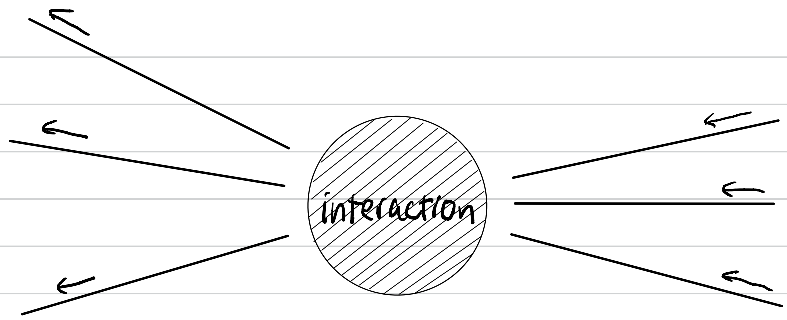
Recap



$t=+\infty$ $\qquad t=0 \qquad$ $t=-\infty$

$$|\psi\rangle_{in} \longleftarrow |\psi, \text{free}\rangle$$

$$|\chi, \text{free}\rangle \longleftarrow |\chi\rangle_{out}$$

$$\langle \chi, \text{free} | S | \psi, \text{free}\rangle := {}_{out}\langle \chi | \psi \rangle_{in}$$

$\mathcal{O} = \mathcal{O}^\dagger$ a scalar opr s.t. $\langle 0 | \mathcal{O}(x) | 0 \rangle = 0$

$\langle p | \mathcal{O}(x) | 0 \rangle = \sqrt{Z}\, e^{i p x}$

- wave packet $f(x)$, $\mathcal{O}_f(t) = \frac{-i}{\sqrt{Z}} \int d^4\mathbf{x}\, f(t,\mathbf{x}) \overleftrightarrow{\partial_t} \mathcal{O}(t,\mathbf{x})$

$$\mathcal{O}_f(t)|0\rangle \xrightarrow{t\to\pm\infty} |f\rangle = |f\rangle_{in} = |f\rangle_{out}$$

$$\mathcal{O}_f(t)^\dagger |0\rangle \xrightarrow{t\to\pm\infty} 0$$

- $f_1, \dots, f_n$ widely separated at $t \to \pm\infty$:

$$|f_1, \dots, f_n\rangle_{in/out} = \lim_{T\to\mp\infty} \mathcal{O}_{f_1}(T) \cdots \mathcal{O}_{f_n}(T)|0\rangle$$

$$\overset{!}{=} \lim_{T_1,\dots,T_n\to\mp\infty} T\, \mathcal{O}_{f_1}(T_1) \cdots \mathcal{O}_{f_n}(T_n)|0\rangle$$

LSZ reduction formula

$$\langle g_1, \cdots, g_n \text{ free} | S | f_1 f_2 \text{ free} \rangle = {}_{out}\langle g_1, \cdots, g_n | f_1 f_2 \rangle_{in} = ?$$

$$\mathcal{O}_f(-T) - \mathcal{O}_f(T) = -\int_{-T}^{T} dt \frac{\partial}{\partial t} \mathcal{O}_f(t)$$

$$= \int_{-T}^{T} dt \frac{i}{\sqrt{Z}} \int_{\mathbb{R}^{d-1}} d^{d-1}\mathbf{x}\, \underbrace{\partial_t \left(f \partial_t \mathcal{O} - \partial_t f \mathcal{O} \right)}_{f \partial_t^2 \mathcal{O} - \underbrace{\partial_t^2 f}_{= (\nabla^2 - m^2) f} \mathcal{O}} (t, \mathbf{x})$$

[as $f(t, \mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$,

spatial partial integration is allowed.

$$= \int_{-T}^{T} dt \int_{\mathbb{R}^{d-1}} d^{d-1}\mathbf{x}\, f(t, \mathbf{x}) \frac{i}{\sqrt{Z}} \underbrace{\left(\partial_t^2 - \nabla^2 + m^2 \right)}_{\partial_x^2 + m^2} \mathcal{O}(t, \mathbf{x})$$

$$=: \int_{-T}^{T} d^d x\, f(x) \frac{i}{\sqrt{Z}} (\partial^2 + m^2) \mathcal{O}(x)$$

Taking its adjoint

$$\mathcal{O}_f(T)^\dagger - \mathcal{O}_f(-T)^\dagger = \int_{-T}^{T} d^d x\, f(x) \frac{i}{\sqrt{Z}} (\partial^2 + m^2) \mathcal{O}(x)$$

Consider $X_{T_1 \cdots T_n, T_1', T_2'} :=$

$$\prod_{i=1}^{n} \int_{-T_i}^{T_i} d^4 y_i \; g_i(y_i)^* \frac{i}{\sqrt{Z}} (\partial_{y_i}^2 + m^2) \prod_{j=1}^{2} \int_{-T_j'}^{T_j'} d^4 x_j \; f_j(x_j) \frac{i}{\sqrt{Z}} (\partial_{x_j}^2 + m^2)$$

$$\langle 0 | T\, \mathcal{O}(y_1) \cdots \mathcal{O}(y_n) \mathcal{O}(x_1) \mathcal{O}(x_2) | 0 \rangle .$$

$$\int_{-T_1}^{T_1} d^4 y_1 \; g_1(y_1)^* \frac{i}{\sqrt{Z}} (\partial_{y_1}^2 + m^2) \langle 0 | T\, \mathcal{O}(y_1) \cdots \mathcal{O}(x_2) | 0 \rangle$$

$$\xrightarrow[T_1 \to \infty]{} \langle 0 | \mathcal{O}_{g_1}(\infty)^\dagger \, T(\mathcal{O}(y_2) \cdots \mathcal{O}(x_2)) | 0 \rangle$$

$$- \langle 0 | T(\mathcal{O}(y_2) \cdots \mathcal{O}(x_2)) \underbrace{\mathcal{O}_{g_1}(-\infty)^\dagger | 0 \rangle}_{= 0}$$

Thus

$$X_{\vec{T}, \vec{T}'} \xrightarrow{T_1 \to \infty} \prod_{i=2}^{n} \int_{-T_i}^{T_i} \cdots \prod_{j=1}^{2} \int_{-T_j'}^{T_j'} \cdots \langle 0 | \mathcal{O}_{g_1}(\infty)^\dagger \, T(\mathcal{O}(y_2) \cdots \mathcal{O}(x_2)) | 0 \rangle$$

Repeating this for $T_2, \cdots, T_n$, we find

$$X_{\vec{T}, \vec{T}'} \xrightarrow{T_1, T_2, \cdots T_n \to \infty} \prod_{j=1}^{2} \int_{-T_j'}^{T_j'} \cdots \underbrace{\langle 0 | \mathcal{O}_{g_1}(\infty)^\dagger \cdots \mathcal{O}_{g_n}(\infty)^\dagger}_{{}_{\text{out}}\langle g_1, \cdots, g_n |} \, T(\mathcal{O}(x_1) \mathcal{O}(x_2)) | 0 \rangle$$

