can be written as

$$q(\tau) = \frac{q_0}{\sqrt{T}} + \sum_{n=1}^{\infty} \left(q_n^c \sqrt{\frac{2}{T}} \cos\left(\frac{2\pi n}{T}\tau\right) + q_n^s \sqrt{\frac{2}{T}} \sin\left(\frac{2\pi n}{T}\tau\right) \right)$$

Again, the motivation for $\frac{1}{\sqrt{T}}$ or $\sqrt{\frac{2}{T}}$ is so that

$$\varphi_0(\tau) = \frac{1}{\sqrt{T}}, \quad \varphi_n^c(\tau) = \sqrt{\frac{2}{T}} \cos\left(\frac{2\pi n}{T}\tau\right), \quad \varphi_n^s(\tau) = \sqrt{\frac{2}{T}} \sin\left(\frac{2\pi n}{T}\tau\right)$$

are orthonormal w.r.t. $(\varphi, \varphi') = \int_0^T d\tau \, \varphi(\tau)^* \varphi'(\tau)$.

$$S_E[q] = \frac{m}{2} \sum_{n=1}^{\infty} \left(\frac{2\pi n}{T}\right)^2 \left((q_n^c)^2 + (q_n^s)^2\right)$$

$$\mathcal{D}q = C \, dq_0 \cdot \prod_{n=1}^{\infty} dq_n^c \, dq_n^s \qquad \text{for some constant } C$$

Note: as q takes values in large but finite interval $[0, L]$, $0 \leq q_0 \leq L\sqrt{T}$

$$Z(S_T^1) = \int C \, dq_0 \cdot \prod_{n=1}^{\infty} dq_n^c \, dq_n^s \, e^{-\frac{m}{2\hbar} \sum_{n=1}^{\infty} \left(\frac{2\pi n}{T}\right)^2 \left((q_n^c)^2 + (q_n^s)^2\right)}$$

$$= C \cdot L\sqrt{T} \cdot \prod_{n=1}^{\infty} \frac{2\pi}{m \left(\frac{2\pi n}{T}\right)^2 / \hbar}$$

$$Z(S^1_T) = C \cdot L\sqrt{T} \cdot \prod_{n=1}^{\infty} \frac{\hbar T^2}{2\pi m n^2}$$

Use the zeta-function regularization:

$$\prod_{n=1}^{\infty} \frac{x}{n} = (2\pi x)^{-\frac{1}{2}} \quad \text{or} \quad \underline{\prod_{n=1}^{\infty} \frac{x^2}{n^2} = (2\pi x)^{-1}}.$$

Define

$$\mathcal{D}q = dq_0 \cdot \prod_{n=1}^{\infty} c\, dq_n^c\, dq_n^s \qquad \text{for some } c$$

$$Z(S^1_T) = L\sqrt{T} \cdot \prod_{n=1}^{\infty} c \frac{\hbar T^2}{2\pi m n^2}$$

$$= L\sqrt{T} \left(2\pi \sqrt{c \cdot \frac{\hbar T^2}{2\pi m}} \right)^{-1}$$

$$= L \sqrt{\frac{m}{c \cdot 2\pi \hbar T}}$$

match with the earlier result with $c = 1$.

② Harmonic oscillator

$$L = \frac{m}{2}\dot{q}^2 - \frac{m\omega^2}{2}q^2, \qquad \hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{q}^2$$

Spectrum $E_n = \hbar\omega(n+\frac{1}{2})$ $\quad n=0,1,2,\cdots$

$\Psi_n(q) =$ Hermite polynomial.

Transition amplitude

A configuration $q(t)$ s.t. $q(t_i)=q_i$, $q(t_f)=q_f$ can be written as

$$q(t) = q_{cl}(t) + \tilde{q}(t)$$

where q_{cl} is a solution of E.O.M. $\ddot{q} = -\omega^2 q$

s.t. $q_{cl}(t_i) = q_i$, $\quad q_{cl}(t_f) = q_f$

$$\left(\text{soln: } q_{cl}(t) = q_i \frac{\sin\omega(t-t_f)}{\sin\omega(t_i-t_f)} + q_f \frac{\sin\omega(t-t_i)}{\sin\omega(t_f-t_i)} \right)$$

and $$\tilde{q}(t) = \sum_{n=1}^{\infty} q_n \sqrt{\frac{2}{t_f-t_i}} \sin\left(\frac{\pi n}{t_f-t_i}(t-t_i)\right).$$

$$S[q] = S[q_{cl} + \tilde{q}] = S[q_{cl}] + S[\tilde{q}]$$

Q: why?

