# Yang-Mills theory (in $d$ spacetime dimensions)

- Specified by gauge group $G$ : a compact Lie group

e.g. $U(N) = \{ N\times N \text{ unitary matrix} \}$ unitary group

$SU(N) = \{ N\times N \text{ unitary}, \det = 1 \}$ special unitary group

e.g. $d=4$, $G = U(1)$ : Maxwell theory.

$\mathfrak{g} = \mathrm{Lie}(G)$ the Lie algebra of $G$ "infinitesimal version of $G$"

e.g. $G = U(N)$ : $\mathfrak{g} = \{ N\times N \text{ antihermitian matrix} \}$

$G = U(1)$ : $\mathfrak{g} = i\mathbb{R} \cong \mathbb{R}$

$G = SU(N)$ : $\mathfrak{g} = \{ N\times N \text{ antihermitian, traceless} \}$

field variable $A_\mu(x)$ : a vector potential with values in $\mathfrak{g}$

(or $A = A_\mu(x) dx^\mu$ : one form ―――― )

field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$

(or $F_A = dA + \frac{1}{2}[A, A] = dA + A^2$ )

the case of matrix Lie algebra

e.g. $d=4$, $G = U(1)$ : $F_{0i} = \partial_0 A_i - \partial_i A_0$ electric field

$F_{ij} = \partial_i A_j - \partial_j A_i$ magnetic field

Yang-Mills action:

$$S[A] = \int -\frac{1}{4e^2} F^{\mu\nu} \cdot F_{\mu\nu} \, d^d x$$

Here "$\cdot$" is a positive definite inner product on $\mathfrak{g}$ which is invariant under the adjoint action of $G$, $X \mapsto g X g^{-1}$ (the infinitesimal version of conjugation $g_t \mapsto g g_t g^{-1}$):

$$g X g^{-1} \cdot g Y g^{-1} = X \cdot Y.$$

e.g. for $G = SU(N)$, a standard choice is $X \cdot Y = -2 \operatorname{Tr} XY$.

$S[A]$ is invariant under a huge symmetry group:

$g(x)$: $G$-valued function on spacetime

$$\rightsquigarrow \quad A_\mu \mapsto A_\mu^g = g^{-1} A_\mu g + g^{-1} \partial_\mu g.$$

Under this, the field strength transforms covariantly,

$$F_{\mu\nu} \mapsto F_{\mu\nu}^g = \partial_\mu A_\nu^g - \partial_\nu A_\mu^g + [A_\mu^g, A_\nu^g] = g^{-1} F_{\mu\nu} g,$$

and thus indeed

$$S[A^g] = \int -\frac{1}{4e^2} g^{-1} F^{\mu\nu} g \cdot g^{-1} F_{\mu\nu} g \, d^d x$$

$$= \int -\frac{1}{4e^2} F^{\mu\nu} \cdot F_{\mu\nu} \, d^d x = S[A].$$

This is a generalization of invariance of Maxwell action under the gauge transformation $A_\mu \mapsto A_\mu + \partial_\mu \lambda$.

Indeed, for $G = U(1)$, $\mathfrak{g} = i\mathbb{R} \cong \mathbb{R}$ and for $g(x) = e^{i\lambda(x)}$,

$$iA_\mu^\lambda = e^{-i\lambda}\, iA_\mu\, e^{i\lambda} + e^{-\lambda}\partial_\mu e^{i\lambda} \Rightarrow A_\mu^\lambda = A_\mu + \partial_\mu \lambda.$$

As in that case, we shall call

$$A_\mu \longmapsto A_\mu^g = g^{-1} A_\mu g + g^{-1}\partial_\mu g$$

the <u>gauge transformation</u> of $A_\mu(x)$ by $g(x)$, and

$$\mathcal{G} := \{\, g(x) \mid G\text{-valued function} \,\}$$

the <u>gauge transformation group</u>. We'd like to regard $A$ and $A^g$ as physically equivalent for any $g \in \mathcal{G}$.
I.e. we would like to physically identify them. If we put

$$\mathcal{A} := \{\, A_\mu(x) \mid \mathfrak{g}\text{-valued vector potential} \,\}$$

the space of physically inequivalent field configurations is the quotient space $\mathcal{A}/\mathcal{G}$.