Further limits :

$$\xrightarrow{T_1' \to \infty} \int_{-T_2'}^{T_2'} d^d x_2\, f_2(x_2) \frac{i}{\sqrt{Z}} (\partial_{x_2}^2 + m^2)$$

$$\left({}_{out}\langle g_1, \cdots, g_n | \mathcal{O}(x_2)\, \mathcal{O}_{f_1}(-\infty) | 0 \rangle - {}_{out}\langle g_1, \cdots, g_n | \mathcal{O}_{f_1}(\infty)\, \mathcal{O}(x_2) | 0 \rangle \right)$$

$$\xrightarrow{T_2' \to \infty} {}_{out}\langle g_1, \cdots g_n | \underbrace{\mathcal{O}_{f_2}(-\infty)\, \mathcal{O}_{f_1}(-\infty) | 0 \rangle}_{|f_1, f_2\rangle_{in}}$$

$$- {}_{out}\langle g_1, \cdots g_n | \underbrace{\mathcal{O}_{f_2}(\infty) \underbrace{\mathcal{O}_{f_1}(-\infty) | 0 \rangle}_{\mathcal{O}_{f_1}(\infty)|0\rangle}}_{|f_1, f_2\rangle_{out}}$$

$$- {}_{out}\langle g_1, \cdots, g_n | \underbrace{\mathcal{O}_{f_1}(\infty) \underbrace{\mathcal{O}_{f_2}(-\infty) | 0 \rangle}_{\mathcal{O}_{f_1}(\infty)|0\rangle}}_{|f_1, f_2\rangle_{out}}$$

$$+ {}_{out}\langle g_1, \cdots, g_n | \underbrace{\mathcal{O}_{f_1}(\infty)\, \mathcal{O}_{f_2}(\infty) | 0 \rangle}_{|f_1, f_2\rangle_{out}}$$

$$= {}_{out}\langle g_1, \cdots, g_n | f_1, f_2 \rangle_{in} - {}_{out}\langle g_1, \cdots, g_n | f_1, f_2 \rangle_{out}$$

$$\cancel{- {}_{out}\langle g_1, \cdots, g_n | f_1, f_2 \rangle} + \cancel{{}_{out}\langle g_1, \cdots, g_n | f_1, f_2 \rangle_{out}}$$

$$= \langle g_1, \cdots, g_n, \text{free} | S | f_1, f_2, \text{free} \rangle - \langle g_1, \cdots, g_n, \text{free} | f_1, f_2, \text{free} \rangle$$

We obtained

$$\langle g_1, \cdots, g_n, \text{free} | S | f_1, f_2, \text{free} \rangle$$

$$= \langle g_1, \cdots, g_n, \text{free} | f_1, f_2, \text{free} \rangle$$

$$+ \prod_{i=1}^{n} \int d^d y_i \; g_i(y_i)^* \frac{i}{\sqrt{Z}} (\partial_{y_i}^2 + m^2) \prod_{j=1}^{2} \int d^d x_j \; f_j(x_j) \frac{i}{\sqrt{Z}} (\partial_{x_j}^2 + m^2)$$

$$\langle 0 | T \, \mathcal{O}(y_1) \cdots \mathcal{O}(y_n) \mathcal{O}(x_1) \mathcal{O}(x_2) | 0 \rangle$$

S-matrix is described by correlation functions.

This is the **LSZ reduction formula**.

Lehmann, Symanzik, Zimmermann

There is also a formula for $\langle g_1, \cdots, g_n, \text{free} | S | f_1, \cdots, f_m, \text{free} \rangle$

which is a bit more complicated.

It simplifies if $\langle g_a | f_b \rangle = 0 \quad \forall a, b$:

$$\langle g_1, \cdots, g_n, \text{free} | S | f_1, \cdots, f_m, \text{free} \rangle$$

$$= \prod_{i=1}^{n} \int d^d y_i \; g_i(y_i)^* \frac{i}{\sqrt{Z}} (\partial_{y_i}^2 + m^2) \prod_{j=1}^{m} \int d^d x_j \; f_j(x_j) \frac{i}{\sqrt{Z}} (\partial_{x_j}^2 + m^2)$$

$$\langle 0 | T \, \mathcal{O}(y_1) \cdots \mathcal{O}(y_n) \mathcal{O}(x_1) \cdots \mathcal{O}(x_m) | 0 \rangle .$$

In Fourier modes :

$$\langle 0| T\, \mathcal{O}(x_1) \cdots \mathcal{O}(x_s) |0\rangle = \int \prod_{i=1}^{s} \frac{d^d p_i}{(2\pi)^d}\, e^{-i p_i x_i}\, G(p_1, \dots, p_s)$$

$$\langle g_1, \dots, g_n \text{ free} |\, (S-1)\, | f_1, f_2, \text{free} \rangle$$

$$= \int \prod_{i=1}^{n} d^d y_i\, \frac{d^d l_i}{(2\pi)^d}\, g_i(y_i)^*\, e^{-i l_i y_i}\, \frac{i}{\sqrt{Z}} (-l_i^2 + m^2)$$
$$\prod_{j=1}^{2} d^d x_j\, \frac{d^d k_j}{(2\pi)^d}\, f_j(x_j)\, e^{i k_j x_j}\, \frac{i}{\sqrt{Z}} (-k_j^2 + m^2)$$
$$G(l_1, \dots, l_n, k_1, k_2)$$

$$\left[\; f(x) = \int \frac{d^{d-1}\mathbb{p}}{(2\pi)^{d-1} 2\omega_{\mathbb{p}}}\, e^{-i p_{\mathbb{p}} x}\, \tilde{f}(\mathbb{p}) \right.$$

$$\left. \int d^d x\, \frac{d^d k}{(2\pi)^d}\, f(x)\, e^{-ikx}\, F(k) = \int \frac{d^{d-1}\mathbb{p}}{(2\pi)^{d-1} 2\omega_{\mathbb{p}}}\, \tilde{f}(\mathbb{p})\, F(k) \Big|_{k = -p_{\mathbb{p}}} \right.$$

