Symmetry in classical mechanics (in Lagrangian)

Suppose $\exists$ a symmetry $q \mapsto q + \delta q$ $\left(\delta q = \epsilon\, u(q, \dot{q}) \right)$

$$\delta L(q, \dot{q}) = \epsilon \frac{d}{dt} \left(\cdots \right) \quad \text{total derivative}$$

Allow variational parameter ϵ to depend on time, $\epsilon(t)$,

s.t. $\epsilon(t_f) = \epsilon(t_i) = 0$:

$$\delta S = \delta \int_{t_i}^{t_f} dt\, L(q, \dot{q}) = \int_{t_i}^{t_f} dt\, \dot{\epsilon}(t)\, Q(q, \dot{q})$$

This $Q = Q(q, \dot{q})$ is called the <u>Noether charge</u>.

<u>Noether's theorem</u> Q is conserved. I.e. it is time-independent for a solution to equation of motion.

<u>proof</u> A solution is s.t. $\delta S = 0$ for $\forall \delta q$ s.t. $\delta q|_{t_f, t_i} = 0$.

For $\forall \epsilon(t)$ s.t. $\epsilon(t_f) = \epsilon(t_i) = 0$, under $q \to q + \epsilon(t)\, u(q, \dot{q})$,

$$0 = \delta S = \int_{t_i}^{t_f} dt\, \dot{\epsilon}(t)\, Q = -\int_{t_i}^{t_f} dt\, \epsilon(t) \frac{dQ}{dt}.$$

$$\therefore \quad \frac{dQ}{dt} = 0 \qquad \underline{\text{Q.E.D.}}$$

Example $L = \frac{m}{2}\dot{q}^2$: a free particle without potential

$\delta q = \epsilon$: translation in q

$$\delta S = \int_{t_i}^{t_f} dt \, \frac{m}{2} 2\dot{q}\dot{\epsilon} = \int_{t_i}^{t_f} dt \, \dot{\epsilon} \, m\dot{q}$$

$\therefore \; Q = m\dot{q}$: momentum.

Example $L = \frac{m}{2}(\dot{q}_1^2 + \dot{q}_2^2) - V(q_1^2 + q_2^2)$

$$g_\alpha : \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \longmapsto \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$$ rotational symmetry

Infinitesimal version:

$$\delta q_1 = -\epsilon q_2, \quad \delta q_2 = \epsilon q_1$$

$$\delta S = \int_{t_i}^{t_f} dt \, \frac{m}{2}\left(2\dot{q}_1(-\dot{\epsilon} q_2) + 2\dot{q}_2(\dot{\epsilon} q_1)\right)$$

$$= \int_{t_i}^{t_f} dt \, \dot{\epsilon} \, m(q_1\dot{q}_2 - q_2\dot{q}_1)$$

$\therefore \; Q = mq_1\dot{q}_2 - mq_2\dot{q}_1$: angular momentum.

<u>Example</u> $L(q,\dot{q})$ general (no explicit t-dependence).

$$\delta q = \epsilon \dot{q} \quad : \text{ time translation.}$$

$$\delta S = \int_{t_i}^{t_f} dt \underbrace{\left(\epsilon \dot{q} \frac{\partial L}{\partial q} + \underbrace{\frac{d}{dt}(\epsilon \dot{q})}_{\epsilon \ddot{q} + \dot{\epsilon}\dot{q}} \frac{\partial L}{\partial \dot{q}} \right)}_{\epsilon \frac{d}{dt} L + \dot{\epsilon} \dot{q} \frac{\partial L}{\partial \dot{q}}}$$

$$= \int_{t_i}^{t_f} dt \, \dot{\epsilon} \left(\dot{q} \frac{\partial L}{\partial \dot{q}} - L \right) dt$$

$$\therefore Q = \dot{q} \frac{\partial L}{\partial \dot{q}}(q,\dot{q}) - L(q,\dot{q}) =: E(q,\dot{q}) \quad \text{energy}$$

c.f. If we solve $\frac{\partial L}{\partial \dot{q}}(q,\dot{q}) \stackrel{!}{=} p$ for $\dot{q}$

and plug the solution $\dot{q} = \dot{q}(p,q)$, then

$$E(q, \dot{q}(p,q)) = \dot{q}(p,q)p - L(q, \dot{q}(p,q))$$

$$= H(p,q) \quad \text{Hamiltonian}$$

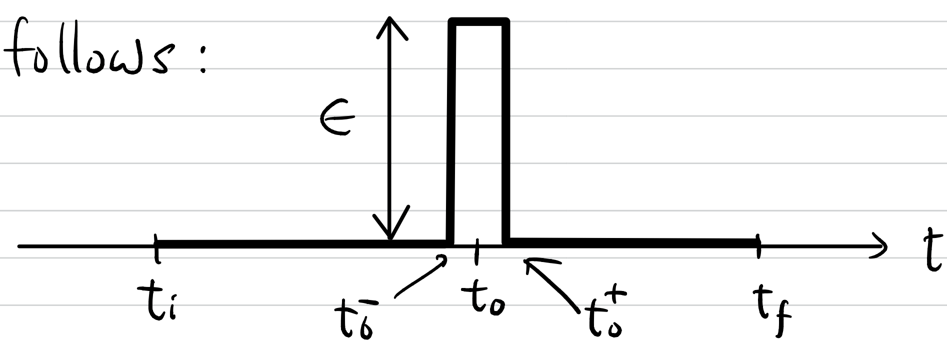
Symmetry in quantum mechanics

Suppose $\exists$ a symmetry $\delta q = \epsilon\, u(q,\dot q)$ of the classical system & it is also a symmetry of the path-integral measure $\mathcal{D}q$.

Apply $\delta q = \epsilon(t)\, u(q,\dot q)$ in the integrand of

$$Z(t_f, q_f ; \mathcal{O}(t_0) ; t_i, q_i) = \int_{q(t_f)=q_f,\ q(t_i)=q_i} \mathcal{D}q \; e^{\frac{i}{\hbar}\int_{t_i}^{t_f} dt\, L(q,\dot q)}\, \mathcal{O}(t_0)$$

for $\epsilon(t)$ as follows:



Note: $\dot\epsilon(t) = \epsilon\,\delta(t - t_0^-) - \epsilon\,\delta(t - t_0^+)$

Ward id

$$0 \overset{\text{Ward id}}{=} \int \delta\left(\mathcal{D}q\, e^{\frac{i}{\hbar}S[q]}\, \mathcal{O}(t_0)\right)$$

$$= \int \mathcal{D}q\, e^{\frac{i}{\hbar}S[q]}\left(\frac{i}{\hbar}\delta S[q]\,\mathcal{O}(t_0) + \delta\mathcal{O}(t_0)\right)$$

$$0 = \int \mathcal{D}q \, e^{\frac{i}{\hbar} S[q]} \left(\frac{i}{\hbar} \delta S[q] \, \mathcal{O}(t_0) + \delta \mathcal{O}(t_0) \right)$$

$$\int_{t_i}^{t_f} dt \, \dot{\epsilon} Q = \epsilon Q(t_0^-) - \epsilon Q(t_0^+)$$

i.e.

$$Z(t_f, q_f ; \delta \mathcal{O}(t_0) ; t_i, q_i)$$

$$= Z\left(t_f, q_f ; \left(\frac{i\epsilon}{\hbar} Q(t_0^+) - \frac{i\epsilon}{\hbar} Q(t_0^-) \right) \mathcal{O}(t_0) ; q_i, t_i \right)$$

$$\rightsquigarrow$$

$$\hat{Z}_{t_f, t_i}(\delta \mathcal{O}(t_0)) = \hat{Z}_{t_f, t_i}\left(\left(\frac{i\epsilon}{\hbar} Q(t_0^+) - \frac{i\epsilon}{\hbar} Q(t_0^-) \right) \mathcal{O}(t_0) \right)$$

Take the limit $t_0^+ \searrow t_0$ and $t_0^- \nearrow t_0$:

$$\widehat{\delta \mathcal{O}} = \frac{i\epsilon}{\hbar} \hat{Q} \circ \hat{\mathcal{O}} - \hat{\mathcal{O}} \circ \frac{i\epsilon}{\hbar} \hat{Q}$$

