

Fermionic quantum mechanics

Consider the classical mechanics with

a pair of anticommuting variables $\psi(t), \bar{\psi}(t)$ and

$$\text{Lagrangian } L = i\bar{\psi}\dot{\psi} - \omega\bar{\psi}\psi$$

where $\omega \in \mathbb{R}$.

Symmetries

① time translation $\psi(t) \rightarrow \psi(t+\delta t), \bar{\psi}(t) \rightarrow \bar{\psi}(t+\delta t)$

$$\leadsto \delta\psi = \epsilon\dot{\psi}, \delta\bar{\psi} = \epsilon\dot{\bar{\psi}}$$

The Noether charge is $E = \omega\bar{\psi}\psi$ (energy)

② phase rotation $\psi(t) \rightarrow e^{-i\alpha}\psi(t), \bar{\psi}(t) \rightarrow e^{i\alpha}\bar{\psi}(t)$.

$$\leadsto \delta\psi = -i\epsilon\psi, \delta\bar{\psi} = i\epsilon\bar{\psi}$$

The Noether charge is $Q = \bar{\psi}\psi$ (fermion number)

Exercise: Show that the Noether charges are as given above.

• L is real (modulo total derivative) under

$$\psi^* = \bar{\psi}, \quad \bar{\psi}^* = \psi :$$

$$L^* = -i\dot{\psi}^* \bar{\psi}^* - \omega \psi^* \bar{\psi}^* = -i\dot{\bar{\psi}} \psi - \omega \bar{\psi} \psi$$

$$= \underbrace{i\bar{\psi}\dot{\psi} - \omega \bar{\psi}\psi}_L + \frac{d}{dt}(-i\bar{\psi}\psi)$$

Let us quantize the system $\left\{ \begin{array}{l} \text{Path-integral ?} \\ \text{Operator ?} \end{array} \right.$

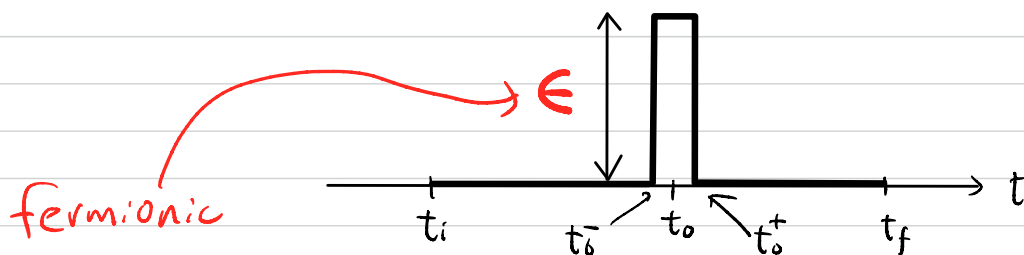
We shall take the "mixed" way: Use the path-integral

to find the commutation relation of $\hat{\psi}$ & $\hat{\bar{\psi}}$

and then find the representation of the algebra.

Consider the variation

$$\delta\psi(t) = \epsilon(t), \quad \delta\bar{\psi}(t) = 0$$



$$\begin{aligned}
\delta S[\psi, \bar{\psi}] &= \delta \int_{t_i}^{t_f} dt (i \bar{\psi} \dot{\psi} - \omega \bar{\psi} \psi) \\
&= \int_{t_i}^{t_f} dt (i \bar{\psi}(t) \dot{\epsilon}(t) - \omega \bar{\psi}(t) \epsilon(t)) \\
&\quad \epsilon \delta(t-t_0^-) - \epsilon \delta(t-t_0^+) \\
&= i \bar{\psi}(t_0^-) \epsilon - i \bar{\psi}(t_0^+) \epsilon - \int_{t_0^-}^{t_0^+} dt \omega \bar{\psi} \epsilon
\end{aligned}$$

Ward identity :

$$\begin{aligned}
0 &= \int \delta(\delta \bar{\psi} \delta \psi) e^{\frac{i}{\hbar} S[\psi, \bar{\psi}]} \psi(t_0) \\
&= \int \delta \bar{\psi} \delta \psi \left(\frac{i}{\hbar} \delta S[\psi, \bar{\psi}] \psi(t_0) + \epsilon \right) \\
&= \int \delta \bar{\psi} \delta \psi \left(-\frac{1}{\hbar} \bar{\psi}(t_0^-) \epsilon \cdot \psi(t_0) + \frac{1}{\hbar} \bar{\psi}(t_0^+) \epsilon \cdot \psi(t_0) \right. \\
&\quad \left. - \frac{i}{\hbar} \int_{t_0^-}^{t_0^+} dt \omega \bar{\psi}(t) \epsilon \cdot \psi(t_0) + \epsilon \right)
\end{aligned}$$

This implies an identity among the operators

$\hat{\psi}, \hat{\bar{\psi}}$ corresponding to $\psi, \bar{\psi}$.