A theory with such an identification of field variables is called a gauge theory. We would like to find a way to quantize gauge theories.

## Infinitesimal gauge transformations

A $\mathfrak{g}$-valued function $\epsilon(x)$ generates a one parameter group of gauge transformations: $g_t(x) = e^{t\epsilon(x)}$:

$$A_\mu \mapsto A_\mu^{g_t} = g_t^{-1} A_\mu g + g_t^{-1} \partial_\mu g.$$

The infinitesimal transformation is

$$\delta_\epsilon A_\mu = \frac{d}{dt} A_\mu^{g_t} \Big|_{t=0} = -\epsilon A_\mu + A_\mu \epsilon + \partial_\mu \epsilon$$

$$= \partial_\mu \epsilon + [A_\mu, \epsilon] =: D_\mu \epsilon \quad \text{covariant derivative.}$$

The space of such $\epsilon(x)$ may be regarded as the Lie algebra of the gauge transformation group.

$$\{\epsilon(x) \mid \mathfrak{g}\text{-valued function}\} = \mathrm{Lie}(\mathcal{G}).$$

# Coupling to matter fields

$\phi(x)$ : a scalar field with values a representation $R$ of $G$,

i.e. a vector space on which $G$ acts linearly.

e.g. $R = \mathbb{C}^N$ for $G = U(N)$ or $SU(N)$ via matrix multiplication.

$R = \mathfrak{g}$ for a general $G$ via adjoint action

$R =$ sum of copies of such, $\mathbb{C}^N \oplus \cdots \oplus \mathbb{C}^N \oplus \mathfrak{g} \oplus \cdots \oplus \mathfrak{g}$.

Gauge transformation by $g \in \mathcal{G}$ :

$$A_\mu \mapsto A_\mu^g, \quad \phi \longmapsto \phi^g = g^{-1}\phi.$$

Infinitesimally, $\delta A_\mu = D_\mu \epsilon$, $\delta\phi = -\epsilon\phi$.

Covariant derivative $D_\mu\phi := \partial_\mu\phi + A_\mu\phi$

Its gauge transformation :

$$D_\mu\phi \mapsto \partial_\mu\phi^g + A_\mu^g\phi^g = \underbrace{\partial_\mu(g^{-1}\phi)}_{\cancel{-g^{-1}\partial_\mu g g^{-1}\phi} + g^{-1}\partial_\mu\phi} + (g^{-1}A_\mu g + \cancel{g^{-1}\partial_\mu g})\, g^{-1}\phi$$

$$= g^{-1}\partial_\mu\phi + g^{-1}A_\mu\phi = g^{-1}D_\mu\phi \qquad \text{homogeneous.}$$

$(\phi_1, \phi_2) \longmapsto \phi_1^\dagger\phi_2$ $G$-invariant inner product on $R$

$$\mathcal{L} = -\frac{1}{4e^2} F^{\mu\nu} \cdot F_{\mu\nu} + (D^\mu \phi)^\dagger D_\mu \phi - f(\phi^\dagger \phi)$$

is gauge invariant.

The system with variable $(A_\mu, \phi)$ and this Lagrangian is the gauge theory of gauge group $G$ with a scalar in a representation $R$ of $G$.

We may also consider a theory with a fermion $\psi$ in a representation $R$ of $G$.

$$\mathcal{L} = -\frac{1}{4e^2} F^{\mu\nu} \cdot F_{\mu\nu} + i\bar{\psi} \not{D}_A \psi - m\bar{\psi}\psi$$

where $\not{D}_A \psi = \gamma^\mu D_\mu \psi = \gamma^\mu (\partial_\mu \psi + A_\mu \psi)$.

e.g. QED with electrons of charge $Q_1, \cdots, Q_{N_f}$:

$$G = U(1), \quad e^{i\lambda} : \psi_i \longmapsto e^{iQ_i\lambda} \psi_i \quad (i = 1, \cdots, N_f)$$

e.g. QCD with color $N_c$ and flavor $N_f$:

$$G = SU(N_c), \quad R = \mathbb{C}^{N_c} \oplus \cdots \oplus \mathbb{C}^{N_c} \quad (N_f \text{ copies})$$

# Quantization of gauge theory (path integral)

In a gauge theory, a field configuration $(A, \phi, \psi, \cdots)$ is identified with its gauge transform $(A^g, \phi^g, \psi^g, \cdots)$

$\mathcal{M}$ = the space of field configurations

$\mathcal{G}$ = the gauge transformation group.