Put $\epsilon \to 1$:

$$\boxed{\widehat{\delta \mathcal{O}} = \frac{i}{\hbar} [\hat{Q}, \hat{\mathcal{O}}]}$$

Ward identity in quantum mechanics

(in operator formalism)

The case of time translation symmetry:

$$\widehat{\frac{d}{dt}\mathcal{O}} = \frac{i}{\hbar}\left[\widehat{H(p,q)}, \hat{\mathcal{O}}\right].$$

On the other hand, using $\hat{H}$ defined by $\hat{Z}_{t_f,t_i} = e^{-\frac{i}{\hbar}(t_f-t_i)\hat{H}}$,

we also know

$$e^{-\frac{i}{\hbar}(t_f-t)\hat{H}}\,\widehat{\frac{d}{dt}\mathcal{O}}\,e^{-\frac{i}{\hbar}(t-t_i)\hat{H}} = \hat{Z}_{t_f,t_i}\left(\frac{d}{dt}\mathcal{O}(t)\right)$$

$$= \frac{d}{dt}\hat{Z}_{t_f,t_i}(\mathcal{O}(t)) = \frac{d}{dt}\left(e^{-\frac{i}{\hbar}(t_f-t)\hat{H}}\,\hat{\mathcal{O}}\,e^{-\frac{i}{\hbar}(t-t_i)\hat{H}}\right)$$

$$= e^{-\frac{i}{\hbar}(t_f-t)\hat{H}}\,\frac{i}{\hbar}[\hat{H},\hat{\mathcal{O}}]\,e^{-\frac{i}{\hbar}(t-t_i)\hat{H}}$$

$$\therefore\ \widehat{\frac{d}{dt}\mathcal{O}} = \frac{i}{\hbar}[\hat{H},\hat{\mathcal{O}}].$$

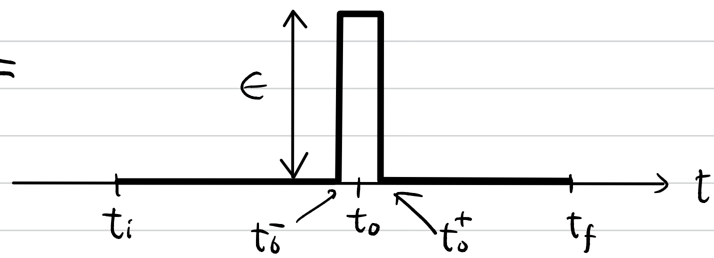
Comparison $\Rightarrow$ $\quad \hat{H} = \widehat{H(p,q)} + \text{c-number}.$

$\hat{H}$ is the operator corresponding to Hamiltonian

(modulo a c-number shift).

The case of q-translation (not a symmetry in general).

Apply $\delta q(t) = \epsilon(t) =$



in the integrand of $Z(t_f, q_f\,;\, q(t_0)\,;\, t_i, q_i)$:

$$0 = \int \delta\left(\mathcal{D}q\, e^{\frac{i}{\hbar} S[q]}\, q(t_0) \right)$$

$$= \int \mathcal{D}q\, e^{\frac{i}{\hbar} S[q]} \left(\frac{i}{\hbar} \delta S[q]\, q(t_0) + \epsilon \right)$$

$$\left(\delta S[q] = \int_{t_i}^{t_f} dt \left(\epsilon(t) \frac{\partial L}{\partial q} + \dot{\epsilon}(t) \frac{\partial L}{\partial \dot{q}} \right) \right.$$

Conjugate momentum p

$$\left. = \int_{t_0^-}^{t_0^+} dt\, \epsilon \frac{\partial L}{\partial q} + \epsilon\, p(t_0^-) - \epsilon\, p(t_0^+) \right)$$

$$= \epsilon \int \mathcal{D}q\, e^{\frac{i}{\hbar} S[q]} \left\{ \frac{i}{\hbar} \left(\int_{t_0^-}^{t_0^+} dt \frac{\partial L}{\partial q} + p(t_0^-) - p(t_0^+) \right) q(t_0) + 1 \right\}$$

Take the limit $t_0^+ \searrow t_0$, $t_0^- \nearrow t_0$:

The part $\int_{t_0^-}^{t_0^+} dt \frac{\partial L}{\partial q} q(t_0)$ vanishes in this limit.

$$\therefore \quad 0 = \frac{i}{\hbar}\left(\hat{q} \circ \hat{p} - \hat{p} \circ \hat{q}\right) + 1$$

$$\boxed{\therefore \quad [\hat{q}, \hat{p}] = i\hbar}$$

The canonical commutation relation!

- Remark on terminology

We used "local observable" for $\mathcal{O}(t)$, but it is common to call it "local operator" even inside path-integral.

We have chosen "observable" to emphasize the distinction between path-integral & operator formalisms.

- It is instructive to do path-integrals in explicit examples. Please do it yourself. For your convenience, a note on it is uploaded.

- In a classical field theory in a general dimension, a continuous symmetry yields a conserved current (Noether current). Just like in quantum mechanics one can derive Ward identity involving Noether current. A note on it will be uploaded.

Fermion path integrals

To describe fermions, we consider path integrals

$$\int d\psi_1 \cdots d\psi_n \, e^{-S_E(\psi_1, \cdots, \psi_n)} f(\psi_1, \cdots, \psi_n)$$

of anticommuting variables $\psi_i \psi_j = -\psi_j \psi_i$.

Algebra & Calculus

Grassmann algebra (graded commutative algebra)

... algebra/$\mathbb{C}$ with bose vs fermi statistics of elements.

If a & b have definite statistics,

$$a \cdot b = (-1)^{|a||b|} \, b \cdot a$$

where $|a| \underset{\text{mod } 2}{\equiv} \begin{cases} 0 & a \text{ even (bosonic)} \\ 1 & a \text{ odd (fermionic)} \end{cases}$

- even elements commute with everything.
- odd elements anticommute with each other.

- Rule of complex conjugation : $(a \cdot b)^* = b^* \cdot a^*$.

- A function $f(\psi) = f(\psi_1, \dots, \psi_n)$ of odd variables $\psi_1, \dots, \psi_n$:

$$\psi_i \psi_j = -\psi_j \psi_i \qquad 1 \le \forall i, j \le n.$$

$$i=j : \quad \psi_i \psi_i = -\psi_i \psi_i \quad \Rightarrow \quad \psi_i^2 = \psi_i \psi_i = 0$$

For finite n, $f(\psi)$ has only finitely many terms

$$f(\psi) = f_0 + \sum_i f_i \psi_i + \sum_{i<j} f_{ij} \psi_i \psi_j + \dots + f_{1\dots n} \psi_1 \cdots \psi_n$$

$$\left(1 + n + \binom{n}{2} + \dots + \binom{n}{n} = 2^n \text{ terms at most.} \right)$$

- <u>Integration</u> of functions of odd variables

$$\int d^n\psi \, f(\psi) = \int d\psi_1 \cdots d\psi_n \, f(\psi_1, \dots, \psi_n)$$

Want : linearity

$$\int d^n\psi \left(f(\psi)\, a + g(\psi)\, b \right) \overset{!}{=} \left(\int d^n\psi \, f(\psi) \right) a + \left(\int d^n\psi \, g(\psi) \right) b,$$

translation invariance

$$\int d^n\psi \, f(\psi + \eta) \overset{!}{=} \int d^n\psi \, f(\psi), \quad \eta = (\eta_1, \dots, \eta_n)$$

independent of $\psi_1, \dots, \psi_n$.

