Renormalization of QED

(We work in Euclidean signature throughout. Suppress "E")

$$\mathcal{L} = \frac{1}{4e^2} F^{\mu\nu} F_{\mu\nu} + \bar{\psi}(-i\not{D}_A + m)\psi + \frac{1}{2e^2}(\partial\cdot A)^2 + \underbrace{\bar{c}\,\partial^2 c}_{\text{decouple} \downarrow \text{drop.}}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\not{D}_A \psi = \gamma^\mu D_\mu \psi = \gamma^\mu(\partial_\mu + iA_\mu)\psi$$

$\exists$ gauge symmetry $\delta A_\mu = -\partial_\mu \alpha$, $\delta\psi = i\alpha\psi$, $\delta\bar{\psi} = \bar{\psi}(-i\alpha)$

which is broken just by the gauge fixing term:

$$\delta\mathcal{L} = \frac{1}{e^2}(-\partial^2\alpha)\,\partial\cdot A.$$

Let us rescale the variables $A_\mu \rightarrow e A_\mu$.

$$\mathcal{L} = \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}(-i\not{\partial} + m)\psi + e\bar{\psi}\not{A}\psi + \frac{1}{2}(\partial\cdot A)^2.$$

$$\delta A_\mu = -\frac{1}{e}\partial_\mu\alpha,\quad \delta\psi = i\alpha\psi,\quad \delta\bar{\psi} = \bar{\psi}(-i\alpha)$$

$$\Rightarrow \quad \delta\mathcal{L} = -\frac{1}{e}\partial^2\alpha\,\partial\cdot A.$$

Note:

$$S=\int d^4x\Big(\underbrace{\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}+\tfrac{1}{2}(\partial\cdot A)^2+\bar\psi(-i\partial\!\!\!/+m)\psi}_{\text{free part}}+\underbrace{e\bar\psi A\!\!\!/\psi}_{\text{interaction}}\Big)$$

$$A_\mu(x)=\int\frac{d^4p}{(2\pi)^4}e^{-ipx}A_\mu(p)$$

$$\psi(x)=\int\frac{d^4p}{(2\pi)^4}e^{-ipx}\psi(p),\quad \bar\psi(x)=\int\frac{d^4x}{(2\pi)^4}e^{-ipx}\bar\psi(p)$$

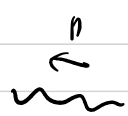
$$=\int\frac{d^4p}{(2\pi)^4}\Big[\tfrac{1}{2}A_\mu(-p)(\delta^{\mu\nu}p^2-\cancel{p^\mu p^\nu})A_\nu(p)+\tfrac{1}{2}A_\mu(-p)\cancel{p^\mu p^\nu}A_\nu(p)$$

$$+\bar\psi(-p)(-p\!\!\!/+m)\psi(p)$$

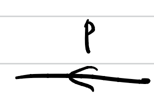
$$+\int\frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\bar\psi(-p-q)e\gamma^\mu\psi(p)A_\nu(q)$$

Free propagators:

$$\overbrace{A_\mu(x)A_\nu(y)}=\int\frac{d^4p}{(2\pi)^4}\frac{\delta_{\mu\nu}}{p^2}e^{-ip(x-y)}$$



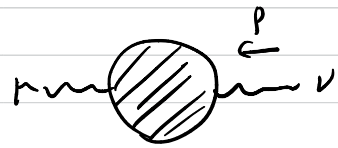
$$\overbrace{\psi(x)\bar\psi(y)}=\int\frac{d^4p}{(2\pi)^4}\frac{e^{-ip(x-y)}}{-p\!\!\!/+m}=\int\frac{d^4p}{(2\pi)^4}\frac{p\!\!\!/+m}{p^2+m^2}e^{-ip(x-y)}$$



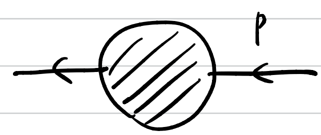
cf $\{\gamma^\mu,\gamma^\nu\}=-2\delta^{\mu\nu}$

Full propagators

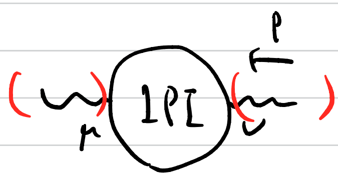
$$\langle A_\mu(x) A_\nu(y) \rangle = \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} G_{\mu\nu}(p)$$



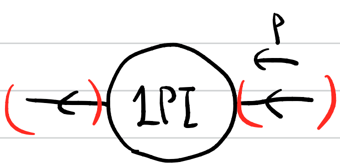
$$\langle \psi(x) \bar{\psi}(y) \rangle = \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} S(p)$$