Take the limit $t_0^+ \rightarrow t_0$, $t_0^- \rightarrow t_0$:

Recalling the time ordered product and noting

$\int_{t_0^-}^{t_0^+} dt \omega \bar{\Psi} \epsilon \cdot \Psi(t_0) \rightarrow 0$ in the limit, we find

$$0 = -\frac{1}{\hbar} \hat{\Psi} \circ \hat{\bar{\Psi}} \epsilon + \frac{1}{\hbar} \hat{\bar{\Psi}} \epsilon \circ \hat{\Psi} + \epsilon$$

$\epsilon, \hat{\Psi}, \hat{\bar{\Psi}}$ fermionic

$$= \left[-\frac{1}{\hbar} \hat{\Psi} \circ \hat{\bar{\Psi}} - \frac{1}{\hbar} \hat{\bar{\Psi}} \circ \hat{\Psi} + 1 \right] \epsilon.$$

$$\therefore \hat{\Psi} \circ \hat{\bar{\Psi}} + \hat{\bar{\Psi}} \circ \hat{\Psi} = \hbar.$$

If we use the notation

$$\{A, B\} := AB + BA \quad (\text{anticommutator})$$

this can be written as

$$\{\hat{\Psi}, \hat{\bar{\Psi}}\} = \hbar.$$

Similarly,

$$\bullet \quad 0 = \int \delta(\delta\bar{\Psi} \delta\Psi e^{\frac{i}{\hbar} S[\Psi, \bar{\Psi}]} \bar{\Psi}(t_0)) \quad \text{for the same } \delta$$

$$\Rightarrow \hat{\bar{\Psi}} \circ \hat{\Psi} = 0$$

$$\bullet \quad 0 = \int \delta(\delta\bar{\Psi} \delta\Psi e^{\frac{i}{\hbar} S[\Psi, \bar{\Psi}]} \Psi(t_0))$$

for $\delta\Psi(t) = 0$ and $\delta\bar{\Psi}(t) = \text{the same } \epsilon(t)$

$$\Rightarrow \hat{\Psi} \circ \hat{\bar{\Psi}} = 0$$

The canonical commutation relation of $\hat{\Psi}$ and $\hat{\bar{\Psi}}$ is

$$\{\hat{\Psi}, \hat{\bar{\Psi}}\} = \hbar,$$

$$\hat{\Psi}^2 = \hat{\bar{\Psi}}^2 = 0.$$

... "Clifford algebra".

Representation of the algebra.

Reality of the variables $\psi^* = \bar{\psi}$, $\bar{\psi}^* = \psi$

\leadsto hermiticity of the operators $\hat{\psi}^\dagger = \hat{\bar{\psi}}$, $\hat{\bar{\psi}}^\dagger = \hat{\psi}$.

The commutation relation reads

$$\{\hat{\psi}, \hat{\psi}^\dagger\} = \hbar$$

$$\{\hat{\psi}, \hat{\psi}\} = \{\hat{\psi}^\dagger, \hat{\psi}^\dagger\} = 0.$$

This is like the algebra of annihilation/creation operators in harmonic oscillators. As in that case, we may prepare a state $|0\rangle$ annihilated by $\hat{\psi}$,

$$\hat{\psi}|0\rangle = 0$$

and build other states by multiplying powers of $\hat{\psi}^\dagger$,

$$\hat{\psi}^\dagger|0\rangle, \hat{\psi}^{\dagger 2}|0\rangle, \hat{\psi}^{\dagger 3}|0\rangle, \dots$$

But, by the relation $\hat{\psi}^{\dagger 2} = 0$, $\hat{\psi}^{\dagger 2}|0\rangle = \hat{\psi}^{\dagger 3}|0\rangle = \dots = 0$.

Only $\hat{\psi}^\dagger|0\rangle$ among them can be non-zero.

Also,

$$\hat{\Psi}(\hat{\Psi}^+|0\rangle) = \underbrace{(\hat{\Psi}\hat{\Psi}^+ + \hat{\Psi}^+\hat{\Psi})}_{\hbar}|0\rangle - \underbrace{\hat{\Psi}^+\hat{\Psi}}_0|0\rangle = \hbar|0\rangle.$$

Thus $\hat{\Psi}^+|0\rangle$ is indeed non-zero. We have a

2-dimensional representation \mathcal{H} , with basis $|0\rangle, \hat{\Psi}^+|0\rangle$.

With respect to this basis, $\hat{\Psi}$ and $\hat{\Psi}^+$ are represented by matrices

$$\hat{\Psi} = \begin{pmatrix} 0 & \hbar \\ 0 & 0 \end{pmatrix}, \quad \hat{\Psi}^+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

[f $|0\rangle$ is normalized as $\| |0\rangle \|^2 = \langle 0|0\rangle = 1$,

$$\| \hat{\Psi}^+|0\rangle \|^2 = \langle 0|\hat{\Psi}\hat{\Psi}^+|0\rangle = \langle 0|\hbar|0\rangle = \hbar.$$

Energy spectrum?

Recall $E = \omega\bar{\Psi}\Psi$.

$$\leadsto \hat{H} = \omega\hat{\Psi}^+\hat{\Psi} = \begin{cases} 0 & \text{on } |0\rangle \\ \hbar\omega & \text{on } \hat{\Psi}^+|0\rangle \end{cases}$$

* There is an operator ordering ambiguity.