- One variable case $f(\psi) = a + \psi b$

$$\int d\psi\,(a+\psi b) \overset{\text{linearity}}{=\!=} \left(\int d\psi \cdot 1\right) a + \left(\int d\psi \cdot \psi\right) b$$

$$\text{transl.inv} \; \Big\|$$

$$\int d\psi\,(a+(\psi+\eta) b) \overset{\text{linearity}}{=\!=} \left(\int d\psi \cdot 1\right)(a+\eta b) + \left(\int d\psi \cdot \psi\right) b$$

$$\therefore \left(\int d\psi \cdot 1\right) \eta b = 0 \qquad \forall_{\text{odd}}\, \eta,\ \forall b.$$

$$\Rightarrow \qquad \boxed{\int d\psi \cdot 1 = 0}$$

- To have non-zero result $\int d\psi \cdot \psi \neq 0$. We set

$$\boxed{\int d\psi \cdot \psi = 1}$$

- $$\int d\psi\,(a + b\psi) = \int d\psi\,(a + (-1)^{|b|}\psi b) \quad \left(\begin{array}{l}\text{if } b \text{ has a} \\ \text{definite stat.}\end{array}\right)$$

$$= (-1)^b\, b$$

i.e. $$\boxed{d\psi\, b = (-1)^b\, b\, d\psi}$$

· Multivariable case : determined by iteration.

$$\int d\psi_1 d\psi_2 \left(f_0 + \psi_1 f_1 + \psi_2 f_2 + \psi_1\psi_2 f_{12} \right)$$

$$= \int d\psi_1 \int d\psi_2 \left(f_0 + \psi_1 f_1 + \psi_2 f_2 + \psi_1\psi_2 f_{12} \right)$$

$$= \int d\psi_1 \left(\underbrace{\int d\psi_2 f_0}_{0} + \underbrace{\int d\psi_2 \psi_1 f_1}_{0} + \underbrace{\int d\psi_2 \psi_2 f_2}_{1} + \underbrace{\int d\psi_2 \psi_1 \psi_2 f_{12}}_{-\psi_1} \right)$$

$$= \underbrace{\int d\psi_1 f_2}_{0} - \underbrace{\int d\psi_1 \psi_1}_{1} f_{12}$$

$$= -f_{12} \qquad \left(\Rightarrow d\psi_1 d\psi_2 = -d\psi_2 d\psi_1 \right)$$

$\vdots$

$$\int d\psi_1 \cdots d\psi_n \left(f_0 + \sum_i \psi_i f_i + \sum_{i<j} \psi_i \psi_j f_{ij} + \cdots + \underbrace{\psi_1 \cdots \psi_n}_{(-1)^{\frac{n(n-1)}{2}} \psi_n \cdots \psi_1} f_{1\cdots n} \right)$$

$$= (-1)^{\frac{n(n-1)}{2}} f_{1\cdots n}$$

$$\int d\psi_1 \cdots d\psi_n \left(f_0 + \sum_i f_i \psi_i + \cdots + f_{1\cdots n} \psi_1 \cdots \psi_n \right)$$

$$= (-1)^{|f_{1\cdots n}| + \frac{n(n-1)}{2}} f_{1\cdots n}$$

Change of variables

$A = (A_{ij})$ $n \times n$ invertible matrix (even)

Commuting case : $x_i = \sum_{j=1}^{n} A_{ij} x_j'$ $(i=1, \cdots, n)$

$$\Rightarrow \quad dx_1 \cdots dx_n = \det A \cdot dx_1' \cdots dx_n'$$

anticommuting case : $\psi_i = \sum_{j=1}^{n} A_{ij} \psi_j'$ $(i=1, \cdots, n)$

$$\Rightarrow \quad d\psi_1 \cdots d\psi_n = (\det A)^{-1} d\psi_1' \cdots d\psi_n'$$

For $f(\psi) = f_0 + \sum_i \underbrace{\psi_i}_{\sum_j A_{ij}\psi_j'} f_i + \cdots + \underbrace{\psi_1 \cdots \psi_n}_{= \det A\, \psi_1' \cdots \psi_n'} f_{1\cdots n}$,

$$\int d\psi_1 \cdots d\psi_n f(\psi) = (-1)^{\frac{n(n-1)}{2}} f_{1\cdots n}$$

$$\int d\psi_1' \cdots d\psi_n' f(\psi) = (-1)^{\frac{n(n-1)}{2}} \det A\, f_{1\cdots n}$$

$$\therefore \int d\psi_1 \cdots d\psi_n f(\psi) = (\det A)^{-1} \int d\psi_1' \cdots d\psi_n' f(\psi)$$

$\square$

"Gaussian" integral

$$\int d\bar{\psi}\, d\psi\, e^{-\bar{\psi}\psi} = \int d\bar{\psi}\, d\psi\, (1-\bar{\psi}\psi) = 1.$$

$$\int d\bar{\psi}\, d\psi\, e^{-\bar{\psi} a \psi} = \int d\bar{\psi}\, d\psi\, (1-\bar{\psi} a \psi) = a.$$

Alternatively, we may use the rule of change of variables

$$\psi = a^{-1}\psi' \quad \Rightarrow \quad d\psi = a\, d\psi',$$

$$\int d\bar{\psi}\, d\psi\, e^{-\bar{\psi} a \psi} = \int d\bar{\psi}\, a\, d\psi'\, e^{-\bar{\psi}\psi'} = a.$$

The latter is useful for generalization to n-pairs :

$$\boxed{\int d\bar{\psi}_1 d\psi_1 \cdots d\bar{\psi}_n d\psi_n\, e^{-\sum_{i,j} \bar{\psi}_i A_{ij} \psi_j} = \det A}$$

c.f. $$\int dx_1 \cdots dx_n = e^{-\frac{1}{2}\sum_{i,j} x_i A_{ij} x_j} = \sqrt{\frac{(2\pi)^n}{\det A}}$$

$$\int d\bar{z}_1 dz_1 \cdots d\bar{z}_n dz_n\, e^{-\sum_{i,j} \bar{z}_i A_{ij} z_j} = \frac{(2\pi i)^n}{\det A}$$

Other useful relations

- $$\int d\psi\,(\psi-\eta)\,f(\psi) \overset{\text{transl.inv}}{=} \int d\psi\,\psi\,f(\psi+\eta)$$
$$= f(\eta)$$

$\therefore$ $\psi-\eta$ is like the δ-function "$\delta(\psi-\eta)$".

- $$\int d\psi_1 d\psi_2\, \psi_2\psi_1\, a = a \qquad \text{for } a\in\mathbb{C}$$
$$\Rightarrow \int \underbrace{(a\psi_2\psi_1)^*}_{\psi_1^*\psi_2^* a^*}(d\psi_1 d\psi_2)^* = a^*$$
$$\|$$
$$\int (d\psi_1 d\psi_2)^*\,\psi_1^*\psi_2^*\,a^*$$

$\therefore$ $(d\psi_1 d\psi_2)^* = d\psi_2^* d\psi_1^*$.

Fermionic quantum mechanics

Consider the classical mechanics with

a pair of anticommuting variables $\psi(t)$, $\bar{\psi}(t)$ and

Lagrangian $\quad L = i\bar{\psi}\dot{\psi} - \omega\bar{\psi}\psi$

where $\omega \in \mathbb{R}$.

Symmetries

① time translation $\quad \psi(t) \to \psi(t+\Delta t)$, $\bar{\psi}(t) \to \bar{\psi}(t+\Delta t)$

$$\rightsquigarrow \quad \delta\psi = \epsilon\dot{\psi}, \quad \delta\bar{\psi} = \epsilon\dot{\bar{\psi}}$$

The Noether charge is $\quad E = \omega\bar{\psi}\psi \quad$ (energy)

② phase rotation $\quad \psi(t) \to e^{-i\alpha}\psi(t)$, $\bar{\psi}(t) \to e^{i\alpha}\bar{\psi}(t)$.

$$\rightsquigarrow \quad \delta\psi = -i\epsilon\psi, \quad \delta\bar{\psi} = i\epsilon\bar{\psi}$$

The Noether charge is $\quad Q = \bar{\psi}\psi \quad$ (fermion number)

Exercise: Show that the Noether charges are as given above.

