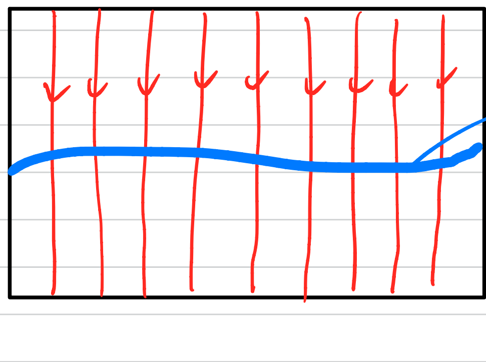
Recap

$$\mathcal{M} = \{ \text{field configuration } (A_\mu(x), \phi(x), \psi(x), \dots) \} \longrightarrow M^n$$

$$\mathcal{G} = \{ G\text{-valued function } g(x) \} \longrightarrow G^{d_G}$$

M



Slice S

$= \{ \chi(\phi) = 0 \}$

gauge fixing condition

M/G

$$Z(f) = \text{``}\int_{M/G} [d\phi]\, e^{-S_E(\phi)} f(\phi)\text{''}$$

$$= \int_M d\phi\, e^{-S_E(\phi)}\, \delta(\underset{\chi(\phi)-\omega}{\chi(\phi)})\, \underbrace{\det(\delta\chi(\phi))}_{\text{F.P. det}} f(\phi)$$

$$= \frac{1}{(2\pi\xi)^{d_G/2}} \int_{M\times\mathcal{G}} d\phi\, d\omega\, e^{-S_E(\phi) - \frac{1}{2\xi}\omega^2} \delta(\chi(\phi)-\omega) \det(\delta\chi(\phi)) f(\phi) \quad (\#)$$

$$= \frac{1}{(2\pi\xi)^{d_G/2}} \int_M d\phi\, e^{-S_E(\phi) - \frac{1}{2\xi}\chi(\phi)^2} \det(\delta\chi(\phi)) f(\phi)$$

(#)

$$Z(f) = \frac{1}{(2\pi\xi)^{d_G/2}} \int_{M\times\mathfrak{g}} d\phi\, d\omega\, e^{-S_E(\phi)-\frac{1}{2\xi}\omega^2} \underbrace{\delta(\chi(\phi)-\omega)}_{=\frac{1}{(2\pi)^{d_G}}\int dB\, e^{iB\cdot(\chi(\phi)-\omega)}} \underbrace{\det(\delta\chi(\phi))}_{=\int d\bar{c}\, dc\, e^{-\bar{c}\cdot\delta_c\chi(\phi)}} f(\phi)$$

$c, \bar{c}$: F.P. ghosts

$$= \frac{1}{(2\pi)^{d_G}} \int_{M\times\mathfrak{g}_b\times\mathfrak{g}_f\times\mathfrak{g}_f} d\phi\, dB\, d\bar{c}\, dc\; e^{-\tilde{S}_E(\phi, B, \bar{c}, c)}$$

$$\tilde{S}_E(\phi, B, \bar{c}, c) = S_E(\phi) + \frac{\xi}{2}B^2 - iB\cdot\chi(\phi) + \bar{c}\cdot\delta_c\chi(\phi)$$

gauge fixed action

The gauge fixed system has a symmetry δ_B called

BRST symmetry

$$\delta_B \Phi = \delta_c \Phi$$

$$\delta_B B = 0$$

$$\delta_B \bar{c} = iB$$

$$\delta_B c = -\frac{1}{2}[c, c]$$

$\cdot\ c = e^a c_a$

$$[c, c] = [e^a c_a, e^b c_b] = [e^a, e^b] c_a c_b$$

If $[e^a, e^b] = e^d f_d^{ab}$,

$$\delta c_a = -\frac{1}{2} f_a^{bc} c_b c_d$$

It is a fermionic symmetry $\begin{cases} \delta_B \text{ bosonic is fermionic} \\ \delta_B \text{ fermionic is bosonic.} \end{cases}$

$$\delta_B(\mathcal{O}_1 \mathcal{O}_2) = \delta_B \mathcal{O}_1 \cdot \mathcal{O}_2 + (-1)^{|\mathcal{O}_1|} \mathcal{O}_1 \cdot \delta_B \mathcal{O}_2$$

$$\delta_B \tilde{S}_E = \cancel{\delta_c S_E(\Phi)}^{\,0} - iB \cdot \cancel{\delta_c \chi(\Phi)} + iB \cdot \cancel{\delta_c \chi(\Phi)}$$

$$- \bar{c}\, \underbrace{\delta c_a}_{-\frac{1}{2} f_a^{bd} c_b c_d} \delta^a \chi(\Phi) + \bar{c}\, c_a \underbrace{\delta_c \delta^a \chi(\Phi)}_{c_b \delta^b \delta^a \chi(\Phi)}$$

$$\left[f_a^{bd} \delta^a \chi = \delta^b \delta^d \chi - \delta^d \delta^b \chi \quad (\because \text{right action}) \right.$$

$$= \frac{1}{2} \bar{c}\, c_b c_d (\delta^b \delta^d - \delta^d \delta^b) \chi(\Phi) + \bar{c}\, c_a c_b \delta^b \delta^a \chi(\Phi) = 0.$$

Remarks

- $\delta_B \circ \delta_B = 0$ (exercise)

  $\mathcal{O}$ is said to be BRST closed when $\delta_B \mathcal{O} = 0$

  BRST exact when $\mathcal{O} = \delta_B(-)$.

  By $\delta_B \circ \delta_B = 0$, BRST exact $\Rightarrow$ BRST closed.