Classically, $\bar{\Psi}\Psi = (1-s)\bar{\Psi}\Psi - s\Psi\bar{\Psi}$ for any s , but they correspond to different operators in the quantum theory:

$$\begin{aligned}\hat{H} &= (1-s)\omega\hat{\Psi}^\dagger\hat{\Psi} - s\omega\hat{\Psi}\hat{\Psi}^\dagger = \omega\hat{\Psi}^\dagger\hat{\Psi} - s\hbar\omega. \\ &= \begin{cases} -s\hbar\omega & \text{on } |0\rangle \\ (1-s)\hbar\omega & \text{on } \hat{\Psi}^\dagger|0\rangle. \end{cases}\end{aligned}$$

e.g. the "symmetric" ordering $s = \frac{1}{2}$:

$$\begin{aligned}\hat{H} &= \frac{1}{2}\omega\hat{\Psi}^\dagger\hat{\Psi} - \frac{1}{2}\omega\hat{\Psi}\hat{\Psi}^\dagger = \frac{\omega}{2}[\hat{\Psi}^\dagger, \hat{\Psi}] \\ &= \begin{cases} -\frac{1}{2}\hbar\omega & \text{on } |0\rangle \\ \frac{1}{2}\hbar\omega & \text{on } \hat{\Psi}^\dagger|0\rangle. \end{cases}\end{aligned}$$

Also $Q = \bar{\Psi}\Psi \rightsquigarrow \hat{Q} = \hat{\Psi}^\dagger\hat{\Psi}$

$$\begin{aligned}\text{or } \hat{Q} &= (1-s)\hat{\Psi}^\dagger\hat{\Psi} - s\hat{\Psi}\hat{\Psi}^\dagger = \hat{\Psi}^\dagger\hat{\Psi} - s\hbar \\ &= \begin{cases} -s\hbar & \text{on } |0\rangle \\ (1-s)\hbar & \text{on } \hat{\Psi}^\dagger|0\rangle. \end{cases}\end{aligned}$$

$\hat{\Psi}$ & $\hat{\Psi}^\dagger$ eigenstates

As a passage to (go back to) the path-integral,

we find eigenstates of $\hat{\Psi}$ and $\hat{\Psi}^\dagger$

(just like eigenstates $|q\rangle$ & $|p\rangle$ for \hat{q} & \hat{p}).

$$\begin{aligned} \text{Suppose } |\psi\rangle &= a|0\rangle + b\hat{\Psi}^\dagger|0\rangle & \text{satisfy } \hat{\Psi}|\psi\rangle &= \psi|\psi\rangle \\ |\bar{\psi}\rangle &= c|0\rangle + d\hat{\Psi}^\dagger|0\rangle & \hat{\Psi}^\dagger|\bar{\psi}\rangle &= \bar{\psi}|\bar{\psi}\rangle. \end{aligned}$$

$$\text{As } \hat{\Psi}|\psi\rangle = (-1)^{|a|} a \hat{\Psi}|0\rangle + (-1)^{|b|} b \hat{\Psi}\hat{\Psi}^\dagger|0\rangle$$

$\hat{\Psi}|0\rangle$

$$\psi|\psi\rangle = \psi a|0\rangle + \psi b\hat{\Psi}^\dagger|0\rangle$$

$$\hat{\Psi}^\dagger|\bar{\psi}\rangle = (-1)^{|c|} c \hat{\Psi}^\dagger|0\rangle + (-1)^{|d|} d \hat{\Psi}^\dagger\hat{\Psi}^\dagger|0\rangle$$

$$\bar{\psi}|\bar{\psi}\rangle = \bar{\psi}c|0\rangle + \bar{\psi}d\hat{\Psi}^\dagger|0\rangle$$

$$\text{we find } \psi a = (-1)^{|b|} b \bar{\psi}, \quad \psi b = 0$$

$$\bar{\psi}c = 0, \quad \bar{\psi}d = (-1)^{|c|} c$$

$$\text{A solution } (a, b) = (1, -\psi/\bar{\psi})$$

$$(c, d) = (\bar{\psi}, -1)$$

$$|\psi\rangle = \left(1 - \frac{\hat{\psi}}{\hbar} \hat{\psi}^\dagger\right) |0\rangle = \left(1 + \frac{1}{\hbar} \hat{\psi}^\dagger \psi\right) |0\rangle,$$

$$|\bar{\psi}\rangle = (\bar{\psi} - \hat{\psi}^\dagger) |0\rangle.$$

} (★)

We have chosen a random normalization, but this turns out to be a useful one.

A more symmetric normalization would be

$$|\psi\rangle = \hbar^{\frac{1}{4}} \left(1 + \frac{1}{\hbar} \hat{\psi}^\dagger \psi\right) |0\rangle, \quad |\bar{\psi}\rangle = \hbar^{-\frac{1}{4}} (\bar{\psi} - \hat{\psi}^\dagger) |0\rangle.$$

This is also good, but we shall use (★)

We also prepare "bra" eigenstates for $\hat{\psi}$ & $\hat{\psi}^\dagger$:

$$\langle\psi| \hat{\psi} = \langle\psi|\psi, \quad \langle\bar{\psi}| \hat{\psi}^\dagger = \langle\bar{\psi}|\bar{\psi}.$$

For these we may take

$$\langle\psi| = |\bar{\psi}\rangle^\dagger = \langle 0| (\bar{\psi} - \hat{\psi}^\dagger)$$

$$\langle\bar{\psi}| = |\psi\rangle^\dagger = \langle 0| \left(1 + \frac{1}{\hbar} \bar{\psi} \hat{\psi}\right).$$

[(⊖) Take the Hermitian conjugate of $\hat{\psi}^\dagger |\bar{\psi}\rangle = \bar{\psi} |\bar{\psi}\rangle$ and $\hat{\psi} |\psi\rangle = \psi |\psi\rangle. \quad \square$]