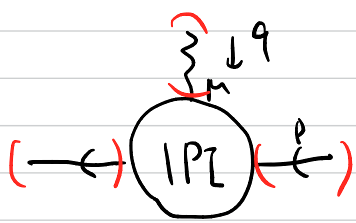
Special 1PI vertices



$$=: \Pi^{\mu\nu}(p)$$

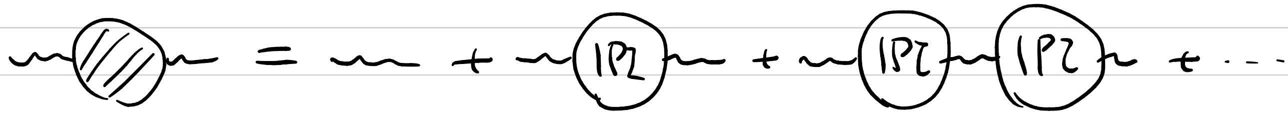


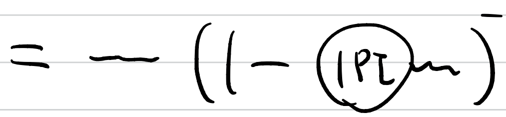
$$=: -\Sigma(p)$$



$$=: -e\Gamma^\mu(p, q)$$

Relationship

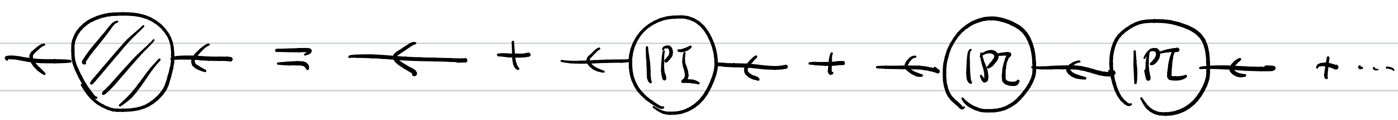


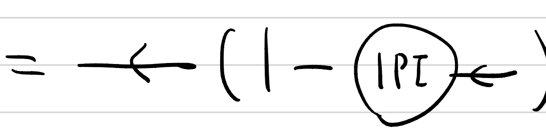
 $= \frac{1}{p^2}\left(1 - \Pi(p)\frac{1}{p^2}\right)^{-1}$

$$= \left[\left(1 - \Pi(p)\frac{1}{p^2}\right)p^2\right]^{-1} = \left(p^2 - \Pi(p)\right)^{-1}$$

$$\therefore \quad G(p) = \left(p^2 - \Pi(p)\right)^{-1}$$

i.e. $\left(\delta^{\mu\nu}p^2 - \Pi^{\mu\nu}(p)\right)G_{\nu\rho}(p) = \delta^{\mu}_{\rho}$

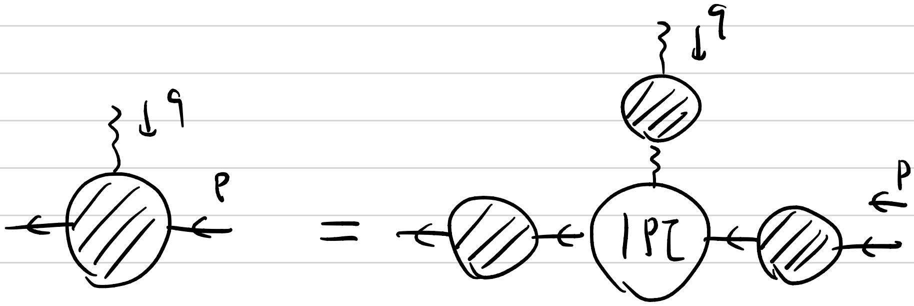


 $= \frac{1}{-\not{p}+m}\left(1 + \Sigma(p)\frac{1}{-\not{p}+m}\right)^{-1}$

$$= \left[\left(1 + \Sigma(p)\frac{1}{-\not{p}+m}\right)(-\not{p}+m)\right]^{-1} = \left(-\not{p}+m+\Sigma(p)\right)^{-1}$$

$$\therefore \quad S(p) = \left(-\not{p}+m+\Sigma(p)\right)^{-1}$$

i.e. $S(p)^{-1} = -\not{p}+m+\Sigma(p)$.



$$\Rightarrow \langle \psi(x)\bar{\psi}(y)A_\nu(z)\rangle$$

$$= \int \frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\, e^{-i(p+q)x+ipy+iqz} \times$$

$$S(p+q)\left(-e\Gamma^\mu(p,q)\right)S(p)\,G_{\mu\nu}(q)$$

Ward identities

For gauge symmetry $\delta A_\mu = -\frac{1}{e}\partial_\mu \alpha$, $\delta\psi = i\alpha\psi$, $\delta\bar\psi = \bar\psi(-i\alpha)$,

$$\delta S = \int d^4x \, \frac{1}{e}(-\partial^2\alpha)\,\partial\cdot A .$$

• $0 = \frac{1}{Z}\int \delta\left(\mathcal{D}\text{fields}\; e^{-S} A_\mu(x)\right)$

$$= \int d^4y \, \frac{1}{e}\,\partial^2\alpha(y)\,\langle \partial\cdot A(y)\, A_\mu(x)\rangle - \frac{1}{e}\,\partial_\mu\alpha(x)$$

$\alpha(x) = e^{-iqx}$

$$= \int d^4y\, \frac{1}{e}\left(-q^2 e^{-iqy}\right) \underbrace{\langle \partial\cdot A(y)\, A_\mu(x)\rangle}_{\int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)}\, ip^\nu G_{\mu\nu}(p)} + \frac{1}{e}\, i q_\mu e^{-iqx}$$

$$= \frac{1}{e}(-q^2)\, e^{-iqx}\, i q^\nu G_{\mu\nu}(q) + \frac{1}{e}\, i q_\mu e^{-iqx}$$

$$\boxed{q^2 G_{\mu\nu}(q)\, q^\nu = q_\mu}$$

$$\left(\delta^{\mu\nu} q^2 - \Pi^{\mu\nu}(q)\right) G_{\nu\rho}(q) = \delta^\mu_\rho$$

$$\boxed{\Pi^{\mu\nu}(q)\, q_\nu = 0}$$

By Euclidean symmetry, $\Pi^{\mu\nu}(q) = \delta^{\mu\nu} X(q^2) + q^\mu q^\nu Y(q^2)$.

$$\Rightarrow \quad q^\mu X(q^2) + q^\mu q^2 Y(q^2) = 0$$

$$\Rightarrow \quad \boxed{\Pi^{\mu\nu}(q) = \left(\delta^{\mu\nu} q^2 - q^\mu q^\nu\right) \Pi(q^2)}$$