- $\tilde{S}_E = S_E - \delta_B\left(\bar{c} \cdot \left(\chi(\phi) - \frac{i\xi}{2} B\right)\right)$

  $\therefore$ The gauge fixing term is BRST exact.

- $\langle \delta_B h \rangle = 0$ by ward identity.

  If $\delta_B f = 0$, then

  $$\langle f \cdot \delta_B h \rangle = (-1)^{|f|} \langle \delta_B (f \cdot h) \rangle = 0.$$

  In particular, if $f_1, \cdots, f_n$ are BRST closed,

  $\langle f_1 \cdots f_n \rangle$ does not change under change of f_i's

  by BRST exact ones, $f_i \to f_i + \delta_B h_i$

These motivate us to consider BRST cohomology :

$$H_{BRST} = \{\text{BRST closed}\} / \{\text{BRST exact}\}$$

A proposal :

Physical observables (states) are BRST cohomology classes (states).

There is another symmetry : ghost number N_{gh}

	φ	B	$\bar{c}$	c
N_{gh}	0	0	-1	1

δ_B increases N_{gh} by 1, $[N_{gh}, \delta_B] = 1$

$\mathcal{F}^i = \{\text{observable of } N_{gh} = i\}$

$\Rightarrow \delta_B : \mathcal{F}^i \to \mathcal{F}^{i+1}$

$$H^i_{BRST}(\mathcal{F}) = \mathrm{Ker}(\delta_B : \mathcal{F}^i \to \mathcal{F}^{i+1}) / \mathrm{Im}(\delta_B : \mathcal{F}^{i-1} \to \mathcal{F}^i).$$

We may also integrate-out B :

$$Z = \frac{1}{(2\pi\xi)^{d_G/2}} \int_{M \times \mathfrak{g} \times \mathfrak{g}} d\phi \, d\bar{c} \, dc \; e^{-\tilde{S}_E(\phi, \bar{c}, c)}$$

$$\tilde{S}_E = S_E(\phi) + \frac{1}{2\xi} \chi(\phi)^2 + \bar{c} \cdot \delta_c \chi(\phi)$$

This is also obtained directly from ① & ②.

This system also has BRST symmetry

$$\delta_B \phi = \delta_c \phi,$$

$$\delta_B \bar{c} = -\frac{1}{\xi} \chi(\phi) \quad \leftarrow \text{from EOM: } B = \frac{i}{\xi} \chi(\phi)$$

$$\delta_B c = -\frac{1}{2}[c, c].$$

But $\delta_B \circ \delta_B = 0$ holds only on-shell

(EOM $\delta_c \chi(\phi) = 0$ is needed).

Back to the case of gauge theory :

$$M \rightsquigarrow \mathcal{M} = \{ (A_\mu(x), \varphi(x), \psi(x), \cdots) \text{ field config.} \}$$

$$G \rightsquigarrow \mathcal{G} = \{ g(x) \mid G\text{-valued function} \}$$

$$\mathfrak{g} \rightsquigarrow \mathrm{Lie}(\mathcal{G}) = \{ \epsilon(x) \mid \mathfrak{g}\text{-valued function} \}$$

As gauge fixing function, we can take

$$\chi[A](x) = \partial^\mu A_\mu(x) \qquad \text{Lorentz gauge}$$

$$\delta_\epsilon \chi[A](x) = \partial^\mu D_\mu \epsilon(x)$$

gauge fixed Lagrangian

$$\tilde{\mathcal{L}}_E = \mathcal{L}_E + \frac{\xi}{2} B^2 - i B \cdot \partial^\mu A_\mu + \bar{c} \cdot \partial^\mu D_\mu c$$

Inverse Wick rotation to real time

(with $B \to iB$, $\bar{c} \to i\bar{c}$ & $\xi \to e^2 \xi$ for convenience)

$$\tilde{\mathcal{L}} = \mathcal{L} + \frac{e^2 \xi}{2} B^2 - B \cdot \partial^\mu A_\mu - i \bar{c} \partial^\mu D_\mu c$$

BRST symmetry :

$$\delta_B A_\mu = D_\mu c \,, \quad \delta_B \phi = -c\phi \,, \quad \delta_B \psi = -c\psi$$

$$\delta_B B = 0$$

$$\delta_B \bar{c} = iB$$

$$\delta_B c = -\frac{1}{2}[c, c]$$

The version where B is integrated out :

$$\tilde{\mathcal{L}} = \mathcal{L} - \frac{1}{2e^2\xi}(\partial^\mu A_\mu)^2 - i\bar{c} \cdot \partial^\mu D_\mu c$$

$$\delta_B \bar{c} = \frac{i}{e^2\xi}\partial^\mu A_\mu \,,$$

$\delta_B(\text{others})$ = same as above.

We may use this as the new starting point for quantization.

For example, we may convert this via Legendre transform to Hamiltonian formulation and then perform the operator quantization.

[※ This is now possible thanks to $-\frac{1}{2e^2\xi}(\partial^\mu A_\mu)^2$:
Without that, A_0 would have no kinetic term and hence no conjugate momentum.]

However A_0 has wrong sign kinetic term (note $\xi > 0$)

$-\frac{1}{2e^2\xi}(\dot{A}_0)^2$ which yields negative norm states.

Also the ghosts with kinetic term $i\dot{\bar{C}}\dot{C}$ also yield zero & negative norm states. [Lec 3, Exercise (c)]

As the existence of such negative/zero norm states indicates, the gauge fixed system has a huge number of unphysical degrees of freedom.