Just like $\int dq |q\rangle\langle q| = \int dp |p\rangle\langle p| = id_{\mathcal{H}}$, we have

$$\int d\psi |\psi\rangle\langle\psi| = \int d\bar{\psi} |\bar{\psi}\rangle\langle\bar{\psi}| = id_{\mathcal{H}}.$$

proof $\int d\psi |\psi\rangle\langle\psi|_0 = \int d\psi \left(1 + \frac{1}{\hbar} \hat{\psi}^+ \psi\right) |0\rangle\langle 0| (\psi - \hat{\psi}) |0\rangle$
0 *0* *0*
 $= \int d\psi \left(1 + \frac{1}{\hbar} \hat{\psi}^+ \psi\right) \psi |0\rangle\langle 0|_0 = |0\rangle.$
0 *0* *0*
 $|0\rangle\langle 0|$ is even

$$\int d\psi |\psi\rangle\langle\psi| \hat{\psi}^+ |0\rangle = \int d\psi \left(1 + \frac{1}{\hbar} \hat{\psi}^+ \psi\right) |0\rangle\langle 0| (\psi - \hat{\psi}) \hat{\psi}^+ |0\rangle$$

$$= \int d\psi \left(1 - \psi \frac{1}{\hbar} \hat{\psi}^+\right) |0\rangle\langle 0| (-\hbar) |0\rangle = \hat{\psi}^+ |0\rangle.$$

$$\int d\bar{\psi} |\bar{\psi}\rangle\langle\bar{\psi}|_0 = \int d\bar{\psi} (\bar{\psi} - \hat{\bar{\psi}}^+) |0\rangle\langle 0| \left(1 + \frac{1}{\hbar} \bar{\psi} \hat{\bar{\psi}}\right) |0\rangle$$

$$= \int d\bar{\psi} (\bar{\psi} - \hat{\bar{\psi}}^+) |0\rangle\langle 0|_0 = |0\rangle.$$

$$\int d\bar{\psi} |\bar{\psi}\rangle\langle\bar{\psi}| \hat{\bar{\psi}}^+ |0\rangle = \int d\bar{\psi} (\bar{\psi} - \hat{\bar{\psi}}^+) |0\rangle\langle 0| \left(1 + \frac{1}{\hbar} \bar{\psi} \hat{\bar{\psi}}\right) \hat{\bar{\psi}}^+ |0\rangle$$

$$= \int d\bar{\psi} (\bar{\psi} - \hat{\bar{\psi}}^+) |0\rangle\langle 0| \bar{\psi} |0\rangle$$

$$= \int d\bar{\psi} (\bar{\psi} - \hat{\bar{\psi}}^+) \bar{\psi} |0\rangle\langle 0|_0 = \hat{\bar{\psi}}^+ |0\rangle.$$

0 *0* *0* *0* *0* *0* *0* *0*
 $|0\rangle\langle 0|$ is even

Q.E.D.

Note also

$$\begin{aligned}\langle \psi_1 | \bar{\psi}_2 \rangle &= \langle 0 | (\psi_1 - \hat{\psi}) (\bar{\psi}_2 - \hat{\psi}^\dagger) | 0 \rangle = \langle 0 | \overbrace{(\psi_1 \bar{\psi}_2 + \hbar)}^{\text{even}} | 0 \rangle \\ &= \hbar + \psi_1 \bar{\psi}_2 = \hbar \left(1 + \frac{1}{\hbar} \psi_1 \bar{\psi}_2 \right) = \hbar e^{\frac{1}{\hbar} \psi_1 \bar{\psi}_2},\end{aligned}$$

$$\begin{aligned}\langle \bar{\psi}_1 | \psi_2 \rangle &= \langle 0 | \left(1 + \frac{1}{\hbar} \bar{\psi}_1 \hat{\psi} \right) \left(1 + \frac{1}{\hbar} \hat{\psi}^\dagger \psi_2 \right) | 0 \rangle \\ &= \langle 0 | \left(1 + \frac{1}{\hbar} \bar{\psi}_1 \psi_2 \right) | 0 \rangle = 1 + \frac{1}{\hbar} \bar{\psi}_1 \psi_2 = e^{\frac{1}{\hbar} \bar{\psi}_1 \psi_2},\end{aligned}$$

$$\begin{aligned}\langle \psi_1 | \psi_2 \rangle &= \langle 0 | (\psi_1 - \hat{\psi}) \left(1 + \frac{1}{\hbar} \hat{\psi}^\dagger \psi_2 \right) | 0 \rangle \\ &= \langle 0 | (\psi_1 - \psi_2) | 0 \rangle = (-1)^{|0\rangle} (\psi_1 - \psi_2),\end{aligned}$$

$$\begin{aligned}\langle \bar{\psi}_1 | \bar{\psi}_2 \rangle &= \langle 0 | \left(1 + \frac{1}{\hbar} \bar{\psi}_1 \hat{\psi} \right) (\bar{\psi}_2 - \hat{\psi}^\dagger) | 0 \rangle \\ &= \langle 0 | (\bar{\psi}_2 - \bar{\psi}_1) | 0 \rangle = -(-1)^{|0\rangle} (\bar{\psi}_1 - \bar{\psi}_2).\end{aligned}$$

The latter two can also be used for derivation of

$$\int d\psi |\psi\rangle \langle \psi| = \int d\bar{\psi} |\bar{\psi}\rangle \langle \bar{\psi}| = \text{id}_{\mathbb{C}}$$

$$\textcircled{?} \int d\psi |\psi\rangle \langle \psi | \psi_1 \rangle = \int d\psi |\psi\rangle (-1)^{|0\rangle} (\psi - \psi_1)$$

$$= \int d\psi (\psi - \psi_1) |\psi\rangle = |\psi_1\rangle \quad \text{etc.}$$

For $A \in \text{End}_{\mathbb{C}} \mathcal{H}$

$$\text{Tr}_{\text{de}} A = \int d\bar{\psi} d\psi \langle -\bar{\psi} | A | \psi \rangle \langle \psi | \bar{\psi} \rangle$$