- $$0 = \frac{1}{Z} \int \delta\left(\mathcal{D}\text{fields } e^{-S} A_{\mu_1}(x_1) A_{\mu_2}(x_2) A_{\mu_3}(x_3) \right)$$

$$= \int d^4y \, \frac{1}{e} \partial^2 \alpha(y) \langle \partial \cdot A(y) A_{\mu_1}(x_1) A_{\mu_2}(x_2) A_{\mu_3}(x_3) \rangle$$

$$- \frac{1}{e} \partial_{\mu_1} \alpha(x_1) \langle A_{\mu_2}(x_2) A_{\mu_3}(x_3) \rangle - 2 \text{ other terms}$$

decomposition into connected parts

$$= \int d^4y \, \frac{1}{e} \partial^2 \alpha(y) \Big\{ \langle \partial \cdot A(y) A_{\mu_1}(x_1) A_{\mu_2}(x_2) A_{\mu_3}(x_3) \rangle_{\text{conn}}$$

$$+ \langle \partial \cdot A(y) A_{\mu_1}(x_1) \rangle \langle A_{\mu_2}(x_2) A_{\mu_3}(x_3) \rangle + 2 \text{ other permutations} \Big\}$$

$$- \frac{1}{e} \partial_{\mu_1} \alpha(x_1) \langle A_{\mu_2}(x_2) A_{\mu_3}(x_3) \rangle - 2 \text{ other terms}$$

$$\Big[\quad \int d^4y \, \partial^2 \alpha(y) \langle \partial \cdot A(y) A_\mu(x) \rangle = \partial_\mu \alpha(x)$$

$$= \int d^4y \, \frac{1}{e} \partial^2 \alpha(y) \langle \partial \cdot A(y) A_{\mu_1}(x_1) A_{\mu_2}(x_2) A_{\mu_3}(x_3) \rangle_{\text{conn}}$$

$$\therefore \quad \langle \partial\cdot A(y)\, A_{\mu_1}(x_1)\, A_{\mu_2}(x_2)\, A_{\mu_3}(x_3) \rangle_{\text{conn}} = 0$$

Similarly $\langle \partial\cdot A(x_1)\, A_{\mu_2}(x_2) \cdots A_{\mu_s}(x_s) \rangle_{\text{conn}} = 0 \quad \forall$ even s.

- Charge conjugation symmetry

S is invariant under

$$A_\mu \to -A_\mu$$

$$\psi \to C\bar{\psi}^T, \quad \bar{\psi} \to \psi^T(-C^{-1})$$

for $C^{-1}\gamma^\mu C = -(\gamma^\mu)^T$

$$\leadsto \boxed{\langle A_{\mu_1}(x_1) \cdots A_{\mu_s}(x_s) \rangle = 0 \quad \text{if } s \text{ is odd.}}$$

- Fermion number symmetry

S is invariant under $A_\mu \to A_\mu$, $\psi \to e^{i\alpha}\psi$, $\bar{\psi} \to \bar{\psi}e^{-i\alpha}$

$$\leadsto \boxed{\langle \psi_{\alpha_1}(x_1) \cdots \psi_{\alpha_s}(x_s)\, \bar{\psi}^{\beta_1}(y_1) \cdots \bar{\psi}^{\beta_t}(y_t) \rangle = 0 \quad \text{if } s \neq t}$$

- Back to gauge symmetry :

$$0 = \frac{1}{Z} \int \delta \left(\mathcal{D}\text{fields}\, e^{-S}\, \psi(x)\, \bar{\psi}(y) \right)$$

$$= \int d^4z\, \frac{1}{e}\, \partial^2 \alpha(z) \left\langle \partial \cdot A(z)\, \psi(x)\, \bar{\psi}(y) \right\rangle + i \left(\alpha(x) - \alpha(y) \right) \left\langle \psi(x) \bar{\psi}(y) \right\rangle$$

$\alpha(x) = e^{-iqx}$

$$= \int d^4z\, \frac{1}{e} \left(-q^2 e^{-iqz} \right) \left\langle \partial \cdot A(z)\, \psi(x)\, \bar{\psi}(y) \right\rangle + i \left(e^{-iqx} - e^{-iqy} \right) \underbrace{\left\langle \psi(x) \bar{\psi}(y) \right\rangle}_{\int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} S(p)}$$

$$\int \frac{d^4p}{(2\pi)^4} \frac{d^4k}{(2\pi)^4}\, e^{-i(p+k)x + ikz + ipy} S(p+k) \left(-e \Gamma^\mu(p,k) \right) S(p)\, G_{\mu\nu}(k)\, i k^\nu$$

$$= q^2 \int \frac{d^4p}{(2\pi)^4}\, e^{-i(p+q)x + ipy} S(p+q)\, \Gamma^\mu(p,q)\, S(p)\, G_{\mu\nu}(q)\, i q^\nu$$

$$+ i \int \frac{d^4p}{(2\pi)^4}\, e^{-i(p+q)x + ipy} \left(S(p) - S(p+q) \right)$$

As $q^2 G_{\mu\nu}(q) q^\nu = q_\mu$, this yields

$$S(p+q)\, \Gamma^\mu(p,q)\, S(p)\, q_\mu = S(p+q) - S(p).$$

Multiplying $S(p+q)^{-1}$ from the left & $S(p)^{-1}$ from the right,

$$q_\mu \Gamma^\mu(p,q) = S(p)^{-1} - S(p+q)^{-1}$$

$$= \left(-\cancel{\not p} + \cancel{m} + \Sigma(p)\right) - \left(-(\cancel{\not p} + \not q) + \cancel{m} + \Sigma(p+q)\right)$$

$$\boxed{\therefore \quad q_\mu \Gamma^\mu(p,q) = \not q + \Sigma(p) - \Sigma(p+q).}$$

Power counting in QED

E_ψ = # external $\psi, \bar{\psi}$ lines, I_ψ = # internal $\overbrace{\psi\bar{\psi}}$'s