$$S[q_{cl}] = \frac{m\omega}{2\sin\omega(t_f - t_i)} \left[(q_f^2 + q_i^2)\cos\omega(t_f - t_i) - 2q_f q_i \right]$$

$$S[\tilde{q}] = \sum_{n=1}^{\infty} \left\{ \frac{m}{2}\left(\frac{\pi n}{t_f - t_i}\right)^2 - \frac{m\omega^2}{2} \right\} q_n^2$$

$$Z(t_f, q_f; t_i, q_i) = \int \mathcal{D}q \, e^{\frac{i}{\hbar} S[q]} \quad q(t_f) = q_f,\ q(t_i) = q_i$$

$$= e^{\frac{i}{\hbar} S[q_{cl}]} \int \underbrace{\mathcal{D}\tilde{q}}_{C \prod_{n=1}^{\infty} dq_n} e^{\frac{i}{\hbar} S[\tilde{q}]}$$

$$= e^{\frac{i}{\hbar} S[q_{cl}]} \underbrace{C \prod_{n=1}^{\infty} \sqrt{\frac{2\pi i \hbar}{m\left(\frac{\pi n}{t_f - t_i}\right)^2 - m\omega^2}}}$$

$$\underbrace{C \prod_{n=1}^{\infty} \sqrt{\frac{2\pi i \hbar (t_f - t_i)}{m(\pi n)^2}}}_{\text{Same as in the free particle in } \mathbb{R}} \prod_{n=1}^{\infty} \sqrt{\frac{1}{1 - \left(\omega(t_f - t_i)/\pi n\right)^2}}$$

- Using the definition of $C\prod_{n=1}^{\infty} dq_n$ determined in the free particle in $\mathbb{R}$,

$$C\prod_{n=1}^{\infty}\sqrt{\frac{2\pi i\hbar(t_f-t_i)}{m(\pi n)^2}} = \sqrt{\frac{m}{2\pi\hbar i(t_f-t_i)}}$$

- For the second ∞-product, use

$$\prod_{n=1}^{\infty}\left(1-\left(\frac{x}{\pi n}\right)^2\right) = \begin{cases} \cdot\ 1 \text{ at } x=0 \\ \cdot\ \text{simple zero at } x=\pi n \ (n\in\mathbb{Z}^\times) \end{cases}$$

$$= \frac{\sin x}{x}.$$

$$\therefore \prod_{n=1}^{\infty}\sqrt{\frac{1}{1-\left(\omega(t_f-t_i)/\pi n\right)^2}} = \sqrt{\frac{\omega(t_f-t_i)}{\sin\omega(t_f-t_i)}}$$

$\therefore\ Z(t_f, q_f ; t_i, q_i)$

$$= e^{\frac{i}{\hbar} S[q_{cl}]} \sqrt{\frac{m}{2\pi\hbar i (t_f - t_i)}} \cdot \sqrt{\frac{\omega(t_f - t_i)}{\sin\omega(t_f - t_i)}}$$

$$= \sqrt{\frac{m\omega}{2\pi\hbar i \sin\omega(t_f - t_i)}}\ e^{\frac{i}{\hbar} S[q_{cl}]}$$

$$= \sqrt{\frac{m\omega}{2\pi\hbar i \sin\omega(t_f - t_i)}}\ e^{\frac{i m\omega}{2\hbar \sin\omega(t_f - t_i)}\left[(q_f^2 + q_i^2)\cos\omega(t_f - t_i) - 2 q_f q_i\right]}.$$

Partition function

... Three ways

(i) In Operator formalism

(ii) Direct Path-integral

(iii) Use $\mathrm{Tr}\, e^{-\frac{T}{\hbar}\hat{H}} = \int dq \langle q | e^{-\frac{T}{\hbar}\hat{H}} | q \rangle$

(i) In operator formalism.

$$\mathrm{Tr}_{\mathcal{H}}\, e^{-\frac{T}{\hbar}\hat{H}} = \sum_{n=0}^{\infty} e^{-\frac{T}{\hbar}\hbar\omega(n+\frac{1}{2})} = e^{-\omega T/2} \sum_{n=0}^{\infty} e^{-\omega T n}$$

$$= e^{-\omega T/2} \cdot \frac{1}{1-e^{-\omega T}} = \frac{1}{e^{\omega T/2} - e^{-\omega T/2}}$$

(ii) Direct path-integral.