⊙ It is enough to show this for $A = \text{id}, \hat{\psi}, \hat{\psi}^+, \hat{\psi}^+ \hat{\psi}$ as they span $\text{End}_{\mathbb{C}} \mathcal{H}$.

$$A = \text{id} : \int d\bar{\psi} d\psi \underbrace{\langle -\bar{\psi} | \psi \rangle}_{e^{-\frac{1}{\hbar} \bar{\psi} \psi}} \underbrace{\langle \psi | \bar{\psi} \rangle}_{\hbar e^{\frac{1}{\hbar} \psi \bar{\psi}}} = \int \hbar d\bar{\psi} d\psi e^{-\frac{2}{\hbar} \bar{\psi} \psi} = 2 = \text{tr id} \quad \checkmark$$

$$A = \hat{\psi} : \int d\bar{\psi} d\psi \underbrace{\langle -\bar{\psi} | \hat{\psi} | \psi \rangle}_{\psi | \psi \rangle} \langle \psi | \bar{\psi} \rangle = \int \hbar d\bar{\psi} d\psi (\pm \psi) e^{-\frac{2}{\hbar} \bar{\psi} \psi} = 0 = \text{tr } \hat{\psi} \quad \checkmark$$

$$A = \hat{\psi}^+ : \int d\bar{\psi} d\psi \underbrace{\langle -\bar{\psi} | \hat{\psi}^+ | \psi \rangle}_{\langle -\bar{\psi} | (-\bar{\psi}) \rangle} \langle \psi | \bar{\psi} \rangle = \int \hbar d\bar{\psi} d\psi (-\bar{\psi}) e^{-\frac{2}{\hbar} \bar{\psi} \psi} = 0 = \text{tr } \hat{\psi}^+ \quad \checkmark$$

$$A = \hat{\psi}^+ \hat{\psi} : \int d\bar{\psi} d\psi \underbrace{\langle -\bar{\psi} | \hat{\psi}^+ \hat{\psi} | \psi \rangle}_{-\bar{\psi} \psi} \langle \psi | \bar{\psi} \rangle = \int \hbar d\bar{\psi} d\psi (-\bar{\psi} \psi) e^{-\frac{2}{\hbar} \bar{\psi} \psi} = \hbar = \text{tr}(\hat{\psi}^+ \hat{\psi}) \quad \checkmark$$

□

We also have

$$\begin{aligned} \text{tr}_{\text{de}} A &= \int d\bar{\Psi} d\Psi \langle \Psi | A | \bar{\Psi} \rangle \langle \bar{\Psi} | -\Psi \rangle \\ &= (-1)^{|\Psi\rangle} \int d\Psi \langle \Psi | A | -\Psi \rangle \\ &= (-1)^{|\Psi\rangle} \int d\bar{\Psi} \langle -\bar{\Psi} | A | \Psi \rangle \end{aligned}$$

If we define $(-1)^{\mathbb{F}}$ by $(-1)^{\mathbb{F}} = \begin{cases} +1 & \text{on } |\psi\rangle \\ -1 & \text{on } \widehat{\Psi}^+ |\psi\rangle, \end{cases}$

$$\text{then } (-1)^{\mathbb{F}} |\Psi\rangle = |-\Psi\rangle$$

$$\hookrightarrow \langle \bar{\Psi} | (-1)^{\mathbb{F}} = \langle -\bar{\Psi} |$$

$$\therefore \text{tr}_{\text{de}} (-1)^{\mathbb{F}} A = \int d\bar{\Psi} d\Psi \langle \bar{\Psi} | A | \Psi \rangle \langle \Psi | \bar{\Psi} \rangle$$

States as wave functions

We can represent states as functions of $\bar{\Psi}$:

$$|\Psi\rangle \leftrightarrow \bar{\Psi}(\bar{\Psi}) = \langle \bar{\Psi} | \Psi \rangle.$$

Note: $\bar{\Psi}(\bar{\Psi})^* = \langle \bar{\Psi} | \Psi \rangle^* = \langle \Psi | \bar{\Psi} \rangle.$

• The inner product of states is represented as

$$\langle \Psi_1 | \Psi_2 \rangle = \langle \Psi_1 | \underbrace{\int d\psi |\psi\rangle}_{\text{even}} \langle \psi | \int d\bar{\Psi} |\bar{\Psi}\rangle \langle \bar{\Psi} | \Psi_2 \rangle$$

$$= \langle \Psi_1 | \underbrace{\int d\bar{\Psi} d\psi |\psi\rangle}_{\text{even}} \langle \psi | \bar{\Psi} \rangle \langle \bar{\Psi} | \Psi_2 \rangle$$

$$= \int d\bar{\Psi} d\psi \underbrace{\langle \Psi_1 | \psi \rangle}_{\bar{\Psi}_1(\bar{\Psi})^*} \underbrace{\langle \psi | \bar{\Psi} \rangle}_{\frac{1}{\hbar} e^{\frac{i}{\hbar} \psi \bar{\Psi}}} \underbrace{\langle \bar{\Psi} | \Psi_2 \rangle}_{\bar{\Psi}_2(\bar{\Psi})}$$

$$= \int \hbar d\bar{\Psi} d\psi \bar{\Psi}_1(\bar{\Psi})^* e^{\frac{i}{\hbar} \psi \bar{\Psi}} \bar{\Psi}_2(\bar{\Psi})$$