E_A = # external A lines, I_A = # internal $\overbrace{AA}$'s

V = # vertices, L = # loops

Superficial degree of divergence

$$D = 4L - I_\psi - 2I_A$$

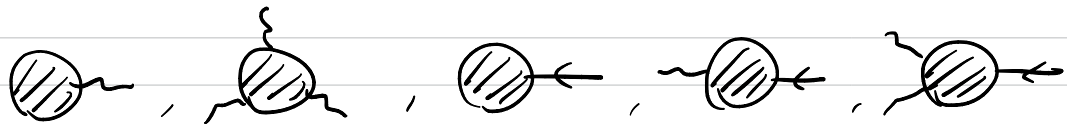
$$L = I - V + 1 = I_\psi + I_A - V + 1$$

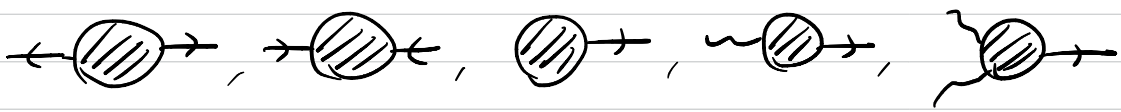
$$V = 2I_A + E_A = \frac{1}{2}(2I_\psi + E_\psi)$$

$$\begin{aligned} D &= 4(I_\psi + I_A - V + 1) - I_\psi - 2I_A \\ &= 3I_\psi + 2I_A - 4V + 4 \\ &= 3\left(V - \tfrac{1}{2}E_\psi\right) + \left(V - E_A\right) - 4V + 4 \\ &= 4 - \frac{3}{2}E_\psi - E_A \end{aligned}$$

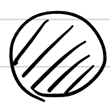
Superficially divergent diagrams

Charge conjugation & fermion # symmetry excludes

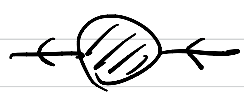


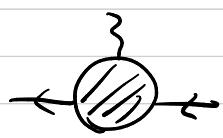
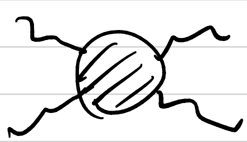


The superficially divergent diagrams are :

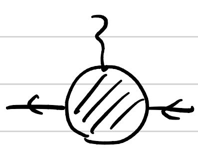
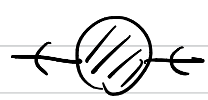
D = 4 : 

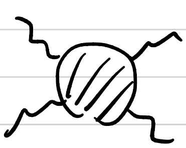
D = 2 : 

D = 1 : 

D = 0 :  , 

 : vacuum energy shift. ~ omit.

 : related to  by Ward identity.

 : not divergent by Ward identity.

see below.

We shall regularize the system respecting the gauge symmetry: the gauge symmetry is broken only by the gauge fixing term,

$$\delta S_{reg} = \int d^4x \, \frac{1}{e} (-\partial^2 \alpha) \, \partial \cdot A .$$

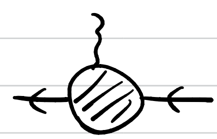
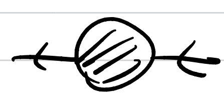
e.g. appropriate Pauli-Villars regularization (later) — Yes!

dimensional regularization — Yes!

random momentum cut-off — No!

Then, the Ward identities we derived remains to hold

- $q_\mu \Gamma^\mu(p,q) = \not{q} + \Sigma(p) - \Sigma(p+q)$.

$\rightsquigarrow$ the divergent (as $\Lambda \to \infty$) part of  is expressed by the divergent part of .

- $\langle \partial \cdot A(y) \, A_{\mu_1}(x_1) \, A_{\mu_2}(x_2) \, A_{\mu_3}(x_3) \rangle_{conn} = 0$

$$\langle A_{\mu_1}(x_1) \cdots A_{\mu_4}(x_4) \rangle_{conn} = \int \prod_{i=1}^{4} \frac{d^4 p_i}{(2\pi)^4} \, e^{-i p_i x_i} \, G^{conn}_{\mu_1 \cdots \mu_4}(p_1, \cdots, p_4)$$

$$\Rightarrow \quad p_1^{\mu_1} \, G^{conn}_{\mu_1 \mu_2 \mu_3 \mu_4}(p_1, p_2, p_3, p_4) = 0 .$$

By Euclidean symmetry,

$$G^{conn}_{\mu_1\cdots\mu_4}(p_1,\cdots,p_4)$$

$$= \delta_{\mu_1\mu_2}\delta_{\mu_3\mu_4} G^{(12)(34)}(p^2) + 2 \text{ other permutations}$$

$p^2 = \{p_i \cdot p_j\}_{1\le i,j\le 4}$

$$+ \sum_{1\le i,j\le 4}\left(p_{i\mu_1} p_{j\mu_2} \delta_{\mu_3\mu_4} G^{(i1)(j2)}(p^2) + 5 \text{ other permutations}\right)$$

$$+ \sum_{1\le i,j,k,l\le 4} p_{i\mu_1} p_{j\mu_2} p_{k\mu_3} p_{l\mu_4} G^{ijkl}(p^2).$$

By dimensional analysis, only $G^{(12)(34)}(p^2)$ can be divergent,

$G^{(ia)(jb)}(p^2)$ & $G^{ijkl}(p^2)$ are finite.