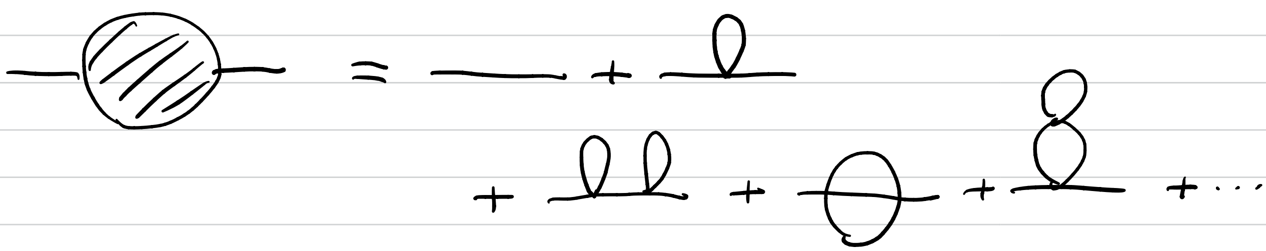
$$= \int \prod_{i=1}^{n} \frac{d^{d-1} \mathbb{q}_i}{(2\pi)^{d-1} 2\omega_{\mathbb{q}_i}}\, \tilde{g}_i(\mathbb{q}_i)^* \prod_{j=1}^{2} \frac{d^{d-1}\mathbb{p}_j}{(2\pi)^{d-1} 2\omega_{\mathbb{p}_j}}\, \tilde{f}_j(\mathbb{p}_j)$$
$$\prod_{i=1}^{n} \frac{-i(l_i^2 - m^2)}{\sqrt{Z}} \prod_{j=1}^{2} \frac{-i(k_j^2 - m^2)}{\sqrt{Z}}\, G(l_1, \dots, l_n, k_1, k_2) \Bigg|_{\substack{l_i = p_{\mathbb{q}_i} \\ k_j = -p_{\mathbb{p}_j}}}$$

S-matrix in perturbation theory

— Diagrammatic expressions

ϕ : an elementary field s.t. $\langle \phi(x) \rangle = 0$

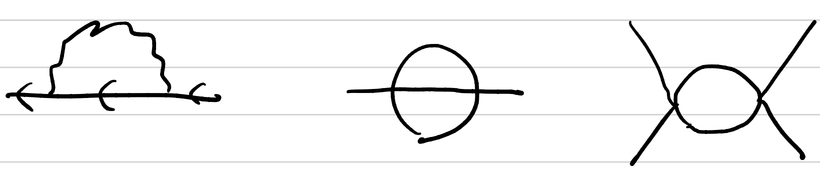
Its two point function $\langle \phi(x)\phi(y) \rangle$ is a sum of connected diagrams



1PI (one particle irreducible) diagram

= a connected diagram which is still connected if an internal line is cut.

e.g.

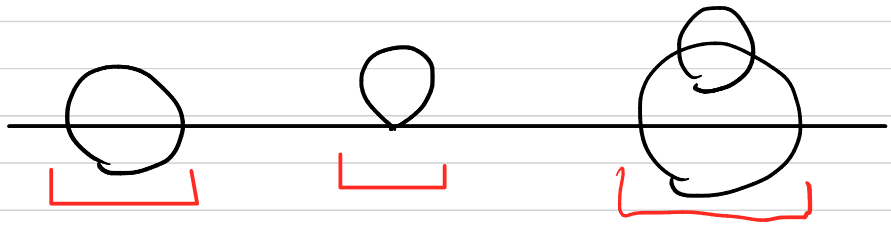


1PI

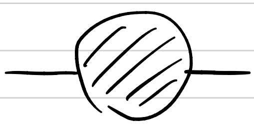
not 1PI

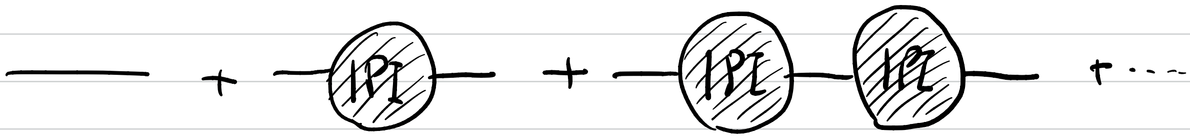
Any connected diagram with two external lines can be decomposed into 1PI blocks :

e.g.



three 1PI blocks



= 

no 1PI   one 1PI   two 1PI's

$$= \frac{i}{p^2-m_0^2} + \frac{i}{p^2-m_0^2}\left(-iM^2(p^2)\right)\frac{i}{p^2-m_0^2}$$

$+i0$ omitted

$$+ \frac{i}{p^2-m_0^2}\left(-iM^2(p^2)\right)\frac{i}{p^2-m_0^2}\left(-iM^2(p^2)\right)\frac{i}{p^2-m_0^2} + \cdots$$

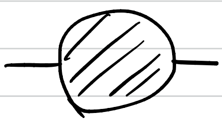
$$= \frac{i}{p^2-m_0^2}\left(1+\frac{M^2(p^2)}{p^2-m_0^2}+\left(\frac{M^2(p^2)}{p^2-m_0^2}\right)^2+\cdots\right)$$

$$= 1\Big/\left(1-\frac{M^2(p^2)}{p^2-m_0^2}\right)$$

$$= \frac{i}{p^2-m_0^2-M^2(p^2)}$$

A zero of $p^2 - m_0^2 - M^2(p^2)$ is mass^2 of a particle.

e.g. if $M^2(p^2) = \alpha p^2 + \beta$


$$= \frac{i}{(1-\alpha)p^2 - m_0^2 - \beta} = \frac{(1-\alpha)^{-1} i}{p^2 - (1-\alpha)^{-1}(m_0^2+\beta)}$$

$$\Rightarrow \quad m^2 = (1-\alpha)^{-1}(m_0^2+\beta), \quad Z = (1-\alpha)^{-1}.$$

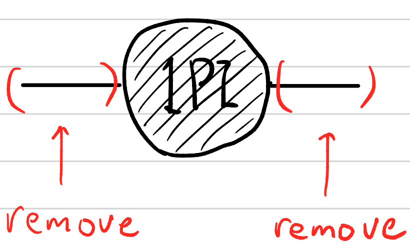
In general, if $p^2 - m_0^2 - M^2(p^2)$ has zeros at $p^2 = m_1^2, m_2^2, \dots$

then, these are mass^2 of particles and

$Z_i = |\langle \Omega, i|\phi(0)|0\rangle|^2$ for the i-th particle is given by

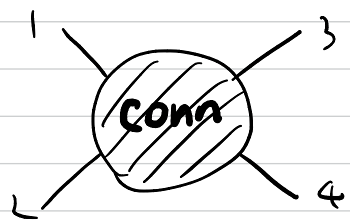
$$Z_i^{-1} = \frac{d}{dp^2}\left(p^2 - M^2(p^2)\right)\Big|_{p^2=m_i^2} = 1 - \frac{dM^2}{dp^2}(m_i^2).$$

Thus the spectrum of particles & their coupling to ϕ can be found by the sum of 1PI diagrams for $\langle \phi(x)\phi(y)\rangle$