• Time evolution of states:

$$\begin{aligned}
 (e^{-i\frac{t_f-t_i}{\hbar}\hat{H}}\Psi)(\bar{\Psi}) &= \langle \bar{\Psi} | e^{-i\frac{t_f-t_i}{\hbar}\hat{H}} | \Psi \rangle \\
 &= \langle \bar{\Psi} | e^{-i\frac{t_f-t_i}{\hbar}\hat{H}} \int d\psi' |\psi'\rangle \langle \psi' | \int d\psi |\psi\rangle \langle \bar{\Psi} | \Psi \rangle \\
 &= \int d\bar{\Psi}' d\psi' \langle \bar{\Psi} | e^{-i\frac{t_f-t_i}{\hbar}\hat{H}} |\psi'\rangle \underbrace{\langle \psi' | \bar{\Psi}' \rangle}_{\hbar e^{\frac{1}{\hbar}\psi'\bar{\Psi}'}} \underbrace{\langle \bar{\Psi}' | \Psi \rangle}_{\Psi(\bar{\Psi})}
 \end{aligned}$$

As in the bosonic case, let us define the transition amplitude by

$$Z(t_f, \bar{\Psi}_f; t_i, \Psi_i) := \langle \bar{\Psi}_f | e^{-i\frac{(t_f-t_i)}{\hbar}\hat{H}} | \Psi_i \rangle.$$

Then the time evolution of states is given by

$$\begin{aligned}
 (e^{-i\frac{t_f-t_i}{\hbar}\hat{H}}\Psi)(\bar{\Psi}_f) \\
 = \int \hbar d\bar{\Psi}_i d\Psi_i Z(t_f, \bar{\Psi}_f; t_i, \Psi_i) e^{\frac{1}{\hbar}\Psi_i\bar{\Psi}_i} \Psi(\bar{\Psi}_i).
 \end{aligned}$$

Path-integral expression for $Z(t_f, \bar{\Psi}_f; t_i, \Psi_i)$

Let us divide $t_f - t_i$ into N pieces, $t_f - t_i = N\epsilon$,

$$e^{-i \frac{t_f - t_i}{\hbar} \hat{H}} = \underbrace{e^{-i \frac{\epsilon}{\hbar} \hat{H}} \cdots e^{-i \frac{\epsilon}{\hbar} \hat{H}}}_N$$

and insert

$$1 = \int d\psi_{j+1} |\psi_{j+1}\rangle \langle \psi_{j+1}| \int d\bar{\psi}_j |\bar{\psi}_j\rangle \langle \bar{\psi}_j|$$

$$= \int d\bar{\psi}_j d\psi_{j+1} |\psi_{j+1}\rangle \langle \psi_{j+1} | \bar{\psi}_j \rangle \langle \bar{\psi}_j|$$

$$= \int \hbar d\bar{\psi}_j d\psi_{j+1} |\psi_{j+1}\rangle e^{-\frac{1}{\hbar} \bar{\psi}_j \psi_{j+1}} \langle \bar{\psi}_j|$$

into the j -th slot ($j=1, 2, \dots, N-1$) and use

$$\langle \bar{\psi}_j | e^{-i \frac{\epsilon}{\hbar} \hat{H}} | \psi_j \rangle = \langle \bar{\psi}_j | (1 - i \frac{\epsilon}{\hbar} H(\bar{\psi}_j, \psi_j) + O(\epsilon^2)) | \psi_j \rangle$$

$$= (1 - \frac{i\epsilon}{\hbar} H(\bar{\psi}_j, \psi_j) + O(\epsilon^2)) \langle \bar{\psi}_j | \psi_j \rangle$$

$$= e^{\frac{1}{\hbar} \bar{\psi}_j \psi_j - \frac{i\epsilon}{\hbar} H(\bar{\psi}_j, \psi_j)} + O(\epsilon^2)$$

Then, $Z(t_f, \bar{\Psi}_f; t_i, \Psi_i)$

$$= \int \prod_{j=1}^{N-1} h d\bar{\Psi}_j d\Psi_{j+1} e^{\frac{1}{\hbar} \bar{\Psi}_f \Psi_N - \frac{i\epsilon}{\hbar} H(\bar{\Psi}_f, \Psi_N)} \\ \cdot e^{-\frac{1}{\hbar} \bar{\Psi}_{N-1} \Psi_N} \cdot e^{\frac{1}{\hbar} \bar{\Psi}_{N-1} \Psi_{N-1} - \frac{i\epsilon}{\hbar} H(\bar{\Psi}_{N-1}, \Psi_{N-1})} \\ \dots \cdot e^{-\frac{1}{\hbar} \bar{\Psi}_1 \Psi_2} \cdot e^{\frac{1}{\hbar} \bar{\Psi}_1 \Psi_1 - \frac{i\epsilon}{\hbar} H(\bar{\Psi}_1, \Psi_1)} + O(N\epsilon^2)$$

$$= \int \prod_{j=1}^{N-1} h d\bar{\Psi}_j d\Psi_{j+1} e^{\frac{1}{\hbar} \bar{\Psi}_N \Psi_N - \frac{i\epsilon}{\hbar} H(\bar{\Psi}_N, \Psi_N)} \\ \cdot \prod_{j=1}^{N-1} e^{-\frac{1}{\hbar} \bar{\Psi}_j (\Psi_{j+1} - \Psi_j) - \frac{i\epsilon}{\hbar} H(\bar{\Psi}_j, \Psi_j)} + O(N\epsilon^2)$$

with $\bar{\Psi}_N = \bar{\Psi}_f, \Psi_1 = \Psi_i$

$$e^{\frac{i}{\hbar} \sum_{j=1}^{N-1} \epsilon \left(i \bar{\Psi}_j \frac{\Psi_{j+1} - \Psi_j}{\epsilon} - H(\bar{\Psi}_j, \Psi_j) \right)}$$