This is the quantum counterpart of the huge gauge symmetry in the classical system: the gauge transformations $(A, \varphi, \psi, \ldots) \mapsto (A^g, \varphi^g, \psi^g, \ldots)$ are regarded as unphysiscal change of field configurations.

The proposal is to take the **BRST cohomology** to select physical degrees of freedom.

For example, the space of physical states is the BRST cohomology of states

$$\mathcal{H}_{phys} := H_{BRST}(\mathcal{H}).$$

It is expected that this consists of positive norm states only.

Hamiltonian formulation of gauge theories

Consider the system without matter fields for simplicity.

$$S[A] = \int -\frac{1}{4e^2} F^{\mu\nu} \cdot F_{\mu\nu} \, d^d x$$

$$= \int d^d x \left(\frac{1}{2e^2} \sum_i F_{0i}^2 - \frac{1}{2e^2} \sum_{i<j} F_{ij}^2 \right) \qquad i,j = 1, \cdots, d-1$$

The system is equivalent to

$$S[\mathbb{A}, \mathbb{E}; A_0] = \int d^d x \left(\sum_i E_i F_{0i} - \frac{e^2}{2} \sum_i E_i^2 - \frac{1}{2e^2} \sum_{i<j} F_{ij}^2 \right).$$

Integrating out $\mathbb{E} = (E_i)_{i=1}^{d-1}$, we obtain the system of $(\mathbb{A}, A_0) = (A_\mu)$ with action $S[A]$.

Inserting $F_{0i} = \dot{A}_i - \partial_i A_0 + [A_0, A_i] = \dot{A}_i - D_i A_0$ and doing partial integration,

$$S[\mathbb{A}, \mathbb{E}; A_0]$$

$$= \int d^d x \left(\sum_i E_i \dot{A}_i - \frac{e^2}{2} \sum_i E_i^2 - \frac{1}{2e^2} \sum_{i<j} F_{ij}^2 + A_0 D_i E_i \right)$$

- $A_0(x)$ is a Lagrange multiplier imposing a constraint

$$\mathbb{D} \cdot \mathbb{E} = 0 \qquad \text{Gauss law.}$$

- $E_i(\mathbb{x})$ is the conjugate momentum of $A_i(\mathbb{x})$.

Their components have Poisson bracket

$$\{A_{ia}(\mathbb{x}), E_{jb}(\mathbb{y})\} = \delta_{ij}\,\delta_{ab}\,\delta(\mathbb{x}-\mathbb{y}).$$

- Hamiltonian is

$$H(\mathbb{E},\mathbb{A}) = \int d^3\mathbb{x}\left(\frac{e^2}{2}\sum_i E_i(\mathbb{x})^2 + \frac{1}{2e^2}\sum_{i<j} F_{ij}(\mathbb{x})^2\right).$$

Let us study the constraint

$$\Phi(\mathbb{x}) := \mathbb{D}\cdot\mathbb{E} = D_i E_i = \partial_i E_i + [A_i, E_i] = 0.$$

For a $\mathfrak{g}$-valued function $\epsilon(\mathbb{x})$ of $\mathbb{x}$, put

$$\Phi(\epsilon) := \int d^{d-1}\mathbb{x}\ \epsilon(\mathbb{x})\cdot\Phi(\mathbb{x}) = -\int d^{d-1}\mathbb{x}\ \mathbb{D}\epsilon(\mathbb{x})\cdot\mathbb{E}.$$

$$\{\Phi(\epsilon), \mathbb{A}(\mathbb{x})\} = \mathbb{D}\epsilon(\mathbb{x})$$ (use this expression)

$$\{\Phi(\epsilon), \mathbb{E}(\mathbb{x})\} = \left\{\int d^{d-1}\mathbb{y}\ \underbrace{\epsilon(\mathbb{y})\cdot[A_i,E_i](\mathbb{y})}_{[E_i,\epsilon](\mathbb{y})\cdot A_i(\mathbb{y})},\ \mathbb{E}(\mathbb{x})\right\}$$

$$= [\mathbb{E},\epsilon](\mathbb{x})$$

$\therefore$ $\Phi(\epsilon)$ generates the gauge transformation by $\epsilon(x)$.

In particular, since H is gauge invariant,

$$\{ \Phi(\epsilon), H \} = 0.$$

Also, as $\Phi = \mathbb{D} \cdot \mathbb{E}$ is covariant,

$$\{ \Phi(\epsilon), \Phi(x) \} = [\Phi, \epsilon](x),$$

and hence

$$\{ \Phi(\epsilon_1), \Phi(\epsilon_2) \} = \{ \Phi(\epsilon_1), \int d^{d-1}x \; \epsilon_2(x) \cdot \Phi(x) \}$$

$$= \int d^{d-1}x \; \underbrace{\epsilon_2(x) \cdot [\Phi, \epsilon_1](x)}_{[\epsilon_1, \epsilon_2](x) \cdot \Phi(x)}$$

$$= \Phi([\epsilon_1, \epsilon_2]).$$

The Hamiltonian system of this type is called the system with a <u>first class constraint</u>.

Constraints on the phase space

$$M = \text{phase space} = \{(\underbrace{q^1, \cdots, q^n}_{q}, \underbrace{p_1, \cdots, p_n}_{p})\}$$

A constraint $\longleftrightarrow$ (q, p) is allowed to be only in a submanifold $N \subset M$

Locally, it is defined by constraint equations

$$\varphi^a(q, p) = 0 \qquad a = 1, \cdots, m \leq 2n$$

N has dimension $2n - m$.

e.g. For $M = \mathbb{R}^{2n}$,

(a) $\varphi = p_n$: $N = \{(q^1, \cdots, q^n, p_1, \cdots, p_{n-1}, 0)\} \cong \mathbb{R}^{2n-1}$.