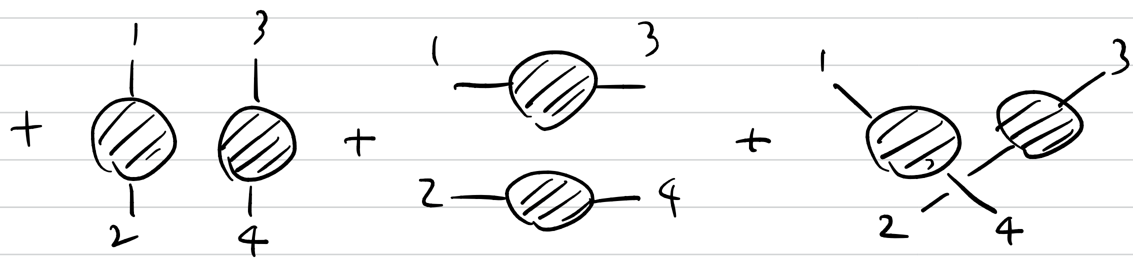


$$= -iM^2(p^2)$$

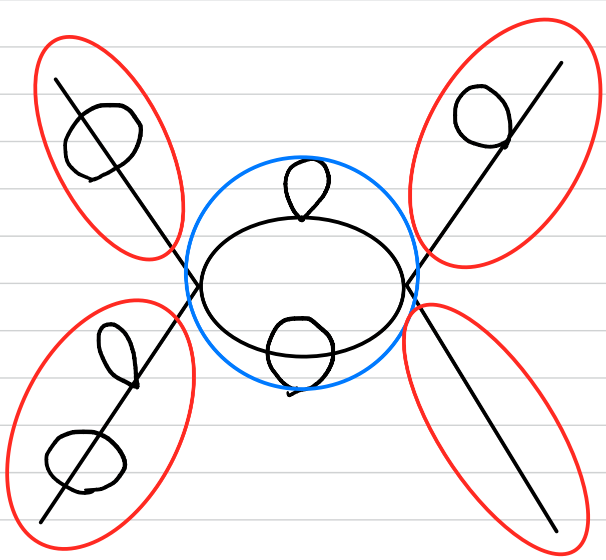
Four point function $\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle$

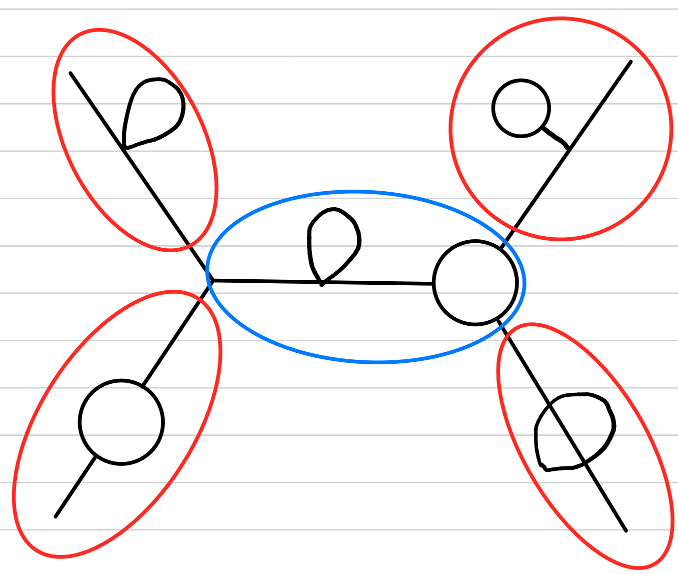
$=$ 

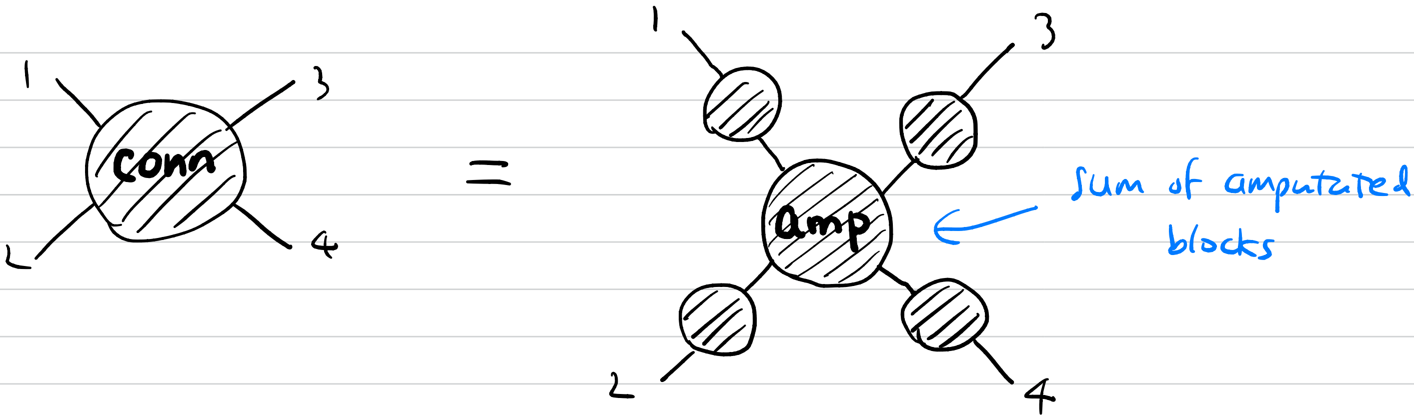
$\leftarrow$ connected



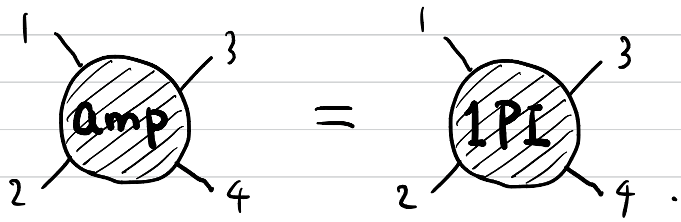
Any connected diagram with four external lines can be decomposed into four external parts and one amputated block