- L is real (modulo total derivative) under

$$\psi^* = \bar{\psi}, \quad \bar{\psi}^* = \psi :$$

$$L^* = -i\dot{\psi}^*\bar{\psi}^* - \omega\psi^*\bar{\psi}^* = -i\dot{\bar{\psi}}\psi - \omega\bar{\psi}\psi$$

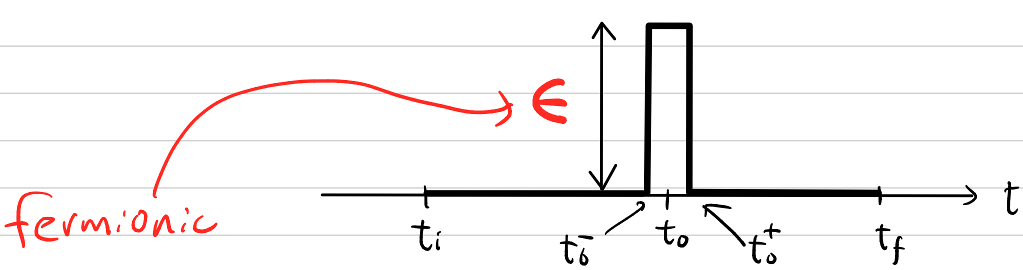
$$= \underbrace{i\bar{\psi}\dot{\psi} - \omega\bar{\psi}\psi}_{L} + \frac{d}{dt}\left(-i\bar{\psi}\psi\right).$$

<u>Let us quantize the system</u> $\begin{cases} \text{Path-integral} & ? \\ \text{Operator} & ? \end{cases}$

We shall take the "mixed" way: Use the path-integral to find the commutation relation of $\hat{\psi}$ & $\hat{\bar{\psi}}$ and then find the representation of the algebra.

Consider the variation

$$\delta\psi(t) = \epsilon(t), \quad \delta\bar{\psi}(t) = 0$$



$$\delta S[\psi,\bar\psi] = \delta \int_{t_i}^{t_f} dt \left(i\bar\psi\dot\psi - \omega\bar\psi\psi \right)$$

$$= \int_{t_i}^{t_f} dt \left(i\bar\psi(t)\underbrace{\dot\epsilon(t)}_{\epsilon\delta(t-t_0^-)-\epsilon\delta(t-t_0^+)} - \omega\bar\psi(t)\epsilon(t) \right)$$

$$= i\bar\psi(t_0^-)\epsilon - i\bar\psi(t_0^+) - \int_{t_0^-}^{t_0^+} dt\, \omega\bar\psi\epsilon$$

Ward identity :

$$0 = \int \delta\left(\mathcal{D}\bar\psi\mathcal{D}\psi\, e^{\frac{i}{\hbar}S[\psi,\bar\psi]} \psi(t_0) \right)$$

$$= \int \mathcal{D}\bar\psi\mathcal{D}\psi \left(\frac{i}{\hbar}\delta S[\psi,\bar\psi]\psi(t_0) + \epsilon \right)$$

$$= \int \mathcal{D}\bar\psi\mathcal{D}\psi \Big(-\frac{1}{\hbar}\bar\psi(t_0^-)\epsilon\cdot\psi(t_0) + \frac{1}{\hbar}\bar\psi(t_0^+)\epsilon\cdot\psi(t_0)$$

$$- \frac{i}{\hbar}\int_{t_0^-}^{t_0^+} dt\, \omega\bar\psi(t)\epsilon\cdot\psi(t_0) + \epsilon \Big)$$

This implies an identity among the operators $\hat\psi$, $\hat{\bar\psi}$ corresponding to ψ, $\bar\psi$.

Take the limit $t_0^+ \searrow t_0$, $t_0^- \nearrow t$:

Recalling the time ordered product and noting

$\int_{t_0^-}^{t_0^+} dt\, \omega \bar{\psi} \epsilon \cdot \psi(t_0) \longrightarrow 0$ in the limit, we find

$$0 = -\frac{1}{\hbar} \hat{\psi} \circ \hat{\bar{\psi}} \epsilon + \frac{1}{\hbar} \hat{\bar{\psi}} \epsilon \circ \hat{\psi} + \epsilon$$

$\epsilon, \hat{\psi}, \hat{\bar{\psi}}$ fermionic

$$= \left[-\frac{1}{\hbar} \hat{\psi} \circ \hat{\bar{\psi}} - \frac{1}{\hbar} \hat{\bar{\psi}} \circ \hat{\psi} + 1 \right] \epsilon .$$

$$\therefore \quad \hat{\psi} \circ \hat{\bar{\psi}} + \hat{\psi} \circ \hat{\bar{\psi}} = \hbar .$$

If we use the notation

$$\{A, B\} := AB + BA \quad \text{(anticommutator)}$$

this can be written as

$$\{\hat{\psi}, \hat{\bar{\psi}}\} = \hbar .$$

Similarly,

- $$0 = \int \delta\left(\mathcal{D}\bar{\psi}\,\mathcal{D}\psi\, e^{\frac{i}{\hbar}S[\psi,\bar{\psi}]}\,\bar{\psi}(t_0)\right) \quad \text{for the same } \delta$$

$$\Rightarrow \quad \hat{\bar{\psi}} \circ \hat{\bar{\psi}} = 0$$

- $$0 = \int \delta\left(\mathcal{D}\bar{\psi}\,\mathcal{D}\psi\, e^{\frac{i}{\hbar}S[\psi,\bar{\psi}]}\,\psi(t_0)\right)$$

for $\delta\psi(t) = 0$ and $\delta\bar{\psi}(t) =$ the same $\epsilon(t)$

$$\Rightarrow \quad \hat{\psi} \circ \hat{\psi} = 0$$

The canonical commutation relation of $\hat{\psi}$ and $\hat{\bar{\psi}}$ is

$$\boxed{\begin{gathered} \{\hat{\psi}, \hat{\bar{\psi}}\} = \hbar, \\ \hat{\psi}^2 = \hat{\bar{\psi}}^2 = 0. \end{gathered}}$$

... "Clifford algebra".

Representation of the algebra.

Reality of the variables $\psi^* = \bar{\psi}$, $\bar{\psi}^* = \psi$

$\rightsquigarrow$ hermiticity of the operators $\hat{\psi}^\dagger = \hat{\bar{\psi}}$, $\hat{\bar{\psi}}^\dagger = \hat{\psi}$.

The commutation relation reads

$$\boxed{\begin{aligned} &\{\hat{\psi}, \hat{\psi}^\dagger\} = \hbar \\ &\{\hat{\psi}, \hat{\psi}\} = \{\hat{\psi}^\dagger, \hat{\psi}^\dagger\} = 0. \end{aligned}}$$

This is like the algebra of annihilation/creation operators in harmonic oscillators. As in that case, we may prepare a state $|0\rangle$ annihilated by $\hat{\psi}$,

$$\hat{\psi}|0\rangle = 0$$

and build other states by multiplying powers of $\hat{\psi}^\dagger$,

$$\hat{\psi}^\dagger|0\rangle, \ \hat{\psi}^{\dagger 2}|0\rangle, \ \hat{\psi}^{\dagger 3}|0\rangle, \ \cdots.$$

But, by the relation $\hat{\psi}^{\dagger 2} = 0$, $\hat{\psi}^{\dagger 2}|0\rangle = \hat{\psi}^{\dagger 3}|0\rangle = \cdots = 0$.

Only $\hat{\psi}^\dagger|0\rangle$ among them can be non-zero.

Also,

$$\hat{\psi}(\hat{\psi}^\dagger|0\rangle) = (\underbrace{\hat{\psi}\hat{\psi}^\dagger + \hat{\psi}^\dagger\hat{\psi}}_{\hbar})|0\rangle - \hat{\psi}^\dagger\underbrace{\hat{\psi}|0\rangle}_{0} = \hbar|0\rangle.$$

Thus $\hat{\psi}^\dagger|0\rangle$ is indeed non-zero. <u>We have a</u> <u>2-dimensional representation</u> $\mathcal{H}$, with basis $|0\rangle$, $\hat{\psi}^\dagger|0\rangle$.