$N \rightarrow \infty$

$N\epsilon = t_f - t_i$: fixed

$$\longrightarrow \int \mathcal{D}\bar{\Psi} \mathcal{D}\Psi e^{\frac{1}{\hbar} \bar{\Psi} \Psi |_{t_f} + \frac{i}{\hbar} \int_{t_i}^{t_f} dt \left(i \bar{\Psi} \dot{\Psi} - H(\bar{\Psi}, \Psi) \right)}$$

$$\bar{\Psi}(t_f) = \bar{\Psi}_f, \Psi(t_i) = \Psi_i$$

Path-integral expression for partition functions

$$\text{Tr} e^{-\frac{T}{\hbar} \hat{H}} = \int d\bar{\Psi} d\Psi \langle -\bar{\Psi} | e^{-\frac{T}{\hbar} \hat{H}} | \Psi \rangle \langle \Psi | \bar{\Psi} \rangle$$

$$\left[\begin{array}{l} \bullet \langle \Psi | \bar{\Psi} \rangle = \hbar e^{-\frac{1}{\hbar} \bar{\Psi} \Psi} \\ \bullet \langle -\bar{\Psi} | e^{-\frac{T}{\hbar} \hat{H}} | \Psi \rangle \\ = \int \prod_{j=1}^{N-1} \hbar d\bar{\Psi}_j d\Psi_{j+1} \cdot e^{\frac{1}{\hbar} \bar{\Psi}_0 \Psi_0 - \frac{\epsilon}{\hbar} H(\bar{\Psi}_0, \Psi_0)} \\ \cdot e^{-\frac{1}{\hbar} \sum_{j=1}^{N-1} \epsilon \left(\bar{\Psi}_j \frac{\Psi_{j+1} - \Psi_j}{\epsilon} + H(\bar{\Psi}_j, \Psi_j) \right)} + O(N\epsilon^2) \\ \text{with } \bar{\Psi}_0 = -\bar{\Psi}, \Psi_1 = \Psi \end{array} \right.$$

$$= \int \hbar d\bar{\Psi} d\Psi \prod_{j=1}^{N-1} \hbar d\bar{\Psi}_j d\Psi_{j+1} e^{-\frac{1}{\hbar} \bar{\Psi} \Psi + \frac{1}{\hbar} \bar{\Psi}_0 \Psi_0 - \frac{\epsilon}{\hbar} H(\bar{\Psi}_0, \Psi_0)} \\ \cdot e^{-\frac{1}{\hbar} \sum_{j=1}^{N-1} \epsilon \left(\bar{\Psi}_j \frac{\Psi_{j+1} - \Psi_j}{\epsilon} + H(\bar{\Psi}_j, \Psi_j) \right)} + O(N\epsilon^2)$$

$$\text{with } \bar{\Psi}_0 = -\bar{\Psi}, \Psi_1 = \Psi$$

Write $\bar{\Psi} = \bar{\Psi}_0 = -\bar{\Psi}_N, \Psi = \Psi_1 = -\Psi_{N+1}$

$$= \int \prod_{j=0}^{N-1} \hbar d\bar{\Psi}_j d\Psi_{j+1} e^{-\frac{1}{\hbar} \sum_{j=1}^N \epsilon \left(\bar{\Psi}_j \frac{\Psi_{j+1} - \Psi_j}{\epsilon} + H(\bar{\Psi}_j, \Psi_j) \right)} + O(N\epsilon^2)$$

$$\text{with } \bar{\Psi}_0 = -\bar{\Psi}_0, \bar{\Psi}_{N+1} = -\Psi_1$$

Let us define $\psi(\epsilon_j), \bar{\psi}(\epsilon_j)$ for $j \in \mathbb{Z}$ by

$$\psi(\epsilon_{j+1}) = \psi_{j+1}, \quad \bar{\psi}(\epsilon_j) = \bar{\psi}_j \quad \text{for } j=0, 1, \dots, N-1$$

$$\& \quad \psi(\epsilon_{j+T}) = \psi(\epsilon_{j+N}) = -\psi(\epsilon_j)$$

$$\bar{\psi}(\epsilon_{j+T}) = \bar{\psi}(\epsilon_{j+N}) = -\bar{\psi}(\epsilon_j)$$

(consistent by $\bar{\psi}_N = -\bar{\psi}_0$ & $\psi_{N+1} = -\psi_1$).