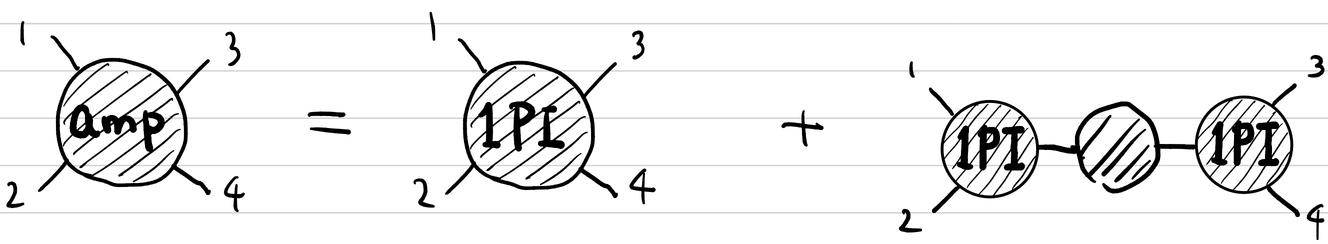


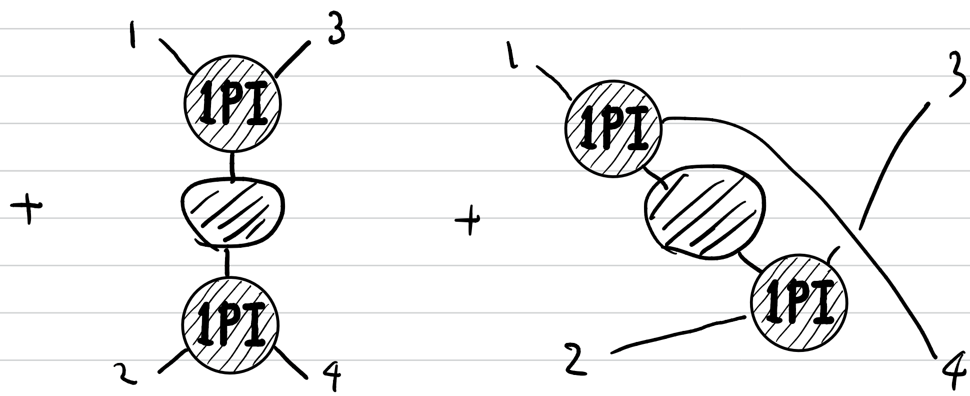


In ϕ^4 theory (or any theory with $\phi \to -\phi$ symmetry)



In general e.g. $U(\phi) = \frac{\lambda_3}{3!}\phi^3 + \frac{\lambda_4}{4!}\phi^4 + \dots$ with $\lambda_3 \neq 0$,





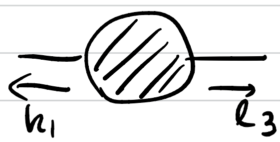
$$\langle g_3, g_4, \text{free} | (S-1) | f_1, f_2, \text{free} \rangle$$

$$= \int \prod_{i=3}^{4} \frac{d^{d-1} \mathbb{P}_i}{(2\pi)^{d-1} 2\omega^i_{\mathbb{P}_i}} \tilde{g}_i(\mathbb{P}_i)^* \prod_{j=1}^{2} \frac{d^{d-1}\mathbb{P}_j}{(2\pi)^{d-1} 2\omega^j_{\mathbb{P}_j}} \tilde{f}_j(\mathbb{P}_j)$$

$$\prod_{i=3}^{4} \frac{-i(\ell_i^2 - m_i^2)}{\sqrt{Z_i}} \prod_{j=1}^{2} \frac{-i(k_j^2 - m_j^2)}{\sqrt{Z_j}} G(\ell_3, \ell_4, k_1, k_2) \Bigg|_{\substack{k_j = -P^j_{\mathbb{P}_j} \\ \ell_i = P^i_{\mathbb{q}_i}}}$$

where $\omega^j_{\mathbb{P}} = \sqrt{\mathbb{P}^2 + m_j^2}$, $P^j_{\mathbb{P}} = (\omega^j_{\mathbb{P}}, \mathbb{P})$.

$$(\ell_3^2 - m_3^2)(k_1^2 - m_1^2)$$

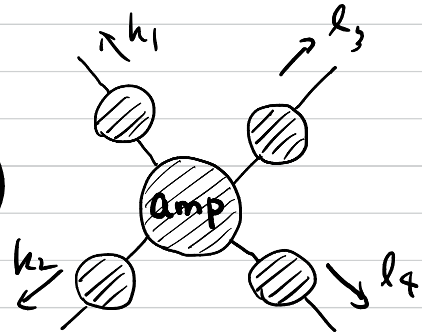


$$\propto \delta(k_1 + \ell_3)(\ell_3^2 - m_3^2)(k_1^2 - m_1^2)\left(\sum_i \frac{i Z_i}{k_1^2 - m_i^2} + \cdots \right)$$

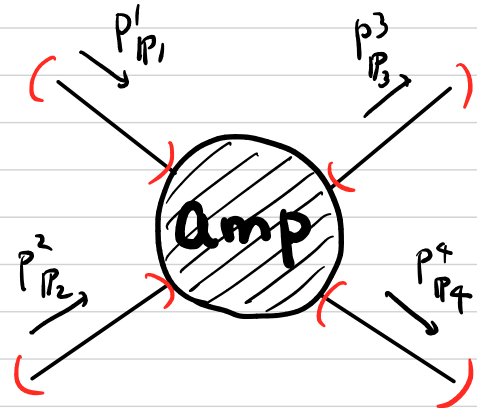
$$\longrightarrow 0 \quad \text{as} \quad \begin{matrix} k_1 \to -P^1_{\mathbb{P}_1} \\ \ell_3 \to P^3_{\mathbb{P}_3} \end{matrix} \quad \left(\Rightarrow \begin{matrix} k_1^2 \to m_1^2 \\ \ell_3^2 \to m_3^2 \end{matrix} \right)$$

Thus, the contribution of disconnected diagrams vanishes.

$$\prod_{l=3}^{4}\left(l_i^2-m_i^2\right)\prod_{j=1}^{2}\left(k_j^2-m_j^2\right)$$



$$\xrightarrow[\;]{\substack{l_i \,\to\, p^i_{\mathbf{p}_i} \\ k_j \,\to\, -p^j_{\mathbf{p}_j}}} \prod_{i=1}^{4} i Z_i \times$$

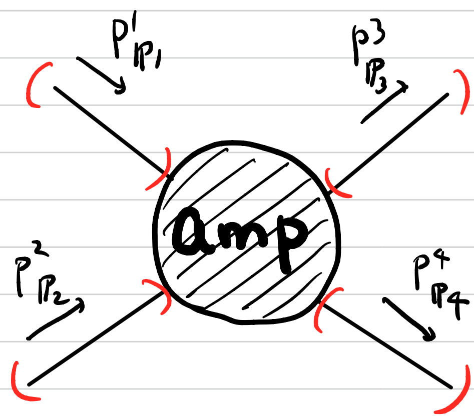


Thus

$$\langle g_3, g_4, \text{free} | (S-1) | f_1, f_2, \text{free} \rangle$$

$$= \int \prod_{i=3}^{4} \frac{d^{d-1}\mathbf{p}_i}{(2\pi)^{d-1} 2\omega^i_{\mathbf{p}_i}} \tilde{g}_i(\mathbf{p}_i)^* \prod_{j=1}^{2} \frac{d^{d-1}\mathbf{p}}{(2\pi)^{d-1} 2\omega^j_{\mathbf{p}_j}} \tilde{f}_j(\mathbf{p}_j)$$

$$\sqrt{Z_1}\sqrt{Z_2}\sqrt{Z_3}\sqrt{Z_4}$$



1PI effective action

Consider a theory of variables $\phi = (\phi_1, \cdots, \phi_N)$

measure $d\phi$ and action $S_E(\phi)$ (omit "E" below).

$$e^{-W(J)} = \int d\phi \; e^{-S(\phi) + J\cdot\phi}$$

Decompose $S(\phi) - J\cdot\phi = \underbrace{\frac{1}{2}\sum_{i,j} \phi_i A_{ij} \phi_j}_{\text{free part}} + \underbrace{\text{else}}_{\text{interaction part}}$

and evaluate $W(J)$ perturbatively.