A periodic configuration $q(\tau+T) = q(\tau)$ can be written as

$$q(\tau) = \frac{q_0}{\sqrt{T}} + \sum_{n=1}^{\infty} \left(q_n^c \sqrt{\frac{2}{T}} \cos\left(\frac{2\pi n}{T}\tau\right) + q_n^s \sqrt{\frac{2}{T}} \sin\left(\frac{2\pi n}{T}\tau\right) \right).$$

For this,

$$S_E[q] = \frac{m\omega^2}{2} q_0^2 + \sum_{n=1}^{\infty} \left(\frac{m}{2}\left(\frac{2\pi n}{T}\right)^2 + \frac{m\omega^2}{2} \right)\left(q_n^{c\,2} + q_n^{s\,2} \right).$$

$$\mathcal{D}q = dq_0 \cdot \prod_{n=1}^{\infty} dq_n^c \, dq_n^s$$ & use ζ-function regularization.

(We know $\mathcal{D}q = dq_0 \cdot \prod_{n=1}^{\infty} c \, dq_n^c \, dq_n^s$ with $c=1$ works!)

$$Z(S^1_T) = \int \mathcal{D}q \; e^{-\frac{1}{\hbar} S_E[q]}$$

$$= \int dq_0 \prod_{n=1}^{\infty} dq_n^c \, dq_n^s \; e^{-\frac{1}{\hbar}\left\{ \frac{m\omega^2}{2} q_0^2 + \sum_{n=1}^{\infty} \frac{m}{2}\left(\left(\frac{2\pi n}{T}\right)^2 + \omega^2\right)\left(q_n^{c\,2} + q_n^{s\,2}\right)\right\}}$$

$$= \int dq_0 \; e^{-\frac{m\omega^2}{2\hbar} q_0^2} \prod_{n=1}^{\infty} \int dq_n^c \, dq_n^s \; e^{-\frac{m}{2\hbar}\left(\left(\frac{2\pi n}{T}\right)^2 + \omega^2\right)\left(q_n^{c\,2} + q_n^{s\,2}\right)}$$

$$= \sqrt{\frac{2\pi}{m\omega^2/\hbar}} \cdot \underbrace{\prod_{n=1}^{\infty} \frac{2\pi}{m\left(\left(\frac{2\pi n}{T}\right)^2 + \omega^2\right)/\hbar}}_{\underbrace{\prod_{n=1}^{\infty} \frac{\hbar T^2}{2\pi m n^2}}_{\text{ζ-fcn reg.}\ \| \ \sqrt{\frac{m}{2\pi \hbar T^2}}} \cdot \underbrace{\prod_{n=1}^{\infty} \frac{1}{1 + \left(\frac{\omega T}{2\pi n}\right)^2}}_{\| \ \prod_{n=1}^{\infty} \frac{1}{1 - \left(\frac{\omega t}{2\pi n}\right)^2}\Big|_{t \to -iT}}}$$

Use $\prod_{n=1}^{\infty}\left(1-\left(\frac{x}{\pi n}\right)^2\right) = \frac{\sin x}{x}$ again:

$$\prod_{n=1}^{\infty}\left(1-\left(\frac{\omega t}{2\pi n}\right)^2\right) = \frac{\sin(\omega t/2)}{\omega t/2} \xrightarrow{t\to -iT} \frac{\sinh(\omega T/2)}{\omega T/2}$$

$$\therefore\ Z(S^1_T) = \sqrt{\frac{2\pi\hbar}{m\omega^2}} \cdot \sqrt{\frac{m}{2\pi\hbar T^2}} \cdot \frac{\omega T/2}{\sinh(\omega T/2)}$$

$$= \frac{1}{2\sinh(\omega T/2)} = \frac{1}{e^{\omega T/2} - e^{-\omega T/2}}$$

(iii) Use the result for $Z(t_f, q_f; t_i, q_i)$ in

$$\mathrm{Tr}_{\mathcal{H}}\left(e^{-\frac{T}{\hbar}\hat{H}}\right) = \int dq\, \underbrace{\langle q| e^{-\frac{T}{\hbar}\hat{H}} |q\rangle}_{Z(t,q;0,q)\big|_{t\to -iT}}$$