The path-integral is over the quotient space $\mathcal{M}/\mathcal{G}$

$$Z = \int_{\mathcal{M}/\mathcal{G}} \underline{\text{measure}}\; e^{-S_E[A, \phi, \psi, \cdots]}$$

$$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \cdots \rangle$$

$$= \frac{1}{Z} \int_{\mathcal{M}/\mathcal{G}} \underline{\text{measure}}\; e^{-S_E[A, \phi, \psi, \cdots]}\, \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \cdots$$

<u>How do we do this?</u>

… Let us do it in a finite dimensional case.

$M$ : a manifold, $\dim M = n$

$G$ : a Lie group acting on $M$, $\dim G = d_G$.

$$g \in G : \phi \in M \longmapsto \phi g \in M$$

( right action : $\phi(gh) = (\phi g) h$

Assume: the action is <u>free</u>, $\phi g = \phi$ for some $\phi \Rightarrow g = 1$.

Suppose a measure $d\phi$ and a function $S_E(\phi)$ on $M$

are $G$-invariant, $d(\phi g) = d\phi$, $S_E(\phi g) = S_E(\phi)$.

Want to consider the gauge theory where

$$\begin{cases} \phi \sim \phi g & \text{identified} \\ f(\phi) & \text{physically meaningful when } G\text{-invariant} \end{cases}$$

<u>Question</u> How do we define <u>measure</u> on $M/G$ for

$$Z = \int_{M/G} \underline{\text{measure}} \; e^{-S_E(\phi)}$$

$$\langle f \rangle = \frac{1}{Z} \int_{M/G} \underline{\text{measure}} \; e^{-S_E(\phi)} f(\phi)$$

?

A naive answer:

$$Z = \frac{1}{\mathrm{Vol}\, G} \int_M d\phi \; e^{-S_E(\phi)}$$

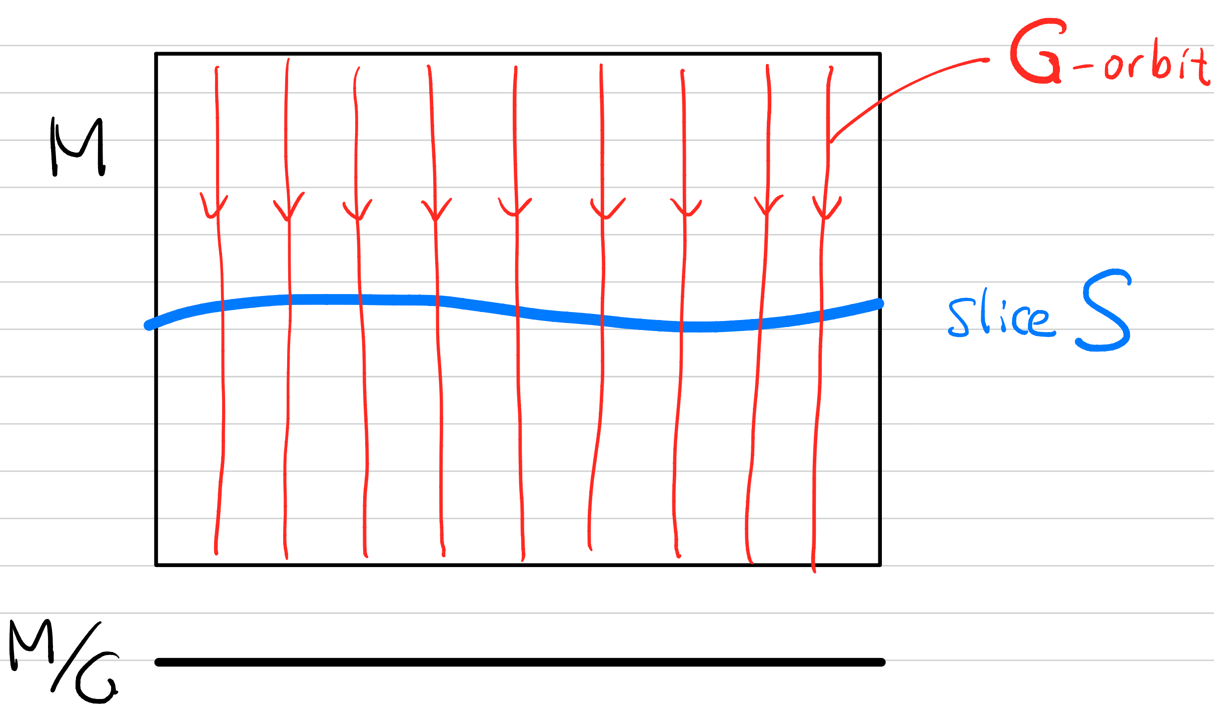
$$\langle f \rangle = \frac{1}{\mathrm{Vol}\, G} \int_M d\phi \; e^{-S_E(\phi)} f(\phi) \Big/ Z$$