※ Everything below is perturbation series but we omit writing "pert" each time.

e.g. $W(J) = W_{pert}(J)$ is the sum of connected diagram.

$$-\frac{\partial}{\partial J_i} W(J) = \frac{\int d\phi \; e^{-S(\phi)+J\cdot\phi} \phi_i}{\int d\phi \; e^{-S(\phi)+J\cdot\phi}} =: \langle \phi_i \rangle_J$$

Solve $\langle \phi_i \rangle_J \overset{!}{=} \phi_i$ $\quad i = 1, \cdots, N$ for J, write the solution

as $J = J(\phi)$ and put

$$\Gamma(\phi) := W(J(\phi)) + J(\phi)\cdot\phi$$

$$\frac{\partial\Gamma(\phi)}{\partial\phi_i} = \cancel{\frac{\partial J_j(\phi)}{\partial\phi_i}\cdot\underbrace{\frac{\partial W}{\partial J_j}(J(\phi))}_{\phi_j}} + \cancel{\frac{\partial J_j(\phi)}{\partial\phi_i}\cdot\phi_j} + J_i(\phi) = J_i(\phi),$$

$$i=1,\dots,N.$$

Thus,

$$\phi_i^* := \langle\phi_i\rangle_{J=0} \quad\Rightarrow\quad J(\phi^*)=0 \quad\therefore\quad \frac{\partial\Gamma}{\partial\phi_i}(\phi^*) = 0.$$

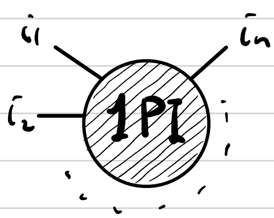
VEV of ϕ at $J=0$ is a critical point of $\Gamma(\phi)$.

Properties of $\Gamma(\phi)$

① It is a generating series of 1PI vertices

$$\Gamma(\phi) = \frac{1}{2}\log\det(A/2\pi) + \frac{1}{2}\sum_{i,j}\phi_i A_{ij}\phi_j - \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{i_1\cdots i_n}\lambda_{1PI}^{i_1\cdots i_n}\phi_{i_1}\cdots\phi_{i_n}$$

where $\lambda_{1PI}^{i_1\cdots i_n}$ is the 1PI vertex defined by

$$\langle\phi_{i_1}\cdots\phi_{i_n}\rangle_{1PI} =$$ 

$$= \sum_{j_1\cdots j_n}\overbrace{\phi_{i_1}\phi_{j_1}}\cdots\overbrace{\phi_{i_n}\phi_{j_n}}\,\lambda_{1PI}^{j_1\cdots j_n}$$

For this reason, $\Gamma(\phi)$ is called 1PI effective action.

② $\Gamma(\varphi) = \frac{1}{2}\log\det(A/2\pi)$ — The sum of 1PI vacuum diagrams of $\mathcal{T}(\varphi)$, the theory with background φ:

$$\begin{cases} \text{variables} & \xi = (\xi_1, \dots, \xi_N) \\ \text{measure} & d_\varphi \xi = d(\varphi + \xi) \\ \text{action} & S_\varphi(\xi) = S(\varphi + \xi) \end{cases}$$

$$\int d_\varphi \xi \; e^{-S_\varphi(\xi)} = \sqrt{\frac{(2\pi)^N}{\det A}} \; e^{\text{connected vacuum diagrams}}$$

$$= e^{-\Gamma(\varphi) + \text{non-1PI conn. vac. diagrams}}$$

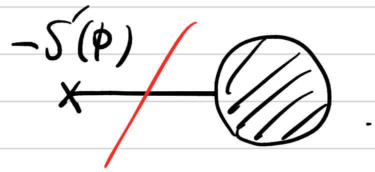
Here we take

$$S_\varphi(\xi) = \underbrace{\frac{1}{2}\xi_i A_{ij}\xi_j}_{\text{free part}} + \underbrace{\text{else}}_{\text{interaction part}}$$

③ This holds for any decomposition of $S_\varphi(\xi)$ into free + interaction. In particular, for the expansion in powers of ξ, we can take the ξ-quadratic part $\frac{1}{2}\sum_{ij}\xi_i\xi_j\partial_i\partial_j S(\varphi)$ as the free part.

$$S_\phi(\xi) = \underbrace{S(\phi)} + \underbrace{S'(\phi)\xi} + \underbrace{\frac{1}{2}S''(\phi)\xi^2}_{\text{free part}} + \underbrace{\frac{1}{3!}S'''(\phi)\xi^3 + \cdots}_{\text{interaction}}$$

- $S(\phi)$ is outside the ξ integral.
- Any diagram involving the vertex $-S'(\phi)\cdot\xi$ is not 1PI:



Thus, we can take only the cubic or higher powers in ξ as the interaction part to produce vertices.

With this understanding,

$$e^{-\Gamma(\phi)} = e^{-S(\phi)} \cdot \sqrt{\frac{(2\pi)^n}{\det S''(\phi)}} \cdot \exp\left(\text{1PI vacuum diagrams}\right).$$

That is,

$$\Gamma(\phi) = S(\phi) + \underbrace{\frac{1}{2}\log\det\left(\frac{S''(\phi)}{2\pi}\right)}_{\frac{1}{2}\operatorname{tr}\log\left(\frac{S''(\phi)}{2\pi}\right)} - \text{1PI vacuum diagrams}.$$

Consequence of 2 :

recover $\hbar$ $\quad \int d_\phi \xi \, e^{-\frac{1}{\hbar} S_\phi(\xi)} = e^{-\frac{1}{\hbar}\Gamma(\phi,\hbar) + \text{others}}$

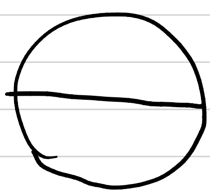
$\rightsquigarrow$ propagator $\propto \hbar$, vertex $\propto \hbar^{-1}$

A 1PI vacuum diagram with # propagator $= P$

vertices $= V$

$\propto \hbar^{P-V} = \hbar^{L-1}$

where $L = P - V + 1$ is # loops

e.g.  $\quad P = 3$, $V = 2$ $\quad L = 3 - 2 + 1 = 2$

Thus, $\Gamma(\phi,\hbar) = \sum_{L=0}^{\infty} \hbar^L \Gamma_L(\phi)$

$\Rightarrow$ $-\Gamma_L(\phi)$ = the sum of 1PI vacuum diagrams with # loops $= L$

($\log \det(A/2\pi\hbar)$ is included in $L = 1$)

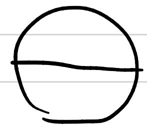
$\therefore$ $\hbar$-expansion = loop expansion.