$Z(t,q;0,q)$

$$= \sqrt{\frac{m\omega}{2\pi\hbar i \sin\omega t}}\ e^{\frac{im\omega}{2\hbar \sin\omega t}\left[2q^2\cos\omega t - 2q^2\right]}$$

For $t \to -iT$, $\quad \sin\omega t \to -i\sinh\omega T$

$\cos\omega t \to \cosh\omega T$

$$2\cosh\omega T - 2 = e^{\omega T} + e^{-\omega T} - 2 = \left(e^{\omega T/2} - e^{-\omega T/2}\right)^2$$

$\therefore\ Z(t, q; 0, q)$

$$\to \sqrt{\frac{m\omega}{2\pi\hbar\sinh\omega T}}\, e^{-\frac{m\omega}{2\hbar\sinh\omega T}\left(e^{\omega T/2} - e^{-\omega T/2}\right)^2 q^2}$$

$$\therefore \int dq\, Z(t, q; 0, q)\Big|_{t\to -iT}$$

$$= \int dq \sqrt{\frac{m\omega}{2\pi\hbar\sinh\omega T}}\, e^{-\frac{m\omega}{2\hbar\sinh\omega T}\left(e^{\omega T/2} - e^{-\omega T/2}\right)^2 q^2}$$

$$= \sqrt{\frac{m\omega}{2\pi\hbar\sinh\omega T}} \cdot \sqrt{\frac{2\pi}{m\omega\left(e^{\omega T/2} - e^{-\omega T/2}\right)^2/\hbar\sinh\omega T}}$$

$$= \frac{1}{e^{\omega T/2} - e^{-\omega T/2}}$$

The results of (i), (ii), (iii) all match. ✓

③ Free particle in a circle $S^1_{2\pi R} = \mathbb{R}/2\pi R\mathbb{Z}$.

$$L = \frac{m}{2}\dot{q}^2 \quad \text{where} \quad q \sim q + 2\pi R.$$

The wave function must be periodic, $\Psi(q+2\pi R) \overset{!}{=} \Psi(q)$.

$$\|\Psi\|^2 := \int_0^{2\pi R} dq\, |\Psi(q)|^2.$$

$$[\hat{q}, \hat{p}] = i\hbar \quad \rightsquigarrow \quad \hat{p} = -i\hbar\frac{d}{dq}, \quad \hat{H} = \frac{1}{2m}\hat{p}^2.$$

$\hat{p} = p$ on $e^{ipq/\hbar}$

this needs to be $q \to q+2\pi R$ periodic

$$\Rightarrow \quad e^{2\pi i pR/\hbar} \overset{!}{=} 1 \quad \text{i.e.} \quad pR/\hbar \overset{!}{\in} \mathbb{Z}$$

$\therefore\ p = \dfrac{\hbar n}{R} \quad (n \in \mathbb{Z})$. Momentum is quantized.

$$\Psi_n(q) = \frac{1}{\sqrt{2\pi R}} e^{inq/R} \quad \left(\begin{array}{c}\text{normalized so that} \\ (\Psi_n, \Psi_m) = \delta_{n,m}\end{array}\right)$$

$$\hat{H}\Psi_n = E_n\Psi_n, \quad E_n = \frac{1}{2m}\left(\frac{\hbar n}{R}\right)^2.$$

transition amplitude

(i) operator :

$$Z(t_f, q_f ; t_i, q_i) = \langle q_f | e^{-\frac{i(t_f-t_i)}{\hbar}\hat{H}} | q_i \rangle$$

$$= \sum_{n\in\mathbb{Z}} \underbrace{\langle q_f | n \rangle}_{\Psi_n(q_f) = \frac{1}{\sqrt{2\pi R}} e^{inq_f/R}} e^{-i\frac{t_f-t_i}{\hbar}\frac{1}{2m}\left(\frac{\hbar n}{R}\right)^2} \underbrace{\langle n | q_i \rangle}_{\Psi_n(q_i)^* = \frac{1}{\sqrt{2\pi R}} e^{-inq_i/R}}$$

$$= \frac{1}{2\pi R} \sum_{n\in\mathbb{Z}} e^{-i\frac{t_f-t_i}{2m}\hbar\left(\frac{n}{R}\right)^2 + i\frac{n(q_f-q_i)}{R}}$$