Ward identity:

$$0 = p_1^{\mu_1} G^{conn}_{\mu_1\cdots\mu_4}(p_1,\cdots,p_4)$$

$$= p_{1\mu_2}\delta_{\mu_3\mu_4} G^{(12)(34)}(p^2) + p_{1\mu_3}\delta_{\mu_2\mu_4} G^{(13)(24)}(p^2) + p_{1\mu_4}\delta_{\mu_2\mu_3} G^{(14)(23)}(p^2)$$

$$+ \sum_{i,j}\left(p_1\cdot p_i \, p_{j\mu_2} \delta_{\mu_3\mu_4} G^{(i1)(j2)}(p^2) + 2 \text{ others}\right) = 0$$

$$+ \sum_{i,j}\left(p_{i\mu_2} p_{j\mu_3} p_{1\mu_4} G^{(i2)(j3)}(p^2) + 2 \text{ others}\right)$$

$$+ \sum_{i,j,k,l} (p_1\cdot p_i)\, p_{j\mu_2} p_{k\mu_3} p_{l\mu_4} G^{ijkl}(p^2)$$

$= 0$

$\mu_3 = \mu_4 \neq \mu_2$:

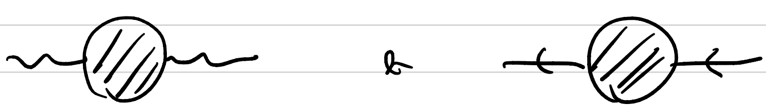
$$0 = P_{1\mu_2} G^{(12)(34)}(p^2) + \sum_{i,j} P_1 \cdot P_i \, P_{j\mu_2} G^{(i1)(j2)}(p^2) = 0$$

$$\Rightarrow \quad G^{(12)(34)}(p^2) = -\sum_i P_1 \cdot P_i \, G^{(i,1)(1,2)}(p^2) \leftarrow \text{finite.}$$

Similarly, $G^{(13)(24)}(p^2)$ and $G^{(14)(23)}(p^2)$ are also finite.

$\therefore$ $G^{\mathrm{conn}}_{\mu_1 \cdots \mu_4}(p_1, \cdots, p_4)$ are finite.

Thus, only



have independent divergences.

One-loop computation

$$\langle A_\rho(x) A_\lambda(y) \rangle_{\text{conn}} = A_\rho(x) A_\lambda(y)$$

$$+ \frac{1}{2} (-e)^2 \underbrace{\int d^4z_1\, d^4z_2 \left\langle A_\rho(x) (\bar\psi \not{A} \psi)(z_1) (\bar\psi \not{A} \psi)(z_2) A_\lambda(y) \right\rangle_{\text{free}}}_{⊛} + \cdots$$

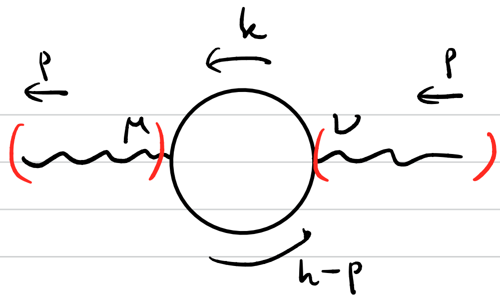
$$⊛ = \frac{e^2}{2} \int d^4z_1\, d^4z_2\, A_\rho(x) A_\mu(z_1) (\bar\psi \gamma^\mu \psi)(z_1) A_\nu(z_2) (\bar\psi \gamma^\nu \psi)(z_2) A_\lambda(y) \times 2$$

move

$$= e^2 \int d^4z_1\, d^4z_2\, \underbrace{A_\rho(x) A_\mu(z_1)}_{\int \frac{d^4p_1}{(2\pi)^4} \frac{\delta_{\rho\mu} e^{-ip_1(x-z_1)}}{p_1^2}} (-\mathrm{tr}) \Big(\gamma^\mu \underbrace{\psi(z_1) \bar\psi(z_2)}_{\int \frac{d^4k_1}{(2\pi)^4} \frac{e^{-ik_1(z_1-z_2)}}{-\not{k}_1 + m}} \gamma^\nu \underbrace{\psi(z_2) \bar\psi(z_1)}_{\int \frac{d^4k_2}{(2\pi)^4} \frac{e^{-ik_2(z_2-z_1)}}{-\not{k}_2 + m}} \Big) \underbrace{A_\nu(z_2) A_\lambda(y)}_{\int \frac{d^4p_2}{(2\pi)^4} \frac{\delta_{\nu\lambda} e^{-ip_2(z_2-y)}}{p_2^2}}$$

$$\left[\begin{array}{l} \int d^4z_1 \Rightarrow (2\pi)^4 \delta(p_1 - k_1 + k_2), \quad \int d^4z_2 \Rightarrow (2\pi)^4 \delta(k_1 - k_2 - p_2) \\ \quad \rightsquigarrow (2\pi)^8 \delta(p_1 - p_2)\, \delta(k_2 - (k_1 - p)) \end{array} \right.$$

$$= \int \frac{d^4p}{(2\pi)^4} \frac{\delta_{\rho\mu} e^{-ipx}}{p^2} \underbrace{e^2 \int \frac{d^4k}{(2\pi)^4} (-\mathrm{tr}) \left(\gamma^\mu \frac{1}{-\not{k} + m} \gamma^\nu \frac{1}{-(\not{k} - \not{p}) + m} \right)}_{=: \Pi_2^{\mu\nu}(p)} \frac{\delta_{\nu\lambda} e^{ipy}}{p^2}$$

$\Pi_2^{\mu\nu}(p) =$ 

$$= -e^2\int\frac{d^4k}{(2\pi)^4}\,\mathrm{tr}\left(\gamma^\mu\frac{1}{\cancel{k}+m}\gamma^\nu\frac{1}{-(\cancel{k-p})+m}\right)$$

$$= -e^2\int\frac{d^4k}{(2\pi)^4}\,\mathrm{tr}\left(\gamma^\mu\frac{\cancel{k}+m}{k^2+m^2}\gamma^\nu\frac{\cancel{k-p}+m}{(k-p)^2+m^2}\right)$$

$$\mathrm{tr}\left(\gamma^\mu(\cancel{k}+m)\gamma^\nu((\cancel{k-p})+m)\right)$$

$$= \mathrm{tr}(\gamma^\mu\gamma^\nu)m^2 + \mathrm{tr}(\gamma^\mu\cancel{k}\gamma^\nu(\cancel{k-p}))$$