(b) $\varphi^1 = q^n$, $\varphi^2 = p_n$: $N = \{(q^1, \cdots, q^{n-1}, 0, p_1, \cdots, p_{n-1}, 0)\} \cong \mathbb{R}^{2n-2}$

(c) $\varphi = \sum_{i=1}^{n} \left((q^i)^2 + (p_i)^2\right) - r$: $N \cong S^{2n-1}$.

The constraint is consistent with the dynamics if the time evolution sends N to itself. That is, if the starting point is in N, it remains so afterwards.

This requires

$$\frac{d\varphi^a}{dt} = \{\varphi^a, H\} \quad \text{vanishes on } N$$

$$\Leftrightarrow \quad \{\varphi^a, H\} = \sum_b C^a_b \varphi^b$$

for some function $C^a_b = C^a_b(q,p)$

(at least in a neighborhood of N).

Two typical cases:

A constraint of <u>first class</u>:

$$\{\varphi^a, \varphi^b\} = 0 \quad \text{on } N$$

A constraint of <u>second class</u>:

$(\{\varphi^a, \varphi^b\})$ is an invertible matrix on N

i.e. $\det\{\varphi^a, \varphi^b\} \neq 0$ on N.

E.g. (a) 1st class, (b) 2nd class, (c) 1st class.

Our main target is 1st class constraint, but let us study the treatment of 2nd class (which will be used also for 1st class).

Treatment of 2nd class constraint

For a 2nd class constraint, the submanifold N itself can be regarded as a phase space, with

Poisson bracket := the <u>Dirac bracket</u> :

For functions f, g on N, take any extensions $\tilde{f}, \tilde{g}$ to a neighborhood of N in M and put

$$\{f, g\}_N := \left(\{\tilde{f}, \tilde{g}\} - \{\tilde{f}, \varphi^a\} D_{ab} \{\varphi^b, \tilde{g}\} \right)\Big|_N$$

where D_{ab} is the inverse matrix of $\{\varphi^a, \varphi^b\}|_N$.

- This does not depend on the choice of extensions.

$\because$ Another choice $\tilde{f}' = \tilde{f} + \Delta\tilde{f}$; $\Delta\tilde{f}|_N = 0$.

$\Rightarrow \Delta\tilde{f} = \sum_a f_a \varphi^a$ for some f_a's

$$\Delta\{f, g\}_N = \left(\{\Delta\tilde{f}, \tilde{g}\} - \{\Delta\tilde{f}, \varphi^b\} D_{bc} \{\varphi^c, \tilde{g}\} \right)\Big|_N$$

$$= \sum_a f_a \left(\{\varphi^a, \tilde{g}\} - \underbrace{\{\varphi^a, \varphi^b\} D_{bc}}_{\delta^a_c} \{\varphi^c, \tilde{g}\} \right)\Big|_N$$

$= 0.$ //

- $\{f, H|_N\}_N = \{\tilde{f}, H\}|_N$.

  Thus, the time evolution is generated by $H|_N$ in the constrained phase space $(N, \{\ ,\ \}_N)$.

- The constrained system can be quantized in the operator formalism in the standard way:

$$[\hat{O}_1, \hat{O}_2] = i\hbar\, \widehat{\{O_1, O_2\}_N}.$$

To be precise, one needs to check that the Dirac bracket $\{\ ,\ \}_N$ has the properties required for Poisson bracket:

(i) antisymmetry: $\{f, g\}_N = -\{g, f\}_N$

(ii) derivation: $\{f, gh\}_N = \{f, g\}_N h + g\{f, h\}_N$

(iii) Jacobi identity: $\{f, \{g, h\}_N\}_N$ + cyclic $= 0$

(iv) non-degeneracy: for any local coordinates $(x^r)_{r=1}^{2n-m}$ on N, $\{x^r, x^s\}_N$ is invertible.

You may try to show these directly. However, there is a conceptually clearer picture in which this is automatic.

It is to view phase spaces as symplectic manifolds:

If ω is the symplectic form on M corresponding to the Poisson bracket $\{\,,\,\}$, and if $\varphi^1 = \cdots = \varphi^m = 0$ is a 2nd class constraint, then, ω restricted to $N = \{\varphi^1 = \cdots = \varphi^m = 0\}$ is non-degenerate, and hence is a symplectic form on N. The Dirac bracket $\{\,,\,\}_N$ is nothing but the Poisson bracket corresponding to $\omega|_N$.

To summarize,

phase space	$\longleftrightarrow$	symplectic manifold
$(M, \{\,,\,\})$	$\longleftrightarrow$	(M, ω)
$(N, \{\,,\,\}_N)$	$\longleftrightarrow$	$(N, \omega\|_N)$.

Given this, the Dirac bracket $\{\,,\,\}_N$ automatically has the required properties (i), (ii), (iii), (iv).