Remark

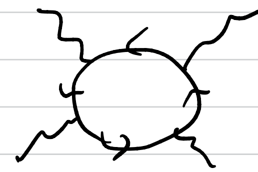
E = # external lines

P = # propagators

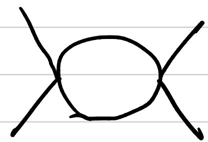
V = # vertices



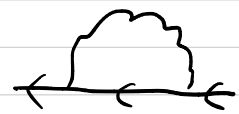
$E=0$
$P=3$
$V=2$



$E=4$
$P=8$
$V=4$



$E=4$
$P=6$
$V=2$



$E=2$
$P=4$
$V=2$

Then # internal lines $I = P - E$

and # loops $L = I - V + 1 = P - E - V + 1$ if connected

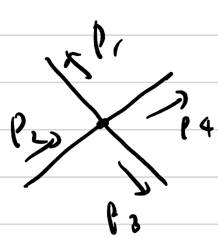
Contribution to partition/correlation function:

$$\int \prod_{v\in V} d^d y_v \int \prod_{e\in E} d^d p_e \, e^{-ip_e(x_e - y_{v(e)})} \int \prod_{i\in I} d^d p_i \, e^{-ip_i(y_{t(i)} - y_{s(i)})} F(p)$$

$$\left[\int d^d y_v \, e^{i\sum_{\ell\in V} \varepsilon_\ell p_\ell y_v} = (2\pi)^d \delta\Big(\sum_{\ell\in V} \varepsilon_\ell p_\ell\Big) \right.$$

sum over lines connected to v

$$\varepsilon_\ell = \begin{cases} +1 & \text{if } \ell \text{ goes out of } v \\ -1 & \text{if } \ell \text{ comes in to } v \end{cases}$$



$\rightsquigarrow \quad (2\pi)^d \delta(p_1 - p_2 + p_3 + p_4)$

$$= \int \prod_{e\in E} d^d p_e \, e^{-i p_e x_e} \int \prod_{i \in I} d^d p_i \underbrace{\prod_{v\in V} (2\pi)^d \delta\Big(\sum_{e\in v} \varepsilon_e p_e\Big)}_{(2\pi)^d \delta\left(\sum_{e\in E} p_e\right) \prod_{v=1}^{V-1} (2\pi)^d \left(\sum_{e\in v} \varepsilon_e p_e\right)}$$

Overall momentum conservation

$\therefore$ net # of momentum integrals

$$= I - (V-1) = L.$$

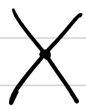
proof of ①

$$-W(J) = -\frac{1}{2}\log\left(\det\frac{A}{2\pi}\right) + \sum D$$

D : connected vacuum diagram
↳ "c.v.d." below

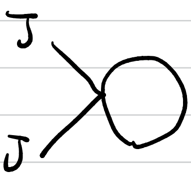
of the perturbation theory with

propagator i —— j $= \phi_i \phi_j = A^{-1}_{ij}$

vertices  etc $= -\frac{\lambda}{4!}\phi^4$ etc

J —— $= J\cdot\phi$

e.g. J —— J $= \frac{1}{2!} J\phi \, J\phi = \frac{1}{2} J_i A^{-1}_{ij} J_j$



$$= \frac{1}{2!} J\phi \, J\phi \, -\frac{\lambda}{4!}\phi\phi\phi\phi \times \binom{4}{2}\cdot 2$$

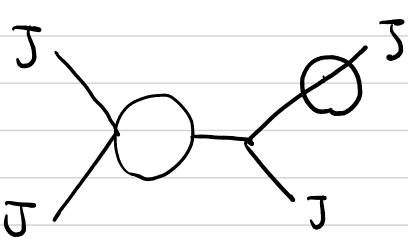
$N_J(D) :=$ # J's in D

$N_{1PI}(D) :=$ # 1PI blocks in D

$N_{s.l.}(D) :=$ # separating lines in D

a separating line := a line s.t. the diagram is disconnected if cut

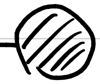
e.g.

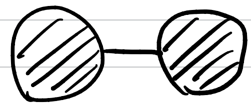


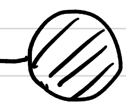
$N_J = 4$

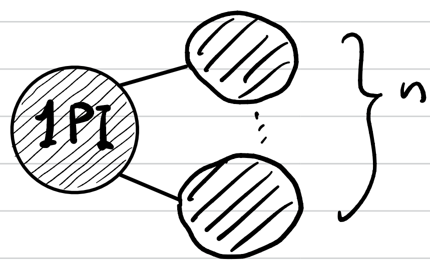
$N_{1PI} = 3$

$N_{sl} = 6$

$\langle \phi_i \rangle_J = i$ —  sum of connected diagrams with one external line ending at i.

- $\displaystyle\sum_{D:\text{c.v.d.}} N_{s.e.}(D)\cdot D =$  $= \frac{1}{2}\langle \phi_i \rangle_J A_{ij} \langle \phi_j \rangle_J$

- $\displaystyle\sum_{D:\text{c.v.d.}} N_J(D)\cdot D = J$ —  $= J_i \langle \phi_i \rangle_J$

- $\displaystyle\sum_{D:\text{c.v.d.}} N_{1PI}(D)\cdot D = \sum_{n=0}^{\infty}$ 

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{i_1\cdots i_n} \lambda_{1PI}^{i_1\cdots i_n} \langle \phi_{i_1} \rangle_J \cdots \langle \phi_{i_n} \rangle_J$$

If we regard 1PI blocks as vertices, then a c.v.d. D is a tree diagram (a diagram without a loop) s.t.