A possible problem: $\mathrm{vol}\, G$ may be infinite

$\int_M d\phi \cdots$ may be infinite.

Suppose we can find a slice $S \subset M$, i.e.

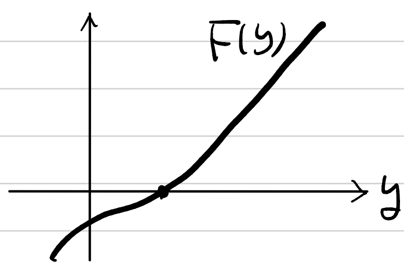
a submanifold s.t. any $G$-orbit has exactly one point in it.



Suppose $S$ is the zero locus of a set of functions of $M$: $\phi \in S \Leftrightarrow \chi_1(\phi) = \cdots = \chi_{d_G}(\phi) = 0$.

$\chi(\phi) = (\chi_a(\phi))_{a=1}^{d_G}$ can be regarded as a function on $M$ with values in $\mathfrak{g} = \mathrm{Lie}(G)$.

Note: For a monotonic function $F(y)$ of a single $y$,




$$\int_{-\infty}^{\infty} \delta(F(y)) \underbrace{dF(y)}_{F'(y)\,dy} = 1.$$

Multivariable case: $y = (y^1, \ldots, y^m)$

$$\int_{\mathbb{R}^m} \prod_{a=1}^{m} \delta(F_a(y)) \cdot \det\left(\frac{\partial F_a(y)}{\partial y^b}\right) d^m y = 1$$

Apply this to the function $F_a(g) = \chi_a(\phi g)$ of $G$ for a fixed $\phi \in M$:

$$\int_G \underbrace{\prod_{a=1}^{d_G} \delta(\chi_a(\phi g))}_{\delta(\chi(\phi g))} \cdot \underbrace{\det\left(\frac{\partial \chi_a(\phi g)}{\partial g_b}\right)}_{\det(\delta^a \chi_a(\phi g))} \cdot dg = 1.$$

insert $1 = \cdots$

$$\int_M d\phi\, e^{-S_E(\phi)} = \int_{M\times G} d\phi\, dg\, e^{-S_E(\phi)}\, \delta(\chi(\phi g))\, \det\big(\delta\chi(\phi g)\big)$$

[ Change the variable $\phi g \to \phi$ and use $G$-invariance of $d\phi$ & $S_E(\phi)$

$$= \int_{M\times G} d\phi\, dg\, e^{-S_E(\phi)}\, \delta(\chi(\phi))\, \det\big(\delta\chi(\phi)\big)$$

$$= \underbrace{\int_G dg}_{\mathrm{Vol}\,G} \int_M d\phi\, e^{-S_E(\phi)}\, \delta(\chi(\phi))\, \det\big(\delta\chi(\phi)\big)$$

$$\therefore Z = \text{“}\frac{1}{\mathrm{Vol}\,G}\int_M d\phi\, e^{-S_E(\phi)}\text{”}$$

$$= \int_M d\phi\, e^{-S_E(\phi)}\, \delta(\chi(\phi))\, \det\big(\delta\chi(\phi)\big).$$

Similarly

$$\langle f \rangle = \text{``} \frac{1}{\mathrm{Vol}\, G} \int_M d\phi \, e^{-S_E(\phi)} f(\phi) \Big/ Z \text{''}$$

$$= \int_M d\phi \, e^{-S_E(\phi)} \delta(\chi(\phi)) \det(\delta\chi(\phi)) f(\phi) \Big/ Z .$$