Reduced phase space for 1st class constraint

Now let us consider the system with a 1st class constraint

$$\varphi^a(q,p) = 0 \qquad a = 1, \cdots, m,$$

$$\{H, \varphi^a\} = \sum_b C^a_b \varphi^b,$$

$$\{\varphi^a, \varphi^b\} = \sum_c C^{ab}_c \varphi^c.$$

Introducing a Lagrange multiplier $\lambda_a(t)$, the action may be written as

$$S = \int_{t_i}^{t_f} dt \left(\sum_i p_i \dot{q}^i - H(q,p) + \sum_a \lambda_a \varphi^a(q,p) \right)$$

[e.g. Yang-Mills theory :

$p_i \to \mathbb{E}(x),\ q^i \to \mathbb{A}(x),\ \lambda_a \to A_0(x),\ \varphi^a \to \Phi(x) = \mathbb{D}\cdot\mathbb{E}(x)$]

Equations of motion (EL eq for $q(t_i)$, $q(t_f)$ fixed):

$$\begin{cases} \dot{q}^i = \dfrac{\partial H}{\partial p_i} + \sum_a \lambda_a \dfrac{\partial \varphi^a}{\partial p_i} \\[2ex] \dot{p}_i = -\dfrac{\partial H}{\partial q^i} - \sum_a \lambda_a \dfrac{\partial \varphi^a}{\partial q^i} \\[2ex] \varphi^a = 0 \end{cases}$$

Rmks

1. $\varphi^a(q,p)=0$ is consistently preserved on $N=\{\varphi^a=0\}\subset M$:

$$\dot{\varphi}^a = \dot{q}^i\frac{\partial\varphi^a}{\partial q^i} + \dot{p}_i\frac{\partial\varphi^a}{\partial p_i} = \underbrace{\{\varphi^a, H\}}_{0 \text{ on } N} + \sum_b \lambda_b \underbrace{\{\varphi^a, \varphi^b\}}_{0 \text{ on } N}$$

2. There is an ambiguity .... $\lambda_a(t)$ is not fixed.

Different $\lambda_a(t)$'s $\rightsquigarrow$ different trajectories

$\rightarrow$ We regard them all physically equivalent.

Physical observables $f(q,p)$ are those which do not change under the change of $\lambda_a(t)$,

$$\dot{f} = \{f, H\} + \sum_b \lambda_b \boxed{\{f, \varphi^b\}}$$

This must vanish on N

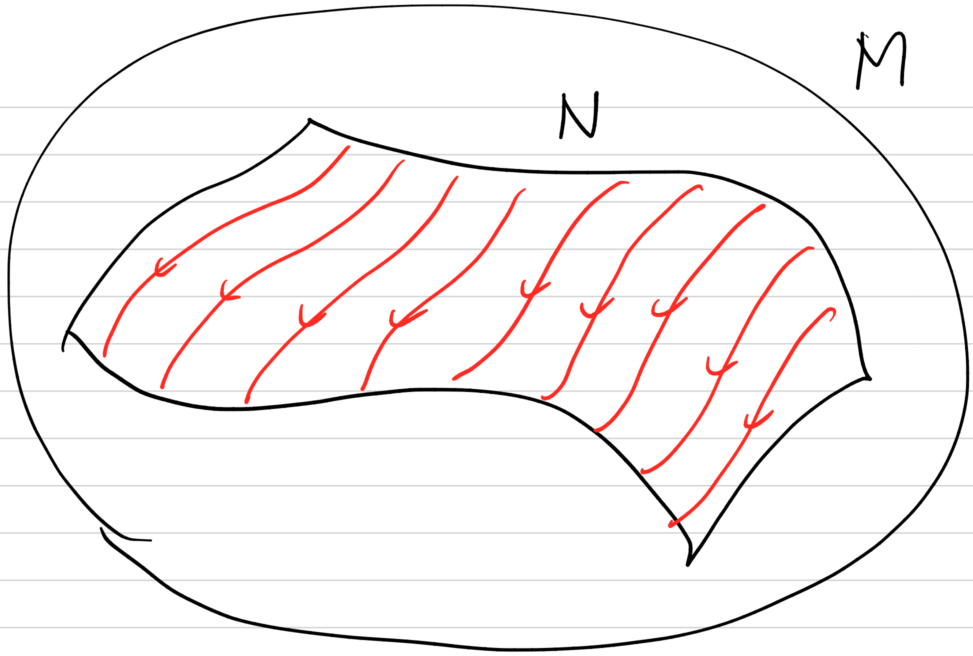
In other words,

$\varphi^a(q,p)$'s generate gauge transformations in M:

$$\delta^a q^i = \{q^i, \varphi^a\}, \quad \delta^a p_i = \{p_i, \varphi^a\}.$$

Since $\delta^a\varphi^b = \{\varphi^b, \varphi^a\} = C^{ba}_c\,\varphi^c$ vanishes on $N=\{\varphi^a=0\}$,

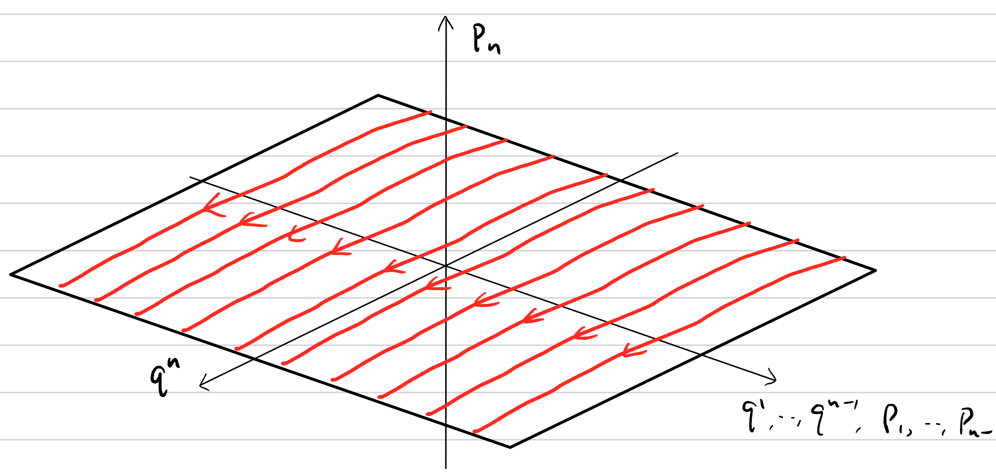
the gauge transformations map points of N to points of N:



Physical observables are functions on N which are invariant under gauge transformations.

e.g. (a) $\varphi = p_n$ in $M = \mathbb{R}^{2n}$

$\delta q^i = \{q^i, p_n\} = \delta^i_n$, $\delta p_i = 0$: translation in q^n.