With respect to this basis, $\hat{\psi}$ and $\hat{\psi}^\dagger$ are represented by matrices

$$\hat{\psi} \doteq \begin{pmatrix} 0 & \hbar \\ 0 & 0 \end{pmatrix}, \quad \hat{\psi}^\dagger \doteq \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

If $|0\rangle$ is normalized as $\| |0\rangle \|^2 = \langle 0|0\rangle = 1$,

$$\| \hat{\psi}^\dagger|0\rangle \|^2 = \langle 0|\hat{\psi}\hat{\psi}^\dagger|0\rangle = \langle 0|\hbar|0\rangle = \hbar.$$

<u>Energy spectrum ?</u>

Recall $E = \omega\bar{\psi}\psi$.

$$\rightsquigarrow \hat{H} = \omega\hat{\psi}^\dagger\hat{\psi} \doteq \begin{cases} 0 & \text{on } |0\rangle \\ \hbar\omega & \text{on } \hat{\psi}^\dagger|0\rangle \end{cases}$$

※ There is an operator ordering ambiguity:

$$\hat{H}_s = (1-s)\omega \hat{\psi}^+ \hat{\psi} - s\omega \hat{\psi}\hat{\psi}^+ = \omega \hat{\psi}^+\hat{\psi} - s\hbar\omega$$

$\uparrow$ minus !

is also possible. For this, $\hat{H}_s = \begin{cases} -s\hbar\omega & \text{on } |0\rangle \\ (1-s)\hbar\omega & \text{on } \hat{\psi}^+|0\rangle. \end{cases}$

e.g. the "symmetric" ordering

$$\hat{H}_{\frac{1}{2}} = \frac{1}{2}\omega \hat{\psi}^+\hat{\psi} - \frac{1}{2}\omega \hat{\psi}\hat{\psi}^+ = \frac{\omega}{2}[\hat{\psi}^+, \hat{\psi}]$$

$$\hat{H}_{\frac{1}{2}} = \begin{cases} -\frac{1}{2}\hbar\omega & \text{on } |0\rangle \\ \frac{1}{2}\hbar\omega & \text{on } \hat{\psi}^+|0\rangle. \end{cases}$$

<u>Also</u> $Q = \bar{\psi}\psi$

$\rightsquigarrow \hat{Q} = \hat{\psi}^+\hat{\psi}$

or $\hat{Q}_s = (1-s)\hat{\psi}^+\hat{\psi} - s\hat{\psi}\hat{\psi}^+ = \hat{\psi}^+\hat{\psi} - s\hbar$

$$= \begin{cases} -s\hbar & \text{on } |0\rangle \\ (1-s)\hbar & \text{on } \hat{\psi}^+|0\rangle. \end{cases}$$

$\hat{\psi}$ & $\hat{\psi}^\dagger$ eigenstates

As a passage to (go back to) the path-integral, we find eigenstates of $\hat{\psi}$ and $\hat{\psi}^\dagger$ (just like eigenstates $|q\rangle$ & $|p\rangle$ for $\hat{q}$ & $\hat{p}$).

Suppose
$$|\psi\rangle = a|0\rangle + b\hat{\psi}^\dagger|0\rangle$$
$$|\bar{\psi}\rangle = c|0\rangle + d\hat{\psi}^\dagger|0\rangle$$
satisfy
$$\hat{\psi}|\psi\rangle = \psi|\psi\rangle$$
$$\hat{\psi}^\dagger|\bar{\psi}\rangle = \bar{\psi}|\bar{\psi}\rangle.$$

As
$$\hat{\psi}|\psi\rangle = (-1)^{|a|}a\underbrace{\hat{\psi}|0\rangle}_{0} + (-1)^{|b|}b\underbrace{\hat{\psi}\hat{\psi}^\dagger|0\rangle}_{\hbar|0\rangle}$$
$$\overset{!}{\|}$$
$$\psi|\psi\rangle = \psi a|0\rangle + \psi b\hat{\psi}^\dagger|0\rangle$$

$$\hat{\psi}^\dagger|\bar{\psi}\rangle = (-1)^{|c|}c\hat{\psi}^\dagger|0\rangle + (-1)^{|d|}d\underbrace{\hat{\psi}^\dagger\hat{\psi}^\dagger}_{0}|0\rangle$$
$$\overset{!}{\|}$$
$$\bar{\psi}|\bar{\psi}\rangle = \bar{\psi}c|0\rangle + \bar{\psi}d\hat{\psi}^\dagger|0\rangle$$

We find
$$\psi a = (-1)^{|b|}b\hbar, \quad \psi b = 0$$
$$\bar{\psi}c = 0, \quad \bar{\psi}d = (-1)^{|c|}c$$

A solution
$$(a, b) = (1, -\psi/\hbar)$$
$$(c, d) = (\bar{\psi}, -1)$$

$$\left.\begin{aligned} |\psi\rangle &= \left(1 - \frac{\psi}{\hbar}\hat{\psi}^{\dagger}\right)|0\rangle = \left(1 + \frac{1}{\hbar}\hat{\psi}^{\dagger}\psi\right)|0\rangle, \\ |\bar{\psi}\rangle &= \left(\bar{\psi} - \hat{\psi}^{\dagger}\right)|0\rangle. \end{aligned}\right\} \bigstar$$

We have chosen a random normalization, but this turns out to be a useful one.

(A more symmetric normalization would be

$$|\psi\rangle = \hbar^{\frac{1}{4}}\left(1 + \frac{1}{\hbar}\hat{\psi}^{\dagger}\psi\right)|0\rangle, \quad |\bar{\psi}\rangle = \hbar^{-\frac{1}{4}}\left(\bar{\psi} - \hat{\psi}^{\dagger}\right)|0\rangle.$$

This is also good, but we shall use $\bigstar$)

We also prepare "bra" eigenstates for $\hat{\psi}$ & $\hat{\psi}^{\dagger}$:

$$\langle\psi|\hat{\psi} = \langle\psi|\psi, \qquad \langle\bar{\psi}|\hat{\psi}^{\dagger} = \langle\bar{\psi}|\bar{\psi}.$$

For these we may take

$$\langle\psi| = |\bar{\psi}\rangle^{\dagger} = \langle 0|(\psi - \hat{\psi})$$

$$\langle\bar{\psi}| = |\psi\rangle^{\dagger} = \langle 0|\left(1 + \frac{1}{\hbar}\bar{\psi}\hat{\psi}\right).$$

[Take the hermitian conjugate of $\hat{\psi}^{\dagger}|\bar{\psi}\rangle = \bar{\psi}|\bar{\psi}\rangle$ and $\hat{\psi}|\psi\rangle = \psi|\psi\rangle$. □]

Just like $\int dq\,|q\rangle\langle q| = \int dp\,|p\rangle\langle p| = \mathrm{id}_{\mathcal{H}}$, we have

$$\boxed{\int d\psi\,|\psi\rangle\langle\psi| = \int d\bar\psi\,|\bar\psi\rangle\langle\bar\psi| = \mathrm{id}_{\mathcal{H}}.}$$

<u>proof</u>

$$\int d\psi\,|\psi\rangle\langle\psi|0\rangle = \int d\psi\,\left(1+\tfrac{1}{\hbar}\hat\psi^\dagger\psi\right)|0\rangle\langle 0|(\psi-\underbrace{\hat\psi)|0\rangle}_{0}$$