(ii) Path-integral :

$q(t)$ s.t. $q(t_i) \sim q_i$, $q(t_f) \sim q_f$ can be written as

$$q(t) = q_i + (q_f - q_i)\frac{t-t_i}{t_f-t_i} + 2\pi R w \frac{t-t_i}{t_f-t_i} + \sum_{n=1}^{\infty} \sqrt{\frac{2}{t_f-t_i}}\, q_n \sin\left(\pi n \frac{t-t_i}{t_f-t_i}\right)$$

where $w \in \mathbb{Z}$ (winding #)

For this,

$$S[q] = (t_f - t_i)\frac{m}{2}\left(\frac{q_f - q_i + 2\pi R w}{t_f - t_i}\right)^2 + \sum_{n=1}^{\infty} \frac{m}{2}\left(\frac{\pi n}{t_f-t_i}\right)^2 q_n^2 .$$

$$\int \mathcal{D}q = \sum_{w\in\mathbb{Z}} \int C \prod_{n=1}^{\infty} dq_n$$

$$Z(t_f, q_f; t_i, q_i) = \sum_{w\in\mathbb{Z}} e^{\frac{i}{\hbar}\frac{m(q_f-q_i+2\pi R w)^2}{2(t_f-t_i)}} \cdot \underbrace{C \prod_{n=1}^{\infty} \int dq_n \, e^{\frac{i}{\hbar}\frac{m}{2}\left(\frac{\pi n}{t_f-t_i}\right)^2 q_n^2}}$$

Same as in free particle in $\mathbb{R}$.

Use the same definition $\rightarrow$ $\|$

$$\sqrt{\frac{m}{2\pi\hbar i(t_f-t_i)}}$$

$$= \sqrt{\frac{m}{2\pi\hbar i(t_f-t_i)}} \sum_{w\in\mathbb{Z}} e^{\frac{i}{\hbar}\frac{m(q_f-q_i+2\pi R w)^2}{2(t_f-t_i)}} .$$

Exercise Show that the results of (i) & (ii) match.

You may use

Poisson resummation formula

$$\sum_{n\in\mathbb{Z}} e^{-\pi a n^2 + 2\pi i b n} = a^{-\frac{1}{2}} \sum_{w\in\mathbb{Z}} e^{-\frac{\pi}{a}(w+b)^2} .$$

Partition function

(i) Operator

$$\mathrm{Tr}_{\mathcal{H}}\, e^{-\frac{T}{\hbar}\hat{H}} = \sum_{n\in\mathbb{Z}} e^{-\frac{T}{\hbar}\frac{1}{2m}\left(\frac{n\hbar}{R}\right)^2} = \sum_{n\in\mathbb{Z}} e^{-\frac{\hbar T}{2m}\left(\frac{n}{R}\right)^2}.$$

(ii) path-integral

A periodic configuration $q(\tau+T) \sim q(\tau)$ can be written as

$$q(\tau) = \frac{1}{\sqrt{T}} q_0 + 2\pi R w \frac{\tau}{T} + \sum_{n=1}^{\infty}\left(q_n^c \sqrt{\frac{2}{T}} \cos\left(\frac{2\pi n\tau}{T}\right) + q_n^s \sqrt{\frac{2}{T}} \sin\left(\frac{2\pi n\tau}{T}\right)\right)$$

where $q_0 \sim q_0 + \sqrt{T}\cdot 2\pi R$ and $w \in \mathbb{Z}$ (winding #).

$$S_E[q] = T\cdot\frac{m}{2}\left(\frac{2\pi R w}{T}\right)^2 + \sum_{n=1}^{\infty}\frac{m}{2}\left(\frac{2\pi n}{T}\right)^2\left((q_n^c)^2 + (q_n^s)^2\right).$$

$$\int \mathcal{D}q = \sum_{w\in\mathbb{Z}} \int dq_0 \prod_{n=1}^{\infty} dq_n^c\, dq_n^s$$

where q_0 integration is on $[0, \sqrt{T}\cdot 2\pi R]$

& ζ-fcn reg is used for $\prod_{n=1}^{\infty}$.