Physical observables are functions of $(q^1, \ldots, q^n, p_1, \ldots, p_{n-1}, 0)$ that are invariant under q^n-translations

... functions of $(q^1, \ldots, q^{n-1}, p_1, \ldots, p_{n-1})$

Define the reduced phase space $M^* = N/\sim$

$x \sim y \iff x$ and y are related by a gauge transformation.

Functions on M^* = gauge invariant functions on N

= functions $\tilde{f}$ on a neighborhood of N in M

s.t. $\{\tilde{f}, \varphi^a\} = f^a_b \varphi^b$

modulo addition of functions vanishing on N.

Theorem $M^* = N/\sim$ has a Poisson bracket :

f, g functions on M^*

$\rightsquigarrow \tilde{f}, \tilde{g}$ s.t. $\{\tilde{f}, \varphi^a\} = f^a_b \varphi^b, \ \{\tilde{g}, \varphi^a\} = g^a_b \varphi^b$

$\{f, g\}_{M^*}$ is represented by $\{\tilde{f}, \tilde{g}\}$.

proof Check points :

1. $\{\tilde{f}, \tilde{g}\}$ defines a function on M^*.
2. independent of the choice of $\tilde{f}, \tilde{g}$.
3. $\{\ ,\ \}_{M^*}$ has required properties for Poisson bracket.

$$① \quad \{\{\tilde f,\tilde g\},\varphi^a\} = \{\underbrace{\{\tilde f,\varphi^a\}}_{f^a{}_b\varphi^b},\tilde g\} + \{\tilde f,\underbrace{\{\tilde g,\varphi^a\}}_{g^a{}_b\varphi^b}\}$$

$$= \{f^a{}_b,\tilde g\}\varphi^b + f^a{}_b\underbrace{\{\varphi^b,\tilde g\}}_{-g^b{}_c\varphi^c} + \{\tilde f, g^a{}_b\}\varphi^b + g^a{}_b\underbrace{\{\tilde f,\varphi^b\}}_{f^b{}_c\varphi^c}$$

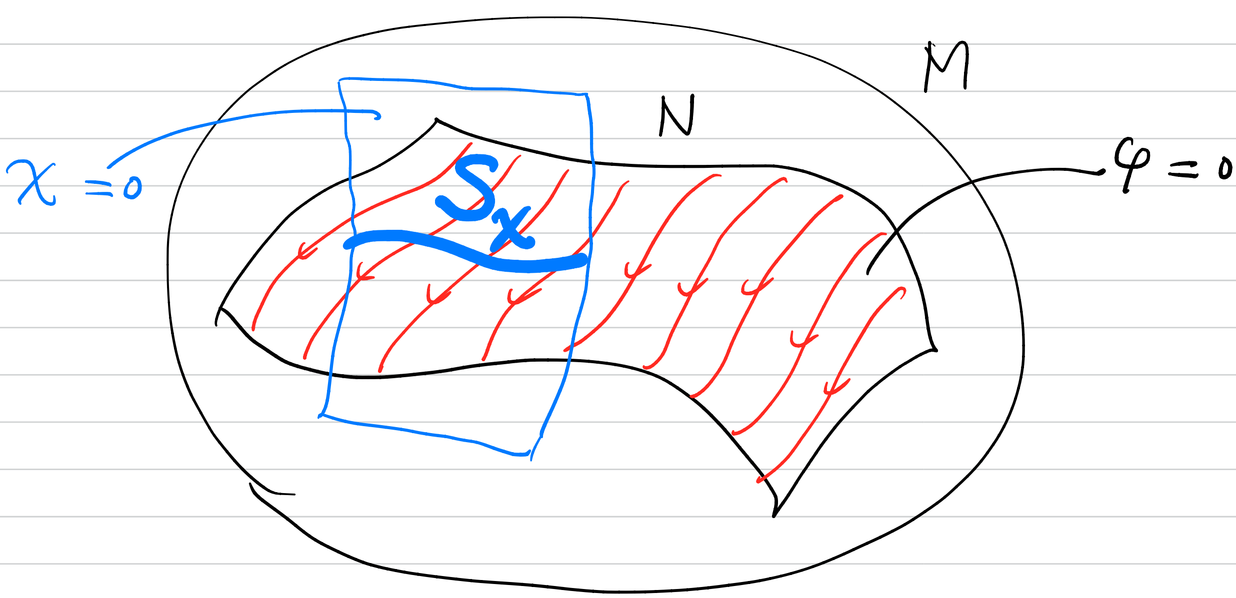
$$= \left(\{f^a{}_b,\tilde g\} - f^a{}_c g^c{}_b + \{\tilde f, g^a{}_b\} + g^a{}_c f^c{}_b\right)\varphi^b \quad \checkmark$$

$$② \quad \tilde f \to \tilde f + \Delta\tilde f \ ; \ \Delta\tilde f = f_a\varphi^a$$

$$\Delta\{\tilde f,\tilde g\} = \{f_a\varphi^a,\tilde g\} = \{f_a,\tilde g\}\varphi^a + f_a\underbrace{\{\varphi^a,\tilde g\}}_{g^a{}_b\varphi^b}$$

$$= \left(\{f_a,\tilde g\} + f_b g^b{}_a\right)\varphi^a \quad \checkmark$$

③ Take a (local) slice S_χ of $N \to N/\sim$ defined by equations $\chi_a(q,p)=0$, $a=1,\cdots,m$, in addition to $\varphi^a(q,p)=0$, $a=1,\cdots,m$.