$\chi(\phi) = (\chi_a(\phi))_{a=1}^{d_G}$ ... gauge fixing function

$\chi_1(\phi) = \cdots = \chi_{d_G}(\phi) = 0$ ... gauge fixing condition

$$\det(\delta\chi(\phi)) = \det\left( \left. \frac{\partial \chi_a(\phi g)}{\partial g_b} \right|_{g=1} \right)$$

$$= \det\left( \left. \frac{\partial \chi_a(\phi e^{\epsilon})}{\partial \epsilon_b} \right|_{\epsilon = 0} \right)$$

$\{e^a\}_{a=1}^{d_G} \subset \mathfrak{g}$ basis

$\epsilon = \sum_a e^a \epsilon_a$

... <u>Faddeev-Popov determinant</u>

The results $Z$, $\langle f \rangle$ do not depend on the choice of gauge fixing condition $\chi(\phi) = 0$.

## Rewriting

① Use independence on the choice of $\chi(\phi)$.

② Use $\det(A_{ij}) = \int \prod_i d\bar{\eta}_i d\eta_i \, e^{-\sum_{i,j} \bar{\eta}_i A_{ij} \eta_j}$

① Replace $\chi(\phi) \to \chi(\phi) - \omega$, $\omega \in \mathfrak{g}$.

Also, $\displaystyle\int_{\mathfrak{g}} d\omega \, e^{-\frac{1}{2\xi}\omega^2} = (2\pi\xi)^{d_G/2}$

$$Z = \int_M d\phi \, e^{-S_E(\phi)} \delta(\chi(\phi)) \det(\delta\chi(\phi))$$

$$1 = \frac{1}{(2\pi\xi)^{d_G/2}} \int_{\mathfrak{g}} d\omega \, e^{-\frac{1}{2\xi}\omega^2}$$

$\chi(\phi) - \omega$

$\delta\chi(\phi)$ intact

$$= \frac{1}{(2\pi\xi)^{d_G/2}} \int_{M \times \mathfrak{g}} d\phi \, d\omega \, e^{-S_E(\phi) - \frac{1}{2\xi}\omega^2} \delta(\chi(\phi) - \omega) \det(\delta\chi(\phi))$$

(#)

$$= \frac{1}{(2\pi\xi)^{d_G/2}} \int_M d\phi \, e^{-S_E(\phi) - \frac{1}{2\xi}(\chi(\phi))^2} \det(\delta\chi(\phi))$$

② $\det(\delta^b \chi_a(\phi)) = \int \prod_{a=1}^{d_G} d\bar{c}^a \, dc_a \, e^{-\sum_{a,b} \bar{c}^a \delta^b \chi_a(\phi) c_b}$

$$\sum_b \delta^b \chi_a(\phi) c_b = \delta_c \chi_a(\phi)$$

infinitesimal transformation of $\chi_a(\phi)$

by $c = \sum_{a=1}^{d_G} e^a c_a$

$$= \int \prod_{a=1}^{d_G} d\bar{c}^a \, dc_a \, e^{-\sum_a \bar{c}^a \delta_c \chi_a(\phi)}$$

$$\overset{\text{or}}{=} \int_{\mathfrak{g}\times\mathfrak{g}} d\bar{c} \, dc \, e^{-\bar{c}\cdot\delta_c \chi(\phi)}$$ in a simplified form.

$c, \bar{c}$ : Faddeev-Popov ghost

Also

$$\delta(\chi(\phi)-\omega) = \int \prod_{a=1}^{d_G} \frac{dB_a}{2\pi} \, e^{iB^a(\chi_a(\phi)-\omega_a)}$$

$$\overset{\text{or}}{=} \frac{1}{(2\pi)^{d_G}} \int_{\mathfrak{g}} dB \, e^{iB\cdot(\chi(\phi)-\omega)}$$

$\therefore \delta(\chi(\phi)-\omega) \det(\delta\chi(\phi))$

$$= \frac{1}{(2\pi)^{d_G}} \int_{\mathfrak{g}\times\mathfrak{g}\times\mathfrak{g}} dB \, d\bar{c} \, dc \, e^{iB\cdot(\chi(\phi-\omega) - \bar{c}\cdot\delta_c\chi(\phi)}$$