$|0\rangle\langle 0|$ is even

$$= \int d\psi\,\left(1+\tfrac{1}{\hbar}\hat\psi^\dagger\underbrace{\psi)\psi}_{0}\,|0\rangle\langle 0|0\rangle = |0\rangle.$$

$$\int d\psi\,|\psi\rangle\langle\psi|\hat\psi^\dagger|0\rangle = \int d\psi\,\left(1+\tfrac{1}{\hbar}\hat\psi^\dagger\psi\right)|0\rangle\langle 0|\overbrace{\psi\,\hat\psi^\dagger}^{0}\ \underbrace{-\hat\psi\,\hat\psi^\dagger}_{-\hbar}\,|0\rangle$$

$$= \int d\psi\,\left(1-\psi\tfrac{1}{\hbar}\hat\psi^\dagger\right)|0\rangle\langle 0|(-\hbar)|0\rangle = \hat\psi^\dagger|0\rangle.$$

$$\int d\bar\psi\,|\bar\psi\rangle\langle\bar\psi|0\rangle = \int d\bar\psi\,(\bar\psi-\hat\psi^\dagger)|0\rangle\langle 0|\left(1+\tfrac{1}{\hbar}\bar\psi\underbrace{\hat\psi\right)|0\rangle}_{0}$$

$$= \int d\bar\psi\,(\bar\psi-\hat\psi^\dagger)|0\rangle\langle 0|0\rangle = |0\rangle.$$

$$\int d\bar\psi\,|\bar\psi\rangle\langle\bar\psi|\hat\psi^\dagger|0\rangle = \int d\bar\psi\,(\bar\psi-\hat\psi^\dagger)|0\rangle\langle 0|\overbrace{\left(\hat\psi^\dagger\right.}^{0}+\tfrac{1}{\hbar}\bar\psi\underbrace{\left.\hat\psi\right)\hat\psi^\dagger}_{\hbar}|0\rangle$$

$$= \int d\bar\psi\,(\bar\psi-\hat\psi^\dagger)|0\rangle\langle 0|\bar\psi|0\rangle$$

$|0\rangle\langle 0|$ is even

$$= \int d\bar\psi\,(\overbrace{\bar\psi\,\bar\psi}^{0}\underbrace{-\hat\psi^\dagger\bar\psi}_{-1})\,|0\rangle\langle 0|0\rangle = \hat\psi^\dagger|0\rangle.$$

<u>Q.E.D.</u>

Note also

$$\langle \psi_1 | \bar{\psi}_2 \rangle = \langle 0 | (\psi_1 - \hat{\psi})(\bar{\psi}_2 - \hat{\psi}^\dagger) | 0 \rangle = \langle 0 | (\overbrace{\psi_1 \bar{\psi}_2}^{\text{even}} + \hbar) | 0 \rangle$$

$$= \hbar + \psi_1 \bar{\psi}_2 = \hbar \left(1 + \frac{1}{\hbar} \psi_1 \bar{\psi}_2 \right) = \hbar\, e^{\frac{1}{\hbar} \psi_1 \bar{\psi}_2},$$

$$\langle \bar{\psi}_1 | \psi_2 \rangle = \langle 0 | (1 + \tfrac{1}{\hbar} \bar{\psi}_1 \hat{\psi})(1 + \tfrac{1}{\hbar} \hat{\psi}^\dagger \psi_2) | 0 \rangle$$

$$= \langle 0 | (1 + \tfrac{1}{\hbar} \bar{\psi}_1 \psi_2) | 0 \rangle = 1 + \tfrac{1}{\hbar} \bar{\psi}_1 \psi_2 = e^{\frac{1}{\hbar} \bar{\psi}_1 \psi_2},$$

$$\langle \psi_1 | \psi_2 \rangle = \langle 0 | (\psi_1 - \hat{\psi}) \left(1 + \tfrac{1}{\hbar} \hat{\psi}^\dagger \psi_2 \right) | 0 \rangle$$

$$= \langle 0 | (\psi_1 - \psi_2) | 0 \rangle = (-1)^{||0\rangle|} (\psi_1 - \psi_2),$$

$$\langle \bar{\psi}_1 | \bar{\psi}_2 \rangle = \langle 0 | (1 + \tfrac{1}{\hbar} \bar{\psi}_1 \hat{\psi})(\bar{\psi}_2 - \hat{\psi}^\dagger) | 0 \rangle$$

$$= \langle 0 | (\bar{\psi}_2 - \bar{\psi}_1) | 0 \rangle = -(-1)^{||0\rangle|} (\bar{\psi}_1 - \bar{\psi}_2).$$

(The latter two can also be used for derivation of

$$\int d\psi\, |\psi\rangle\langle\psi| = \int d\bar{\psi}\, |\bar{\psi}\rangle\langle\bar{\psi}| = \mathrm{id}_{\mathcal{H}}$$

[$$\int d\psi\, |\psi\rangle\langle\psi|\psi_1\rangle = \int d\psi\, |\psi\rangle (-1)^{||0\rangle|} (\psi - \psi_1)$$

$$\overset{||\psi\rangle| = ||0\rangle|}{=} \int d\psi\, (\psi - \psi_1) |\psi\rangle = |\psi_1\rangle \quad \text{etc.}$$])

For $A \in \mathrm{End}_{\mathbb{C}} \mathcal{H}$

$$\mathrm{Tr}_{\mathcal{H}} A = \int d\bar{\psi}\, d\psi \, \langle -\bar{\psi} | A | \psi \rangle \langle \psi | \bar{\psi} \rangle$$

It is enough to show this for $A = \mathrm{id}, \hat{\psi}, \hat{\psi}^{\dagger}, \hat{\psi}^{\dagger}\hat{\psi}$ as they span $\mathrm{End}_{\mathbb{C}} \mathcal{H}$.

$A = \mathrm{id}$: $\displaystyle\int d\bar{\psi}\, d\psi \, \underbrace{\langle -\bar{\psi} | \psi \rangle}_{e^{-\frac{1}{\hbar}\bar{\psi}\psi}} \underbrace{\langle \psi | \bar{\psi} \rangle}_{\hbar e^{\frac{1}{\hbar}\psi\bar{\psi}}} = \int \hbar \, d\bar{\psi}\, d\psi \, e^{-\frac{2}{\hbar}\bar{\psi}\psi} = 2$

$= \mathrm{tr}\, \mathrm{id}$ ✓

$A = \hat{\psi}$: $\displaystyle\int d\bar{\psi}\, d\psi \, \langle -\bar{\psi} | \underbrace{\hat{\psi} | \psi \rangle}_{\psi | \psi \rangle} \langle \psi | \bar{\psi} \rangle = \int \hbar \, d\bar{\psi}\, d\psi \, (\pm \psi) \, e^{-\frac{2}{\hbar}\bar{\psi}\psi} = 0$

$= \mathrm{tr}\, \hat{\psi}$ ✓

$A = \hat{\psi}^{\dagger}$: $\displaystyle\int d\bar{\psi}\, d\psi \, \underbrace{\langle -\bar{\psi} | \hat{\psi}^{\dagger}}_{\langle -\bar{\psi} | (-\bar{\psi})} | \psi \rangle \langle \psi | \bar{\psi} \rangle = \int \hbar \, d\bar{\psi}\, d\psi \, (\mp \bar{\psi}) \, e^{-\frac{2}{\hbar}\bar{\psi}\psi} = 0$

$= \mathrm{tr}\, \hat{\psi}^{\dagger}$ ✓

$A = \hat{\psi}^{\dagger}\hat{\psi}$: $\displaystyle\int d\bar{\psi}\, d\psi \, \langle -\bar{\psi} | \underbrace{\hat{\psi}^{\dagger}\hat{\psi}}_{-\bar{\psi}\psi} | \psi \rangle \langle \psi | \bar{\psi} \rangle = \int \hbar \, d\bar{\psi}\, d\psi \, (-\bar{\psi}\psi) \, e^{-\frac{2}{\hbar}\bar{\psi}\psi}$

$= \hbar = \mathrm{tr}(\hat{\psi}^{\dagger}\hat{\psi})$ ✓

□

We also have

$$\mathrm{tr}_{\mathcal{H}} A = \int d\bar{\psi}\, d\psi \, \langle \psi | A | \bar{\psi} \rangle \langle \bar{\psi} | -\psi \rangle$$

$$= (-1)^{||0\rangle|} \int d\psi \, \langle \psi | A | -\psi \rangle$$

$$= (-1)^{||0\rangle|} \int d\bar{\psi} \, \langle -\bar{\psi} | A | \psi \rangle$$

If we define $(-1)^F$ by $(-1)^F = \begin{cases} +1 & \text{on } |0\rangle \\ -1 & \text{on } \hat{\psi}^\dagger |0\rangle \end{cases}$,

then $(-1)^F |\psi\rangle = |-\psi\rangle$

& $\langle \bar{\psi} | (-1)^F = \langle -\bar{\psi} |$.

$$\therefore \quad \boxed{\mathrm{tr}_{\mathcal{H}} (-1)^F A = \int d\bar{\psi}\, d\psi \, \langle \bar{\psi} | A | \psi \rangle \langle \psi | \bar{\psi} \rangle}$$