- $\mathrm{tr}(\gamma^\mu\gamma^\nu) = \frac{1}{2}\mathrm{tr}\{\gamma^\mu,\gamma^\nu\} = -\mathrm{tr}\,\delta^{\mu\nu} = -4\,\delta^{\mu\nu}$
- $\mathrm{tr}(\gamma^\mu\gamma^\rho\gamma^\nu\gamma^\lambda) = \frac{1}{2}\mathrm{tr}(\underbrace{\gamma^\mu\gamma^\rho\gamma^\nu\gamma^\lambda}) + \frac{1}{2}\mathrm{tr}(\cancel{\gamma^\rho\gamma^\nu}\gamma^\lambda\gamma^\mu)$

$$\{\gamma^\mu,\gamma^\rho\}\gamma^\nu\gamma^\lambda - \gamma^\rho\{\gamma^\mu,\gamma^\nu\}\gamma^\lambda + \gamma^\rho\gamma^\nu\{\gamma^\mu,\gamma^\lambda\} - \cancel{\gamma^\rho\gamma^\nu}\gamma^\lambda\gamma^\mu$$

$$= -\delta^{\mu\rho}\mathrm{tr}(\gamma^\nu\gamma^\lambda) + \delta^{\mu\nu}\mathrm{tr}(\gamma^\rho\gamma^\lambda) - \delta^{\mu\lambda}\mathrm{tr}(\gamma^\rho\gamma^\nu)$$

$$= 4\delta^{\mu\rho}\delta^{\nu\lambda} - 4\delta^{\mu\nu}\delta^{\rho\lambda} + 4\delta^{\mu\lambda}\delta^{\rho\nu}$$

$$= -4\delta^{\mu\nu}m^2 + 4k^\mu(k-p)^\nu - 4\delta^{\mu\nu}k\cdot(k-p) + 4k^\nu(k-p)^\mu$$

$$= 4\left(-\delta^{\mu\nu}(m^2 + k\cdot(k-p)) + k^\mu(k-p)^\nu + k^\nu(k-p)^\mu\right)$$

$$\Pi_2^{\mu\nu}(p) = -4e^2 \int \frac{d^4k}{(2\pi)^4} \frac{-\delta^{\mu\nu}(m^2 + k\cdot(k-p)) + k^\mu (k-p)^\nu + k^\nu (k-p)^\mu}{(k^2+m^2)((k-p)^2+m^2)}$$

- $$\frac{1}{(k^2+m^2)((k-p)^2+m^2)} = \int_0^1 \frac{dx}{\big(\underbrace{(1-x)(k^2+m^2)+x((k-p)^2+m^2)}_{k^2-2xpk+xp^2+m^2}\big)^2}$$

$$= (\underbrace{k-xp}_{\ell})^2 + \underbrace{x(1-x)p^2+m^2}_{\Delta}$$

- denominator $= -\delta^{\mu\nu}\left(m^2 + (\ell+xp)\cdot(\ell+(x-1)p)\right)$

$$+ (\ell+xp)^\mu (\ell+(x-1)p)^\nu + (\ell+xp)^\nu (\ell+(x-1)p)^\mu$$

$$= -\delta^{\mu\nu}(m^2 - x(1-x)p^2 + \ell^2) + 2\ell^\mu\ell^\nu - 2x(1-x)p^\mu p^\nu$$

+ ℓ-linear terms

$$\ell^\mu \ell^\nu \simeq \frac{1}{4}\delta^{\mu\nu}\ell^2$$

equality after $\int d^4\ell$ (provided convergent).

$$\simeq -\delta^{\mu\nu}\Big(\underbrace{m^2+x(1-x)p^2}_{\Delta} + \frac{\ell^2}{2}\Big) + 2x(1-x)\left(\delta^{\mu\nu}p^2 - p^\mu p^\nu\right)$$

If the integral were convergent, $\Pi_2^{\mu\nu}(p)$ would be

$$-4e^2 \int_0^1 dx \int \frac{d^4\ell}{(2\pi)^4} \frac{-\delta^{\mu\nu}(\Delta + \frac{\ell^2}{2}) + 2x(1-x)(\delta^{\mu\nu}p^2 - p^\mu p^\nu)}{(\ell^2+\Delta)^2},$$

but this is quadratically & logarithmically divergent.

Pauli-Villars regularization :

above $=: \Pi_2^{\mu\nu}(p,m)$

$$\Pi^{\mu\nu}_{2\,PV}(p) = \sum_i C_i \, \Pi_2^{\mu\nu}(p, M_i) \qquad (C_0 = 1,\ M_0 = m)$$

so that the integral is convergent.

$\Leftrightarrow$ introduce regulator fields $(\psi_i, \bar{\psi}_i)$ with Lagrangian

$$\Delta\mathcal{L} = \sum_i \bar{\psi}_i \left(-i\partial\!\!\!/ + \sqrt{C_i}\, e A\!\!\!/ + M_i\right)\psi_i$$

The system preserves gauge invariance : the gauge transformation

$$\delta A_\mu = -\partial_\mu \alpha, \quad \delta\psi = i\alpha\psi, \quad \delta\bar{\psi} = \bar{\psi}(-i\alpha),$$

also does $\delta\psi_i = i\sqrt{C_i}\,\alpha\,\psi_i$, $\delta\bar{\psi}_i = \bar{\psi}_i\,(-i\sqrt{C_i}\,\alpha)$.