Insert this in (#) :

$$Z = \frac{1}{(2\pi\xi)^{d_G/2}} \int_{M \times \mathfrak{g}} d\phi \, d\omega \; e^{-S_E(\phi) - \frac{1}{2\xi}\omega^2}$$

$$\times \frac{1}{(2\pi)^{d_G}} \int_{\mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}} dB \, d\bar{c} \, dc \; e^{iB\cdot(\chi(\phi - \omega)) - \bar{c}\cdot\delta_c \chi(\phi)}$$

Perform $\omega$-integral

$$\int d\omega \; e^{-\frac{1}{2\xi}\omega^2 - iB\cdot\omega} = (2\pi\xi)^{d_G/2} \, e^{-\frac{\xi}{2}B^2}$$

We end up with

$$Z = \frac{1}{(2\pi)^{d_G}} \int_{M \times \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}} d\phi \, dB \, d\bar{c} \, dc \; e^{-\tilde{S}_E(\phi, B, \bar{c}, c)}$$

$$\tilde{S}_E = S_E(\phi) + \frac{\xi}{2}B^2 - iB\cdot\chi(\phi) + \bar{c}\cdot\delta_c\chi(\phi)$$

... gauge fixed action

Similarly for $\langle f \rangle$.

The gauge fixed system has a symmetry $\delta_B$ called

BRST symmetry

$$\delta_B \phi = \delta_c \phi$$
$$\delta_B B = 0$$
$$\delta_B \bar{c} = iB$$
$$\delta_B c = -\frac{1}{2}[c, c]$$

$\cdot\ c = e^a c_a$

$$[c, c] = [e^a c_a, e^b c_b] = [e^a, e^b] c_a c_b$$

If $[e^a, e^b] = e^d f_d^{ab}$,

$$\delta c_a = -\frac{1}{2} f_a^{bc} c_b c_d$$

It is a fermionic symmetry $\begin{cases} \delta_B \text{ bosonic is fermionic} \\ \delta_B \text{ fermionic is bosonic.} \end{cases}$

$$\delta_B(\mathcal{O}_1 \mathcal{O}_2) = \delta_B \mathcal{O}_1 \cdot \mathcal{O}_2 + (-1)^{|\mathcal{O}_1|} \mathcal{O}_1 \cdot \delta_B \mathcal{O}_2$$

$$\delta_B \tilde{S}_E = \cancel{\delta_c S_E(\phi)}^{\,0} - iB\cdot \cancel{\delta_c \chi(\phi)} + iB \cdot \cancel{\delta_c \chi(\phi)}$$
$$- \bar{c}\, \underbrace{\delta c_a}_{-\frac{1}{2} f_a^{bd} c_b c_d} \delta^a \chi(\phi) + \bar{c}\, c_a \underbrace{\delta_c \delta^a \chi(\phi)}_{c_b \delta^b \delta^a \chi(\phi)}$$

$$\left[ f_a^{bd} \delta^a \chi = \delta^b \delta^d \chi - \delta^d \delta^b \chi \quad (\because \text{right action}) \right.$$

$$= \frac{1}{2} \bar{c}\, c_b c_d (\delta^b \delta^d - \delta^d \delta^b) \chi(\phi) + \bar{c}\, c_a c_b \delta^b \delta^a \chi(\phi) = 0.$$

**Remarks**

- $\delta_B \circ \delta_B = 0$ (exercise)

  $\mathcal{O}$ is said to be **BRST closed** when $\delta_B \mathcal{O} = 0$

  **BRST exact** when $\mathcal{O} = \delta_B(-)$.

  By $\delta_B \circ \delta_B = 0$, BRST exact $\Rightarrow$ BRST closed.

- $\tilde{S}_E = S_E - \delta_B\left(\bar{c}\cdot\left(\chi(\phi) - \frac{i\xi}{2}B\right)\right)$

  $\because$ The gauge fixing term is BRST exact.

- $\langle \delta_B h \rangle = 0$ by ward identity.

  If $\delta_B f = 0$, then

  $$\langle f \cdot \delta_B h \rangle = (-1)^{|f|} \langle \delta_B (f \cdot h) \rangle = 0.$$

  In particular, if $f_1, \cdots, f_n$ are BRST closed,

  $\langle f_1 \cdots f_n \rangle$ does not change under change of $f_i$'s

  by BRST exact ones, $f_i \to f_i + \delta_B h_i$

These motivate us to consider BRST cohomology :

$$H_{BRST} = \{\text{BRST closed}\} / \{\text{BRST exact}\}$$

A proposal :

Physical observables (states) are BRTS cohomology classes (states).