$$Z(S^1_T) = \int \mathcal{D}q \, e^{-\frac{1}{\hbar} S_E[q]}$$

$$= \sum_{w\in\mathbb{Z}} \sqrt{T}\cdot 2\pi R \cdot e^{-\frac{m}{2\hbar}\frac{(2\pi R w)^2}{T}} \underbrace{\prod_{n=1}^{\infty} \int dq_n^c \, dq_n^s \, e^{-\frac{m}{2\hbar}\left(\frac{2\pi n}{T}\right)^2 (q_n^{c\,2} + q_n^{s\,2})}}_{\text{same as in free particle in } \mathbb{R}}$$

ζ-fcn reg $\rightarrow$ $\|$

$$\sqrt{\frac{m}{2\pi\hbar T^2}}$$

$$= 2\pi R \cdot \sqrt{\frac{m}{2\pi\hbar T}} \sum_{w\in\mathbb{Z}} e^{-\frac{m}{2\hbar T}(2\pi R w)^2}$$

<u>Exercise</u> Show that the results of (i) & (ii) match

(Again, Poisson resummation formula can be used.)

(iii) We may also use the result for $Z(t_f, q_f ; t_i, q_i)$ in

$$\mathrm{Tr}\left(e^{-\frac{T}{\hbar}\hat{H}}\right) = \int_{S^1_{2\pi R}} dq \, \underbrace{\langle q | e^{-\frac{T}{\hbar}\hat{H}} | q \rangle}_{Z(t, q ; 0, q)\big|_{t \to -iT}}$$

Recall

$$Z(t,q;0,q)\Big|_{t\to -iT}$$

$$= \begin{cases} \dfrac{1}{2\pi R}\displaystyle\sum_{n\in\mathbb{Z}} e^{-\frac{\hbar T}{2m}\left(\frac{n}{R}\right)^2} & \text{op.} \\ \sqrt{\dfrac{m}{2\pi\hbar T}}\displaystyle\sum_{w\in\mathbb{Z}} e^{-\frac{m(2\pi R w)^2}{2\hbar T}} & \text{P.I.} \end{cases}$$

$$\therefore \int_{S^1_{2\pi R}} dq \; Z(t,q;0,q)\Big|_{t\to -iT}$$

$$= 2\pi R \cdot Z(t,q;0,q)\Big|_{t\to -iT}$$

= nothing but the results in (i) & (ii).

Remark

In the limit $R \to \infty$, the circle $S^1_{2\pi R}$ decompactifies to the line $\mathbb{R}$.

Let us see what happens to the partition function

$$Z(S^1_T) = \begin{cases} \sum_{n \in \mathbb{Z}} e^{-\frac{\hbar T}{2m}\left(\frac{n}{R}\right)^2} & (\mathrm{OP}) \\ 2\pi R \sqrt{\frac{m}{2\pi\hbar T}} \sum_{w \in \mathbb{Z}} e^{-\frac{m(2\pi R w)^2}{2\hbar T}} & (\mathrm{P.I.}) \end{cases}$$

Of course, it diverges in the limit. But the two different expressions tell us different reasons (which must be equivalent):

- In (OP), all $n \in \mathbb{Z}$ contributes with weight 1, and the divergence is because the number of states is ∞. In fact, even below a finite energy $E = \frac{1}{2m}\left(\frac{\hbar n}{R}\right)^2 \leq E_0$, # of states is ∞ in the limit (the spectrum becomes continuous).

- In (P.I.), as $R \to \infty$, the $w=0$ term is dominant. $Z(S^1_T) \sim 2\pi R \sqrt{\frac{m}{2\pi\hbar T}}$, and the divergence is because the volume is ∞. (Recall that $L \cdot \sqrt{\frac{m}{2\pi\hbar T}}$ is the regularized version of $Z(S^1_T)$ for the particle in a line.)

3 Particle in S^1 with a theta term

$$q \sim q + 2\pi R$$

$$L = \frac{m}{2}\dot{q}^2 + \theta\dot{q}$$

Called theta term

Exercises :

1. Find the Hamitonian $H(p, q)$.
2. Find the spectrum.
3. Find the Euclidean Lagrangian $L_E(q, \dot{q})$.
   (Warning : it is not real !)
4. Compute $Z(t_f, q_f, t_i, q_i)$ in Operator & path-integral.
   (check that the result match.)
5. Compute $\mathrm{Tr}_{\mathcal{H}}(e^{-\frac{T}{\hbar}\hat{H}}) = Z(S^1_T)$
   in Operator & path-integral.
   (check that the result match.)