States as wave functions

We can represent states as functions of $\bar{\psi}$:

$$|\Psi\rangle \longleftrightarrow \Psi(\bar{\psi}) = \langle \bar{\psi} | \Psi \rangle.$$

Note: $\Psi(\bar{\psi})^* = \langle \bar{\psi} | \Psi \rangle^* = \langle \Psi | \psi \rangle.$

- The inner product of states is represented as

$$\langle \Psi_1 | \Psi_2 \rangle = \langle \Psi_1 | \underbrace{\int d\psi \, |\psi\rangle\langle\psi|}_{\text{even}} \int d\bar{\psi} \, |\bar{\psi}\rangle\langle\bar{\psi}| \Psi_2 \rangle$$

$$= \langle \Psi_1 | \int \underbrace{d\bar{\psi}\, d\psi}_{\text{even}} \, |\psi\rangle\langle\psi|\bar{\psi}\rangle\langle\bar{\psi}|\Psi_2\rangle$$

$$= \int d\bar{\psi}\, d\psi \, \underbrace{\langle \Psi_1 | \psi \rangle}_{\Psi_1(\bar{\psi})^*} \underbrace{\langle \psi | \bar{\psi} \rangle}_{\hbar e^{\frac{1}{\hbar}\psi\bar{\psi}}} \underbrace{\langle \bar{\psi} | \Psi_2 \rangle}_{\Psi_2(\bar{\psi})}$$

$$= \int \hbar \, d\bar{\psi}\, d\psi \; \Psi_1(\psi)^* \, e^{\frac{1}{\hbar}\psi\bar{\psi}} \, \Psi_2(\bar{\psi})$$

- Time evolution of states:

$$\left(e^{-i\frac{t_f-t_i}{\hbar}\hat{H}}\Psi\right)(\bar{\psi}) = \langle\bar{\psi}|e^{-i\frac{t_f-t_i}{\hbar}\hat{H}}|\Psi\rangle$$

$$= \langle\bar{\psi}|e^{-i\frac{t_f-t_i}{\hbar}\hat{H}}\int d\psi'\,|\psi'\rangle\langle\psi'|\int d\bar{\psi}'\,|\bar{\psi}'\rangle\langle\bar{\psi}'|\Psi\rangle$$

$$= \int d\bar{\psi}'d\psi'\,\langle\bar{\psi}|e^{-i\frac{t_f-t_i}{\hbar}\hat{H}}|\psi'\rangle\underbrace{\langle\psi'|\bar{\psi}'\rangle}_{\hbar e^{\frac{1}{\hbar}\psi'\bar{\psi}'}}\underbrace{\langle\bar{\psi}'|\Psi\rangle}_{\Psi(\bar{\psi}')}$$

As in the bosonic case, let us define the transition amplitude by

$$Z(t_f,\bar{\psi}_f\,;\,t_i,\psi_i) := \langle\bar{\psi}_f|e^{-i\frac{(t_f-t_i)}{\hbar}\hat{H}}|\psi_i\rangle.$$

Then the time evolution of states is given by

$$\left(e^{-i\frac{t_f-t_i}{\hbar}\hat{H}}\Psi\right)(\bar{\psi}_f)$$

$$= \int \hbar\, d\bar{\psi}_i d\psi_i\; Z(t_f,\bar{\psi}_f\,;\,t_i,\psi_i)\,e^{\frac{1}{\hbar}\psi_i\bar{\psi}_i}\,\Psi(\bar{\psi}_i).$$

Path-integral expression for $Z(t_f, \bar{\psi}_f ; t_i, \psi_i)$

Let us divide $t_f - t_i$ into N pieces, $t_f - t_i = N\epsilon$,

$$e^{-i\frac{t_f - t_i}{\hbar}\hat{H}} = \underbrace{e^{-\frac{i\epsilon}{\hbar}\hat{H}} \cdots\cdots e^{-\frac{i\epsilon}{\hbar}\hat{H}}}_{N}$$

and insert

$$\begin{aligned} \mathrm{id} &= \int d\psi_{j+1} |\psi_{j+1}\rangle\langle\psi_{j+1}| \int d\bar{\psi}_j |\bar{\psi}_j\rangle\langle\bar{\psi}_j| \\ &= \int d\bar{\psi}_j\, d\psi_{j+1} |\psi_{j+1}\rangle\langle\psi_{j+1}|\bar{\psi}_j\rangle\langle\bar{\psi}_j| \\ &= \int \hbar\, d\bar{\psi}_j\, d\psi_{j+1} |\psi_{j+1}\rangle\, e^{-\frac{1}{\hbar}\bar{\psi}_j\psi_{j+1}} \langle\bar{\psi}_j| \end{aligned}$$

into the j-th slot $(j = 1, 2, \cdots, N-1)$ and use

$$\begin{aligned} \langle\bar{\psi}_j| e^{-i\frac{\epsilon}{\hbar}\hat{H}} |\psi_j\rangle &= \langle\bar{\psi}_j| \left(1 - i\frac{\epsilon}{\hbar} H(\bar{\psi}_j, \psi_j) + O(\epsilon^2)\right) |\psi_j\rangle \\ &= \left(1 - \frac{i\epsilon}{\hbar} H(\bar{\psi}_j, \psi_j) + O(\epsilon^2)\right) \langle\bar{\psi}_j|\psi_j\rangle \\ &= e^{\frac{1}{\hbar}\bar{\psi}_j\psi_j - \frac{i\epsilon}{\hbar} H(\bar{\psi}_j, \psi_j)} + O(\epsilon^2) \end{aligned}$$

Then, $Z(t_f, \bar{\psi}_f ; t_i, \psi_i)$

$$= \int \prod_{j=1}^{N-1} \hbar\, d\bar{\psi}_j\, d\psi_{j+1}\; e^{\frac{1}{\hbar}\bar{\psi}_f \psi_N - \frac{i\epsilon}{\hbar} H(\bar{\psi}_f, \psi_N)}$$

$$\cdot\, e^{-\frac{1}{\hbar}\bar{\psi}_{N-1}\psi_N} \cdot e^{\frac{1}{\hbar}\bar{\psi}_{N-1}\psi_{N-1} - \frac{i\epsilon}{\hbar} H(\bar{\psi}_{N-1}, \psi_{N-1})}$$

$$\cdots\cdots \cdot\, e^{-\frac{1}{\hbar}\bar{\psi}_1 \psi_2} \cdot e^{\frac{1}{\hbar}\bar{\psi}_1 \psi_i - \frac{i\epsilon}{\hbar} H(\bar{\psi}_1, \psi_i)} + O(N\epsilon^2)$$

$$= \int \prod_{j=1}^{N-1} \hbar\, d\bar{\psi}_j\, d\psi_{j+1}\; e^{\frac{1}{\hbar}\bar{\psi}_N \psi_N - \frac{i\epsilon}{\hbar} H(\bar{\psi}_N, \psi_N)}$$

$$\cdot\, \underbrace{\prod_{j=1}^{N-1} e^{-\frac{1}{\hbar}\bar{\psi}_j (\psi_{j+1} - \psi_j) - \frac{i\epsilon}{\hbar} H(\bar{\psi}_j, \psi_j)}}_{= e^{\frac{i}{\hbar}\sum_{j=1}^{N-1} \epsilon \left(i \bar{\psi}_j \frac{\psi_{j+1} - \psi_j}{\epsilon} - H(\bar{\psi}_j, \psi_j) \right)}} + O(N\epsilon^2)$$

with $\bar{\psi}_N := \bar{\psi}_f, \quad \psi_1 := \psi_i$

$N \to \infty$

$N\epsilon = t_f - t_i$: fixed

$$\longrightarrow \int \mathcal{D}\bar{\psi}\, \mathcal{D}\psi\; e^{\frac{1}{\hbar}\bar{\psi}\psi\big|_{t_f} + \frac{i}{\hbar}\int_{t_i}^{t_f} dt \left(i \bar{\psi}\dot{\psi} - H(\bar{\psi}, \psi) \right)}$$