Since the equations $\chi_1 = \cdots = \chi_m = 0$ must be maximally violated by the gauge transformations $\{-, \varphi^a\}$,

$$\det \{\chi_a, \varphi^b\} \neq 0 \quad \text{on } S_\chi .$$

Write $\{\Phi^A\}_{A=1}^{2m}$ for $\{\chi_a\}_{a=1}^{m} \cup \{\varphi^a\}_{a=1}^{m}$.

$$\{\Phi^A, \Phi^B\}\Big|_{S_\chi} = \begin{pmatrix} \{\chi_a, \chi_b\} & \{\chi_a, \varphi^b\} \\ \{\varphi^a, \chi_b\} & 0 \end{pmatrix}\Bigg|_{S_\chi} =: \begin{pmatrix} X & Y \\ -Y^T & 0 \end{pmatrix}$$

is invertible as $Y_a{}^b = \{\chi_a, \varphi^b\}\big|_{S_\chi}$ is invertible.

This means that $S_\chi = \{\Phi^A = 0,\ A = 1, \dots, 2m\}$ is a 2nd class constraint. In particular, the Dirac bracket $\{\,,\,\}_{S_\chi}$ is defined on S_χ.

<u>Claim</u> $\{\,,\,\}_{S_\chi} = \{\,,\,\}_{M^*}$ under $S_\chi \overset{\text{local}}{\cong} M^*$.

☺ Let f & g be functions on S_χ. They can be extended to gauge invariant functions on N and then to functions $\tilde{f}, \tilde{g}$ defined on a neighborhood of N in M s.t.

$\{\tilde{f}, \varphi^a\} = f^a_a \varphi^b$ and $\{\tilde{g}, \varphi^a\} = g^a_b \varphi^b$. Then

$$\{f, g\}_{S_\chi} = \left(\{\tilde{f}, \tilde{g}\} - \{\tilde{f}, \Phi^A\} D_{AB} \{\Phi^B, \tilde{g}\}\right)\Big|_{S_\chi}.$$

$$\left[\ \{\tilde{f}, \varphi^a\}\big|_{S_\chi} = \{\varphi^b, \tilde{g}\}\big|_{S_\chi} = 0\right.$$

$$= \left(\{\tilde{f}, \tilde{g}\} - \{\tilde{f}, \chi_a\} D_{\chi_a \chi_b} \{\chi_b, \tilde{g}\}\right)\Big|_{S_\chi}$$

$$\left[\ \begin{pmatrix} X & Y \\ -Y^T & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & -(Y^{-1})^T \\ Y^{-1} & Y^{-1} X (Y^{-1})^T \end{pmatrix}\right.$$

$$\Rightarrow \quad \underline{D_{\chi_a \chi_b} = 0}$$

$$= \{\tilde{f}, \tilde{g}\}\big|_{S_\chi}.$$

On the other hand, $\{\tilde{f}, \tilde{g}\}$ represents $\{f, g\}_{M^*}$. //

Since $\{\ ,\ \}_{S_\chi}$ has the properties required for Poisson bracket, $\{\ ,\ \}_{M^*}$ also does. ✓

Q.E.D.

Once again, viewing phase spaces as symplectic manifold makes things more transparent.

A phase space $(M, \{\,,\,\})$ with a 1st class constraint

$\varphi^1 = \cdots = \varphi^m = 0$ (with some assumption)

is a symplectic manifold (M, ω) with an action of a Lie group G with a "moment map μ."

The reduced phase space corresponds to the "symplectic quotient $\mu^{-1}(0)/G$".

There is no need of extension of functions $f \mapsto \tilde{f}$ nor choice of local slice S_x.

Now, the system can be quantized in the operator formalism:

$$[\hat{\mathcal{O}}_1, \hat{\mathcal{O}}_2] = i\hbar \widehat{\{\mathcal{O}_1, \mathcal{O}_2\}_{M^*}}.$$

If we can find a global slice S_χ, we just have to quantize S_χ with its Dirac bracket $\{\ ,\ \}_{S_\chi}$:

$$[\hat{\mathcal{O}}_1, \hat{\mathcal{O}}_2] = i\hbar \widehat{\{\mathcal{O}_1, \mathcal{O}_2\}_{S_\chi}}.$$

<u>The case $\{\chi_a, \chi_b\} = 0$</u>

In this case, we may find canonical coordinates of M in which the latter m p-coordinates are $\chi_1, \cdots, \chi_m$:

$$(q^1, \cdots, q^{n-m}, q^{n-m+1}, \cdots, q^n, p_1, \cdots, p_{n-m}, \underset{\chi_1}{\underset{\shortparallel}{p_{n-m+1}}}, \cdots, \underset{\chi_m}{\underset{\shortparallel}{p_n}})$$

Then, $$\{\varphi^a, \chi_b\} = \frac{\partial \varphi^a}{\partial q^i} \underbrace{\frac{\partial \chi_b}{\partial p_i}}_{\delta^i_{n-m+b}} - \frac{\partial \varphi^a}{\partial p_i} \underbrace{\frac{\partial \chi_b}{\partial q^i}}_{0} = \frac{\partial \varphi^a}{\partial q^{n-m+b}}$$

$1 \leq a \leq m$

$n-m+1 \leq n-m+b \leq n$

Since this is invertible, we may take

$$(q^1, \cdots, q^{n-m}, \varphi^1, \cdots, \varphi^m, p_1, \cdots, p_{n-m}, \chi_1, \cdots, \chi_m)$$

as (not necessarily canonical) coordinates of M.