Then,

$$\Pi^{\mu\nu}_{2\,PV}(p) = -4e^2 \int_0^1 dx \int \frac{d^4 l}{(2\pi)^4} \sum_i C_i \frac{-\delta^{\mu\nu}\left(\Delta_i + \frac{l^2}{2}\right) + 2x(1-x)\left(\delta^{\mu\nu}p^2 - p^\mu p^\nu\right)}{\left(l^2 + \Delta_i\right)^2}$$

with $\Delta_i = x(1-x)p^2 + M_i^2$

$$\int \frac{d^4 \ell}{(2\pi)^4} f(\ell^2) = \frac{\mathrm{Vol}(S^3)}{2(2\pi)^4} \int_0^\infty \ell^2 d\ell^2 f(\ell^2) = \frac{1}{(4\pi)^2} \int_0^\infty t\,dt\, f(t),$$

$$\frac{\mathrm{Vol}(S^3)}{2(2\pi)^4} = \frac{1}{(4\pi)^2}$$

$$\Pi^{\mu\nu}_{2,PV}(p) = -\frac{4e^2}{(4\pi)^2} \int_0^1 dx \int_0^\infty t\,dt \sum_i C_i \frac{-\delta^{\mu\nu}\left(\Delta_i + \frac{t}{2}\right) + 2x(1-x)(\delta^{\mu\nu} p^2 - p^\mu p^\nu)}{(t+\Delta_i)^2}$$

- $$\frac{dt\, t^2}{(t+\Delta_i)^2} = dt \frac{(t+\Delta_i - \Delta_i)^2}{(t+\Delta_i)^2} = dt\left(1 - \frac{2\Delta_i}{t+\Delta_i} + \frac{\Delta_i^2}{(t+\Delta_i)^2}\right)$$
$$= d\left(t - 2\Delta_i \log(t+\Delta_i) - \frac{\Delta_i^2}{t+\Delta_i}\right)$$

- $$\frac{dt\, t}{(t+\Delta_i)^2} = \frac{dt}{t+\Delta_i} - \frac{dt\,\Delta_i}{(t+\Delta_i)^2} = d\left(\log(t+\Delta_i) + \frac{\Delta_i}{t+\Delta_i}\right)$$

$$= -\frac{4e^2}{(4\pi)^2} \int_0^1 dx \Bigg[-\frac{1}{2}\delta^{\mu\nu} \sum_i C_i \left(t - 2\Delta_i \log(t+\Delta_i) - \frac{\Delta_i^2}{t+\Delta_i}\right)$$
$$+ \sum_i C_i \left(-\delta^{\mu\nu}\Delta_i + 2x(1-x)(\delta^{\mu\nu}p^2 - p^\mu p^\nu)\right)\left(\log(t+\Delta_i) + \frac{\Delta_i}{t+\Delta_i}\right)\Bigg]_0^\infty$$

The integral is convergent provided

$$\sum_i C_i = 0 \quad \text{and} \quad \sum_i C_i M_i^2 = 0.$$

This is possible with two regulator fields: $i = 0, 1, 2$.

Then,

$$\Pi^{\mu\nu}_{2PV}(p) = -\frac{4e^2}{(4\pi)^2}\int_0^1 dx\Big[\frac{1}{2}\delta^{\mu\nu}\sum_i C_i\left(-2\Delta_i \cancel{\log\Delta_i} - \cancel{\Delta_i}\right)$$

$$-\sum_i C_i\left(-\cancel{\delta^{\mu\nu}\Delta_i} + 2x(1-x)(\delta^{\mu\nu}p^2 - p^\mu p^\nu)\right)\left(\log\Delta_i + \cancel{1}\right)\Big]$$

$$= \frac{8e^2}{(4\pi)^2}\left(\delta^{\mu\nu}p^2 - p^\mu p^\nu\right)\int_0^1 dx\; x(1-x)\sum_i C_i \log\Delta_i$$

$$\sum_i C_i\log\Delta_i = \log\left(m^2 + x(1-x)p^2\right) + \sum_{i=1,2} C_i \log\left(M_i^2 + x(1-x)p^2\right)$$

$$= \log\left(m^2+x(1-x)p^2\right) + \sum_{i=1,2} C_i\log M_i^2 + O\left(\frac{p^2}{M_1^2}, \frac{p^2}{M_2^2}\right)$$

Define $M := M_1^{-C_1} M_2^{-C_2}$ (mass scale ☺ $-C_1 - C_2 = C_0 = 1$).

Then $\sum_i C_i \log\Delta_i = \log\left(\frac{m^2+x(1-x)p^2}{M^2}\right) + O\left(\frac{p^2}{M_1^2}, \frac{p_2^2}{M_2^2}\right)$

$$\Pi^{\mu\nu}_{2PV}(p) = \left(\delta^{\mu\nu}p^2 - p^\mu p^\nu\right)\Pi_{2PV}(p^2)\;;$$

$$\Pi_{2PV}(p^2) = -\frac{8e^2}{(4\pi)^2}\int_0^1 dx\; x(1-x)\log\left(\frac{M^2}{m^2+x(1-x)p^2}\right)$$

$$\langle \psi(x)\bar{\psi}(y)\rangle_{conn} = \psi(x)\bar{\psi}(y)$$

$$+ \underbrace{\frac{1}{2}(-e)^2\int d^4z_1\, d^4z_2 \left\langle \psi(x)\,(\bar{\psi}\cancel{A}\psi)(z_1)\,(\bar{\psi}\cancel{A}\psi)(z_2)\,\bar{\psi}(y)\right\rangle_{free}}_{\star} + \cdots$$

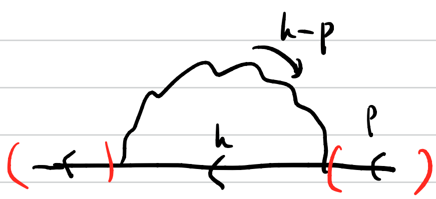
$$\star = \frac{e^2}{2}\int d^4z_1\, d^4z_2\ \psi(x)\,(\bar{\psi}\cancel{A}\psi)(z_1)\,(\bar{\psi}\cancel{A}\psi)(z_2)\,\bar{\psi}(y) \times 2$$