$$E = N_J, \quad V = N_{1PI}, \quad P = N_{s.e.}$$

$$\therefore\ 0 = L = P - E - V + 1 = N_{s.e.} - N_J - N_{1PI} + 1$$

$$\therefore\ -N_{s.e.}(D) + N_J(D) + N_{1PI}(D) = 1.$$

$$-W(J) + \frac{1}{2}\log\det\left(\frac{A}{2\pi}\right) = \sum_{D:\text{c.v.d}} D$$

$$= \sum_{D:\text{c.v.d.}} \left(-N_{s.e.}(D) + N_J(D) + N_{1PI}(D)\right)\cdot D$$

$$= -\frac{1}{2}\langle\phi_i\rangle_J A_{ij}\langle\phi_j\rangle_J + J_i\langle\phi_i\rangle_J + \sum_{n=0}^{\infty}\frac{1}{n!}\lambda_{1PI}^{i_1\cdots i_n}\langle\phi_{i_1}\rangle_J\cdots\langle\phi_{i_n}\rangle_J$$

—— (☆)

Set $J = J(\phi)$ here : $\langle\phi_i\rangle_{J(\phi)} = \phi_i$

$$-W(J(\phi)) + \frac{1}{2}\log\det\left(\frac{A}{2\pi}\right)$$

$$= -\frac{1}{2}\phi_i A_{ij}\phi_j + J(\phi)\cdot\phi + \sum_{n=0}^{\infty}\frac{1}{n!}\lambda_{1PI}^{i_1\cdots i_n}\phi_{i_1}\cdots\phi_{i_n}$$

$$\therefore \Gamma(\phi) = W(J(\phi)) + J(\phi)\cdot\phi$$

$$= \frac{1}{2}\log\det\left(\frac{A}{2\pi}\right) + \frac{1}{2}\phi_i A_{ij}\phi_j - \sum_{n=0}^{\infty}\frac{1}{n!}\lambda_{1PI}^{i_1\cdots i_n}\phi_{i_1}\cdots\phi_{i_n}$$

//

<u>proof of ②</u>

$$e^{-\Gamma(\phi)} = e^{-W(J(\phi)) - J(\phi)\cdot\phi}$$

$$= \int d\phi' \, e^{-S(\phi') + J(\phi)\cdot\phi' - J(\phi)\cdot\phi}$$

$$\overset{\phi' = \phi + \xi}{=} \int d_\phi \xi \, e^{-S_\phi(\xi) + J(\phi)\cdot\xi} = e^{-W^\phi(J(\phi))}.$$

$W^\phi, \langle \cdots \rangle_J^\phi$ etc $:=$ $W, \langle \cdots \rangle_J$ etc for $J(\phi)$.

$$Z^\phi \cdot \langle \xi_i \rangle^\phi_{J(\phi)} = \int d_\phi \xi \, e^{-S_\phi(\xi) + J(\phi)\cdot\xi} \xi_i$$

$$= \int d\phi' \, e^{-S(\phi') + J(\phi)\cdot(\phi' - \phi)} (\phi'_i - \phi_i)$$

$$= e^{-J(\phi)\cdot\phi} \left(\langle \phi_i \rangle_{J(\phi)} - \phi_i \right) = 0$$

Apply (☆) to $J(\phi)$ with $J = J(\phi)$:

$$-\Gamma(\phi) = -W^\phi(J(\phi))$$

$$\overset{(☆)}{=} -\frac{1}{2}\log\det\left(\frac{A}{2\pi}\right) - \frac{1}{2} \overset{0}{\langle \phi_i \rangle^\phi_{J(\phi)}} A_{ij} \overset{0}{\langle \phi_j \rangle^\phi_{J(\phi)}}$$

$$+ J_i(\phi) \overset{0}{\langle \phi_i \rangle^\phi_{J(\phi)}} + \sum_{n=0}^{\infty} \frac{1}{n!} \lambda^{\phi \, i_1 \cdots i_n}_{1PI} \overset{0}{\langle \phi_{i_1} \rangle^\phi_{J(\phi)}} \cdots \overset{0}{\langle \phi_{i_n} \rangle^\phi_{J(\phi)}}$$

Only $n=0$ remains!

$$= -\frac{1}{2}\log\det\left(\frac{A}{2\pi}\right) + \underbrace{\lambda^{\text{vac}\,\phi}_{1PI}}_{\text{sum of 1PI vacuum diagrams of } J(\phi).}$$

sum of 1PI vacuum diagrams of $J(\phi)$. //

<u>proof of ③</u> (The case $N=1$ for simplicity)

$$S(\phi) = \frac{1}{2}(a+b)\phi^2 - P(\lambda,\phi)$$

polynomial of ϕ with parameter λ

Perturbation theory P1 : $S = \underbrace{\frac{1}{2}(a+b)\phi^2}_{\text{free}} - \underbrace{P(\lambda,\phi)}_{\text{interaction}}$

vs

Perturbation theory P2 : $S = \underbrace{\frac{1}{2}a\phi^2}_{\text{free}} + \underbrace{\frac{1}{2}b\phi^2 - P(\lambda,\phi)}_{\text{interaction}}$

In P1, $a+b \in \mathbb{C}$, $\mathrm{Re}(a+b)>0$, λ : formal parameter

In P2, $a \in \mathbb{C}$, $\mathrm{Re}(a)>0$, b & λ : formal parameters.

For comparison to make sense, $a, b \in \mathbb{C}$, $|b| < \mathrm{Re}\, a$,

$f(a+b) = (a+b)^{-r}$ in P1 is regarded as

$\sum_{m=0}^{\infty} \frac{1}{m!} f^{(m)}(a)\, b^m$ in P2.

With this understanding

$$Z_{P1} = \sum_{n=0}^{\infty} \frac{1}{n!} \int d\phi \; e^{-\frac{1}{2}(a+b)\phi^2} P(\lambda,\phi)^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \underbrace{\int d\phi \; e^{-\frac{1}{2}a\phi^2} \sum_{m=0}^{\infty} \frac{1}{m!} \left(-\tfrac{1}{2}b\phi^2\right)^m P(\lambda,\phi)^n}$$

absolutely convergent for $|b| < \mathrm{Re}\, a$

$$= \sum_{n,m=0}^{\infty} \frac{1}{n!} \frac{1}{m!} \int d\phi \; e^{-\frac{1}{2}a\phi^2} \left(-\tfrac{1}{2}b\phi^2\right)^m P(\lambda,\phi)^n$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{m=0} \binom{k}{m} \int d\phi \; e^{-\frac{1}{2}a\phi^2} \left(-\tfrac{1}{2}b\phi^2\right)^m P(\lambda,\phi)^{k-m}$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \int d\phi \; e^{-\frac{1}{2}a\phi^2} \underbrace{\sum_{m=0} \binom{k}{m} \left(-\tfrac{1}{2}b\phi^2\right)^m P(\lambda,\phi)^{k-m}}_{\left(-\frac{1}{2}b\phi^2 + P(\lambda,\phi)\right)^k}$$

$$= Z_{P2}$$

//