There is another symmetry : ghost number $N_{gh}$

| | $\varphi$ | $B$ | $\bar{c}$ | $c$ |
|---|---|---|---|---|
| $N_{gh}$ | 0 | 0 | $-1$ | 1 |

$\delta_B$ increases $N_{gh}$ by 1, $[N_{gh}, \delta_B] = 1$

$$\mathcal{F}^i = \{\text{observable of } N_{gh} = i\}$$

$$\Rightarrow \quad \delta_B : \mathcal{F}^i \to \mathcal{F}^{i+1}$$

$$H^i_{BRST}(\mathcal{F}) = \mathrm{Ker}(\delta_B : \mathcal{F}^i \to \mathcal{F}^{i+1}) / \mathrm{Im}(\delta_B : \mathcal{F}^{i-1} \to \mathcal{F}^i).$$

We may also integrate-out B :

$$Z = \frac{1}{(2\pi\xi)^{d_G/2}} \int_{M \times \mathfrak{g} \times \mathfrak{g}} d\phi \, d\bar{c} \, dc \; e^{-\tilde{S}_E(\phi, \bar{c}, c)}$$

$$\tilde{S}_E = S_E(\phi) + \frac{1}{2\xi} \chi(\phi)^2 + \bar{c} \cdot \delta_c \chi(\phi)$$

This is also obtained directly from ① & ②.

This system also has BRST symmetry

$$\delta_B \phi = \delta_c \phi,$$

$$\delta_B \bar{c} = -\frac{1}{\xi} \chi(\phi) \quad \leftarrow \text{from EOM: } B = \frac{i}{\xi} \chi(\phi)$$

$$\delta_B c = -\frac{1}{2}[c, c].$$

But $\delta_B \circ \delta_B = 0$ holds only on-shell

i.e. using EOM $\delta_c \chi(\phi) = 0$.

Back to the case of gauge theory :

$$M \rightsquigarrow \mathcal{M} = \{(A_\mu(x), \phi(x), \psi(x), \cdots) \text{ field config.}\}$$

$$G \rightsquigarrow \mathcal{G} = \{ g(x) \mid G\text{-valued function} \}$$

$$\mathfrak{g} \rightsquigarrow \mathrm{Lie}(\mathcal{G}) = \{ \epsilon(x) \mid \mathfrak{g}\text{-valued function} \}$$

As gauge fixing function, we can take

$$\chi[A](x) = \partial^\mu A_\mu(x) \qquad \text{Lorentz gauge}$$

$$\delta_\epsilon \chi[A](x) = \partial^\mu D_\mu \epsilon(x)$$

gauge fixed Lagrangian

$$\tilde{\mathcal{L}}_E = \mathcal{L}_E + \frac{\xi}{2} B^2 - i B \cdot \partial^\mu A_\mu + \bar{c} \cdot \partial^\mu D_\mu c$$

Inverse Wick rotation to real time

(with $B \to iB$, $\bar{c} \to i\bar{c}$ & $\xi \to e^2 \xi$ for convenience)

$$\tilde{\mathcal{L}} = \mathcal{L} + \frac{e^2 \xi}{2} B^2 - B \cdot \partial^\mu A_\mu - i \bar{c}\, \partial^\mu D_\mu c$$

BRST symmetry :

$$\delta_B A_\mu = D_\mu c \,, \quad \delta_B \phi = -c\phi, \quad \delta_B \psi = -c\psi$$

$$\delta_B B = 0$$

$$\delta_B \bar{c} = iB$$

$$\delta_B c = -\frac{1}{2}[c,c]$$

The version where $B$ is integrated out :

$$\tilde{\mathcal{L}} = \mathcal{L} - \frac{1}{2e^2\xi}(\partial^\mu A_\mu)^2 - i\,\bar{c}\cdot\partial^\mu D_\mu c$$

$$\delta_B \bar{c} = \frac{i}{e^2\xi}\partial^\mu A_\mu \,,$$

$\delta_B(\text{others})$ = same as above.