$$= e^2\int d^4z_1\, d^4z_2\ \underbrace{\psi(x)\bar{\psi}(z_1)}_{\int\frac{d^4p_1}{(2\pi)^4}\frac{e^{-ip_1(x-z_1)}}{-\cancel{p}_1+m}}\gamma^\mu\,\underbrace{\psi(z_1)\bar{\psi}(z_2)}_{\int\frac{d^4k_1}{(2\pi)^4}\frac{e^{-ik_1(z_1-z_2)}}{-\cancel{k}_1+m}}\gamma^\nu\,\underbrace{\psi(z_2)\bar{\psi}(y)}_{\int\frac{d^4p_2}{(2\pi)^4}\frac{e^{-ip_2(z_2-y)}}{-\cancel{p}_2+m}}\cdot\underbrace{A_\mu(z_1)A_\nu(z_2)}_{\int\frac{d^4h_2}{(2\pi)^4}\frac{\delta_{\mu\nu}e^{-ih_2(z_1-z_2)}}{h_2^2}}$$

$$(2\pi)^4\delta(p_1-k_1-h_2)\,(2\pi)^4\delta(k_1-p_2+h_2)$$

$$= (2\pi)^8\,\delta(p_1-p_2)\,\delta(k_2+h_1-p_2)$$

$$= \int\frac{d^4p}{(2\pi)^4}\frac{e^{-ipx}}{-\cancel{p}+m}\ \underbrace{e^2\int\frac{d^4k}{(2\pi)^4}\gamma^\mu\frac{1}{-\cancel{k}+m}\gamma^\nu\frac{\delta_{\mu\nu}}{(k-p)^2}}_{=:\,-\Sigma_2(p)}\ \frac{e^{ipy}}{-\cancel{p}+m}$$



$$\Sigma_2(p) = -e^2 \int \frac{d^4k}{(2\pi)^4} \gamma^\mu \frac{1}{-\not{k}+m} \gamma^\nu \frac{\delta_{\mu\nu}}{(k-p)^2}$$

$$= -e^2 \int \frac{d^4k}{(2\pi)^4} \frac{\gamma^\mu(\not{k}+m)\gamma_\mu}{(k^2+m^2)(k-p)^2}$$

$$\left[\begin{array}{l} \gamma^\mu(\not{k}+m)\gamma_\mu = 2\not{k} - 4m \\ \left(\begin{array}{l} \gamma^\mu\gamma_\mu = -4 \\ \gamma^\mu\gamma^\rho\gamma_\mu = \{\gamma^\mu, \gamma^\rho\}\gamma_\mu - \gamma^\rho\gamma^\mu\gamma_\mu \\ \qquad = -2\delta^{\mu\rho}\gamma_\mu + 4\gamma^\rho = 2\gamma^\rho \end{array}\right) \end{array}\right.$$

$$= e^2 \int \frac{d^4k}{(2\pi)^4} \frac{-2\not{k}+4m}{(k^2+m^2)(k-p)^2}$$

<u>Pauli-Villars regularization</u>

$$\frac{1}{(k-p)^2} \longrightarrow \frac{1}{(k-p)^2} - \frac{1}{(k-p)^2+\Lambda^2}$$

$\Leftrightarrow$ introduce a regulator field B_μ with Lagrangian

$$\Delta\mathcal{L} = \frac{1}{2} B^\mu(-\partial^2+\Lambda^2)B_\mu + ie\bar{\psi}\not{B}\psi$$

gauge invariance is preserved with $\delta B = 0$.

$M^2 = 0$ or Λ^2

- $$\frac{1}{(k^2+m^2)((k-p)^2+M^2)} = \int_0^1 \frac{dx}{\big(\underbrace{(1-x)(k^2+m^2)+x((k-p)^2+M^2)}\big)^2}$$

$$k^2 - 2xpk + xp^2 + (1-x)m^2 + xM^2$$

$$= (\underbrace{k-xp}_{\ell})^2 + \underbrace{x(1-x)p^2 + (1-x)m^2 + xM^2}_{\Delta}$$

- denominator $= -2(\not{\ell} + x\not{p}) + 4m \simeq -2x\not{p} + 4m$

$$\Sigma_{2,PV}(p) = \sum_{i=0}^{1} C_i \Sigma_2(p,\Lambda_i) \; ; \; (C_0,\Lambda_0) = (1,0), \; (C_1,\Lambda_1) = (-1,\Lambda)$$

$$= 2e^2 \int_0^1 dx \int \frac{d^4\ell}{(2\pi)^4} \sum_i C_i \frac{-x\not{p} + 2m}{(\ell^2+\Delta_i)^2}$$

$$= \frac{2e^2}{(4\pi)^2} \int_0^1 dx \int_0^\infty t\,dt \sum_i C_i \frac{-x\not{p}+2m}{(t+\Delta_i)^2}$$

$$= \frac{2e^2}{(4\pi)^2} \int_0^1 dx \left[\sum_i C_i \left(\log(t+\Delta_i) + \frac{\Delta_i}{t+\Delta_i} \right) \right]_0^\infty (-x\not{p}+2m)$$

$$= -\frac{2e^2}{(4\pi)^2} \int_0^1 dx \sum_i C_i \left(\log\Delta_i + \cancel{1} \right) (-x\not{p}+2m)$$

$$= -\frac{2e^2}{(4\pi)^2} \int_0^1 dx \log\left(\frac{\Delta_0}{\Delta_1}\right) (-x\not{p}+2m)$$

$$\log \frac{\Delta_0}{\Delta_1} = \log\left(\frac{x(1-x)p^2 + (1-x)m^2}{x(1-x)p^2 + (1-x)m^2 + x\Lambda^2} \right)$$

$$= \log\left(\frac{x(1-x)p^2 + (1-x)m^2}{x\Lambda^2} \right) + O\left(\frac{p^2}{\Lambda^2}, \frac{m^2}{\Lambda^2} \right)$$

$$\Sigma_{2,PV}(p) = \frac{2e^2}{(4\pi)^2} \int_0^1 dx \, (-x\not{p} + 2m) \log\left(\frac{x\Lambda^2}{x(1-x)p^2 + (1-x)m^2} \right)$$