$\bar{\psi}(t_f) = \bar{\psi}_f, \quad \psi(t_i) = \psi_i$

Path-integral expression for partition functions

$$\mathrm{Tr}_{\mathcal{H}}\, e^{-\frac{T}{\hbar}\hat{H}} = \int d\bar{\psi}\, d\psi\, \langle -\bar{\psi}|\, e^{-\frac{T}{\hbar}\hat{H}} |\psi\rangle\langle\psi|\bar{\psi}\rangle$$

- $\langle\psi|\bar{\psi}\rangle = \hbar\, e^{-\frac{1}{\hbar}\bar{\psi}\psi}$
- $\langle -\bar{\psi}| e^{-\frac{T}{\hbar}\hat{H}} |\psi\rangle$

$$= \int \prod_{j=1}^{N-1} \hbar\, d\bar{\psi}_j\, d\psi_{j+1} \cdot e^{\frac{1}{\hbar}\bar{\psi}_N\psi_N - \frac{\epsilon}{\hbar}H(\bar{\psi}_N,\psi_N)} \cdot e^{-\frac{1}{\hbar}\sum_{j=1}^{N-1}\epsilon\left(\bar{\psi}_j \frac{\psi_{j+1}-\psi_j}{\epsilon} + H(\bar{\psi}_j,\psi_j)\right)} + O(N\epsilon^2)$$

with $\bar{\psi}_N := -\bar{\psi}$, $\psi_1 := \psi$

$$= \int \hbar\, d\bar{\psi}\, d\psi \prod_{j=1}^{N-1} \hbar\, d\bar{\psi}_j\, d\psi_{j+1}\, e^{-\frac{1}{\hbar}\bar{\psi}\psi + \frac{1}{\hbar}\bar{\psi}_N\psi_N - \frac{\epsilon}{\hbar}H(\bar{\psi}_N,\psi_N)} \cdot e^{-\frac{1}{\hbar}\sum_{j=1}^{N-1}\epsilon\left(\bar{\psi}_j \frac{\psi_{j+1}-\psi_j}{\epsilon} + H(\bar{\psi}_j,\psi_j)\right)} + O(N\epsilon^2)$$

with $\bar{\psi}_N := -\bar{\psi}$, $\psi_1 := \psi$

Write $\bar{\psi} = \bar{\psi}_0 = -\bar{\psi}_N$, $\psi = \psi_1 = -\psi_{N+1}$

$$= \int \prod_{j=0}^{N-1} \hbar\, d\bar{\psi}_j\, d\psi_{j+1}\, e^{-\frac{1}{\hbar}\sum_{j=1}^{N}\epsilon\left(\bar{\psi}_j \frac{\psi_{j+1}-\psi_j}{\epsilon} + H(\bar{\psi}_j,\psi_j)\right)} + O(N\epsilon^2)$$

with $\bar{\psi}_N := -\bar{\psi}_0$, $\psi_{N+1} := -\psi_1$

Let us define $\psi(\epsilon j), \bar{\psi}(\epsilon j)$ for $j \in \mathbb{Z}$ by

$$\psi(\epsilon(j+1)) = \psi_{j+1}, \quad \bar{\psi}(\epsilon j) = \bar{\psi}_j \quad \text{for } j = 0, 1, \cdots, N-1$$

$$\& \quad \psi(\epsilon j + T) = \psi(\epsilon(j+N)) = -\psi(\epsilon j)$$

$$\bar{\psi}(\epsilon j + T) = \bar{\psi}(\epsilon(j+N)) = -\bar{\psi}(\epsilon j)$$

(Consistent by $\bar{\psi}_N = -\bar{\psi}_0$ & $\psi_{N+1} = -\psi_1$).

Then, the exponent is

$$-\frac{1}{\hbar} \sum_{j \in \mathbb{Z}/N\mathbb{Z}} \epsilon \left(\bar{\psi}(\epsilon j) \frac{\psi(\epsilon j + \epsilon) - \psi(\epsilon j)}{\epsilon} + H(\bar{\psi}(\epsilon j), \psi(\epsilon j)) \right)$$

$$\xrightarrow[N\epsilon = T]{N \to \infty} -\frac{1}{\hbar} \int_{\mathbb{R}/T\mathbb{Z}} d\tau \underbrace{\left(\bar{\psi}(\tau) \frac{d\psi(\tau)}{d\tau} + H(\bar{\psi}(\tau), \psi(\tau)) \right)}_{L_E(\bar{\psi}, \psi, \frac{d\psi}{d\tau})}$$

$$\therefore \operatorname{Tr}_{\mathcal{H}} e^{-\frac{T}{\hbar}\hat{H}}$$

$$= \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \, e^{-\frac{1}{\hbar} \int_{\mathbb{R}/T\mathbb{Z}} d\tau \, L_E(\bar{\psi}, \psi, \frac{d\psi}{d\tau})}$$

$$\bar{\psi}(\tau + T) = -\bar{\psi}(\tau), \quad \psi(\tau + T) = -\psi(\tau)$$

antiperiodic

$$\mathrm{Tr}_{\mathcal{H}} (-1)^F e^{-\frac{T}{\hbar}\hat{H}} = \int d\bar{\psi}\, d\psi \, \langle \bar{\psi} | e^{-\frac{T}{\hbar}\hat{H}} | \psi \rangle \langle \psi | \bar{\psi} \rangle$$

$$= \int \hbar\, d\bar{\psi}\, d\psi \prod_{j=1}^{N-1} \hbar\, d\bar{\psi}_j\, d\psi_{j+1} \; e^{-\frac{1}{\hbar}\bar{\psi}\psi + \frac{1}{\hbar}\bar{\psi}_N \psi_N - \frac{\epsilon}{\hbar} H(\bar{\psi}_N, \psi_N)} \cdot e^{-\frac{1}{\hbar}\sum_{j=1}^{N-1} \epsilon \left(\bar{\psi}_j \frac{\psi_{j+1} - \psi_j}{\epsilon} + H(\bar{\psi}_j, \psi_j) \right)} + O(N\epsilon^2)$$

with $\bar{\psi}_N := \bar{\psi}$, $\psi_1 := \psi$

Write $\bar{\psi} = \bar{\psi}_0 = -\bar{\psi}_N$, $\psi = \psi_1 = -\psi_{N+1}$

$$= \int \prod_{j=0}^{N-1} \hbar\, d\bar{\psi}_j\, d\psi_{j+1} \; e^{-\frac{1}{\hbar}\sum_{j=1}^{N} \epsilon \left(\bar{\psi}_j \frac{\psi_{j+1} - \psi_j}{\epsilon} + H(\bar{\psi}_j, \psi_j) \right)} + O(N\epsilon^2)$$

with $\bar{\psi}_N := \bar{\psi}_0$, $\psi_{N+1} := \psi_1$

$$= \int \prod_{j \in \mathbb{Z}/N\mathbb{Z}} \hbar\, d\bar{\psi}(\epsilon j)\, d\psi(\epsilon j + \epsilon) \; e^{-\frac{1}{\hbar}\sum_{j \in \mathbb{Z}/N\mathbb{Z}} \epsilon \left(\bar{\psi}(\epsilon j) \frac{\psi(\epsilon j + \epsilon) - \psi(\epsilon j)}{\epsilon} + H(\bar{\psi}(\epsilon j), \psi(\epsilon j)) \right)} + O(N\epsilon^2)$$

$$\bar{\psi}(\epsilon j + T) = \bar{\psi}(\epsilon j), \quad \psi(\epsilon j + T) = \psi(\epsilon j)$$

$$\xrightarrow[N\epsilon = T]{N \to \infty} \int \mathcal{D}\bar{\psi}\, \mathcal{D}\psi \; e^{-\frac{1}{\hbar}\int_{\mathbb{R}/T\mathbb{Z}} d\tau \, L_E\left(\bar{\psi}, \psi, \frac{d\psi}{d\tau}\right)}$$

$$\bar{\psi}(\tau + T) = \bar{\psi}(\tau), \quad \psi(\tau + T) = \psi(\tau)$$

periodic