Then, the exponent is

$$-\frac{1}{\hbar} \sum_{j \in \mathbb{Z}/N\mathbb{Z}} \left(\bar{\psi}(\epsilon_j) \frac{\psi(\epsilon_{j+1}) - \psi(\epsilon_j)}{\epsilon} + H(\bar{\psi}(\epsilon_j), \psi(\epsilon_j)) \right)$$

$$\xrightarrow[N\epsilon = T]{N \rightarrow \infty} -\frac{1}{\hbar} \int_{\mathbb{R}/T\mathbb{Z}} d\tau \left(\bar{\psi}(\tau) \frac{d\psi(\tau)}{d\tau} + H(\bar{\psi}(\tau), \psi(\tau)) \right)$$

$L_{\epsilon}(\bar{\psi}, \psi, \frac{1}{\hbar} \frac{d\psi}{d\tau})$

$$\therefore \text{Tr}_{\mathcal{H}} e^{-\frac{T}{\hbar} \hat{H}}$$

$$= \int \mathcal{D}\bar{\psi} \mathcal{D}\psi e^{-\frac{1}{\hbar} \int_{\mathbb{R}/T\mathbb{Z}} d\tau L_{\epsilon}(\bar{\psi}, \psi, \frac{d\psi}{d\tau})}$$

$$\bar{\psi}(\tau+T) = -\bar{\psi}(\tau), \quad \psi(\tau+T) = -\psi(\tau)$$

antiperiodic

$$\text{Tr} e^{(-)^F e^{-\frac{T}{\hbar} \hat{H}}} = \int d\bar{\Psi} d\Psi \langle \bar{\Psi} | e^{-\frac{T}{\hbar} \hat{H}} | \Psi \rangle \langle \Psi | \bar{\Psi} \rangle$$

$$= \int \hbar d\bar{\Psi} d\Psi \prod_{j=1}^{N-1} \hbar d\bar{\Psi}_j d\Psi_{j+1} e^{-\frac{1}{\hbar} \bar{\Psi} \Psi + \frac{1}{\hbar} \bar{\Psi}_0 \Psi_0 - \frac{\epsilon}{\hbar} H(\bar{\Psi}_0, \Psi_0)} \\ \cdot e^{-\frac{1}{\hbar} \sum_{j=1}^{N-1} \epsilon \left(\bar{\Psi}_j \frac{\Psi_{j+1} - \Psi_j}{\epsilon} + H(\bar{\Psi}_j, \Psi_j) \right)} + O(N\epsilon^2)$$

with $\bar{\Psi}_0 = \bar{\Psi}$, $\Psi_1 = \Psi$

Write $\bar{\Psi} = \bar{\Psi}_0 = -\bar{\Psi}_N$, $\Psi = \Psi_1 = -\Psi_{N+1}$

$$= \int \prod_{j=0}^{N-1} \hbar d\bar{\Psi}_j d\Psi_{j+1} e^{-\frac{1}{\hbar} \sum_{j=1}^N \epsilon \left(\bar{\Psi}_j \frac{\Psi_{j+1} - \Psi_j}{\epsilon} + H(\bar{\Psi}_j, \Psi_j) \right)} + O(N\epsilon^2)$$

with $\bar{\Psi}_0 = \bar{\Psi}_0$, $\bar{\Psi}_{N+1} = \Psi_1$

$$= \int \prod_{j \in \mathbb{Z}/N\mathbb{Z}} \hbar d\bar{\Psi}(\epsilon_j) d\Psi(\epsilon_{j+\epsilon}) e^{-\frac{1}{\hbar} \sum_{j \in \mathbb{Z}/N\mathbb{Z}} \epsilon \left(\bar{\Psi}(\epsilon_j) \frac{\Psi(\epsilon_{j+\epsilon}) - \Psi(\epsilon_j)}{\epsilon} + H(\bar{\Psi}(\epsilon_j), \Psi(\epsilon_j)) \right)} \\ + O(N\epsilon^2)$$

$$\bar{\Psi}(\epsilon_{j+\tau}) = \bar{\Psi}(\epsilon_j), \quad \Psi(\epsilon_{j+\tau}) = \Psi(\epsilon_j)$$

$$N \rightarrow \infty$$

$$N\epsilon = T$$

$$\rightarrow \int \mathcal{D}\bar{\Psi} \mathcal{D}\Psi e^{-\frac{1}{\hbar} \int_{\mathbb{R}/T\mathbb{Z}} d\tau L_\epsilon(\bar{\Psi}, \Psi, \frac{d\Psi}{d\tau})}$$

$$\bar{\Psi}(\tau+T) = \bar{\Psi}(\tau), \quad \Psi(\tau+T) = \Psi(\tau)$$

periodic

To summarize,

$$Z(t_f, \bar{\Psi}_f; t_i, \Psi_i)$$

$$= \int \mathcal{D}\bar{\Psi} \mathcal{D}\Psi e^{\frac{1}{\hbar} \bar{\Psi} \Psi|_{t_f} + \frac{i}{\hbar} \int_{t_i}^{t_f} dt (i \bar{\Psi} \dot{\Psi} - H(\bar{\Psi}, \Psi))},$$

$$\bar{\Psi}(t_f) = \bar{\Psi}_f, \quad \Psi(t_i) = \Psi_i$$

$$\text{Tr}_{\mathcal{H}} e^{-\frac{T}{\hbar} \hat{H}} = \int \mathcal{D}\bar{\Psi} \mathcal{D}\Psi e^{-\frac{1}{\hbar} \int_{\mathbb{R}/T\mathbb{Z}} d\tau L_E}$$

$$\bar{\Psi}(\tau+T) = -\bar{\Psi}(\tau), \quad \Psi(\tau+T) = -\Psi(\tau),$$

antiperiodic

$$\text{Tr}_{\mathcal{H}} (-1)^F e^{-\frac{T}{\hbar} \hat{H}} = \int \mathcal{D}\bar{\Psi} \mathcal{D}\Psi e^{-\frac{1}{\hbar} \int_{\mathbb{R}/T\mathbb{Z}} d\tau L_E}$$

$$\bar{\Psi}(\tau+T) = \bar{\Psi}(\tau), \quad \Psi(\tau+T) = \Psi(\tau).$$

periodic