Then, as coordinates of $S_\chi = \{ \varphi^1 = \cdots = \varphi^m = \chi_1 = \cdots = \chi_m = 0 \}$,

we may take $(q^1, \cdots, q^{n-m}, p_1, \cdots, p_{n-m})$.

<u>Claim</u> These are canonical coordinates of S_χ with respect to the Dirac bracket $\{\ ,\ \}_{S_\chi}$.

☺ Let f, g be functions on S_χ and as their extensions $\tilde{f}, \tilde{g}$ to M, let us take

$$\tilde{f}(q, p) = f(q^1, \cdots, q^{n-m}, p_1, \cdots, p_{n-m})$$
$$\tilde{g}(q, p) = g(q^1, \cdots, q^{n-m}, p_1, \cdots, p_{n-m})$$

$$\{f, g\}_{S_\chi} = \left(\{\tilde{f}, \tilde{g}\} - \{\tilde{f}, \Phi^A\} D_{AB} \{\Phi^B, \tilde{g}\} \right) \Big|_{S_\chi}.$$

As $\{\chi_a, \chi_b\} = 0$, $D = \begin{pmatrix} 0 & -(Y^{-1})^T \\ Y^{-1} & 0 \end{pmatrix}$,

but $\{\tilde{f}, \chi_a\} = \{\chi_b, \tilde{g}\} = 0$.

$$\therefore \{f, g\}_{S_\chi} = \{\tilde{f}, \tilde{g}\}\Big|_{S_\chi} = \sum_{r=1}^{n} \frac{\partial f}{\partial q^r}\frac{\partial g}{\partial p_r} - \frac{\partial f}{\partial p_r}\frac{\partial g}{\partial q^r} \quad //$$

In this case, the operator quantization takes a particularly simple form:

$$[\hat{q}^r, \hat{p}_s] = i\hbar \delta^r_s \qquad 1 \leq r, s \leq n-m.$$

$$[\hat{q}^r, \hat{q}^s] = [\hat{p}_r, \hat{p}_s] = 0$$

<u>Example</u> Maxwell theory (free $U(1)$ gauge theory)

The Gauss law constraint is

$$\Phi(\mathbf{x}) = \nabla \cdot \mathbb{E}(\mathbf{x}) = 0.$$

As the slice, we can take the Coulomb gauge

$$\chi(\mathbf{x}) = \nabla \cdot \mathbb{A}(\mathbf{x}) = 0,$$

which does satisfy $\{\chi(\mathbf{x}), \chi(\mathbf{y})\} = 0$. Then the modes of $\mathbb{E}(\mathbf{x})$ and $\mathbb{A}(\mathbf{x})$ satisfying Gauss law $\nabla \cdot \mathbb{E}(\mathbf{x}) = 0$ and Coulomb gauge $\nabla \cdot \mathbb{A}(\mathbf{x}) = 0$ form canonical conjugate pairs and can be quantized in the canonical way.

We may also do path-integral quantization:

Notation $q^* = (q^1, \dots, q^{n-m})$, $p^* = (p_1, \dots, p_{n-m})$

$q^! = (q^{n-m+1}, \dots, q^n)$, $p^! = (p_{n-m+1}, \dots, p_n) = (\chi_1, \dots, \chi_m) = \chi$

$(q^*, p^*) \in S_\chi$ determines the value of $q^!$

$$\Rightarrow \quad q^! = q^!(q^*, p^*).$$

$$Z(t_f, q_f^*; t_i, q_i^*) = \int_{q^*(t_f) = q_f^*,\ q^*(t_i) = q_i^*} \mathcal{D}q^* \mathcal{D}p^* \, e^{\frac{i}{\hbar}\int_{t_i}^{t_f} dt \underbrace{\left(p_r \dot{q}^r - H|_{S_\chi}\right)}}$$

$$= \int_{q^*(t_f) = q_f^*,\ q^*(t_i) = q_i^*} \mathcal{D}q^* \mathcal{D}p^* \mathcal{D}q^! \mathcal{D}p^! \prod_t \delta(\underbrace{p^!(t)}_{\chi(q(t),p(t))}) \underbrace{\delta\left(q^!(t) - q^!(q^*(t), p^*(t))\right)}_{\delta(\varphi(q(t),p(t)))\det\left(\underbrace{\frac{\partial \varphi^a}{\partial q^{n-m+b}}}_{\{\varphi^a, \chi_b\}}(q(t),p(t))\right)} \cdot e^{\frac{i}{\hbar}\int_{t_i}^{t_f} dt\,(p_i \dot{q}^i - H)}$$

$$= \int_{q^*(t_f) = q_f^*,\ q^*(t_i) = q_i^*} \mathcal{D}q \, \mathcal{D}p \, \mathcal{D}\lambda \prod_t \delta(\chi(q(t), p(t))) \det\left(\{\varphi^a, \chi_b\}(q(t), p(t))\right) e^{\frac{i}{\hbar}\int_{t_i}^{t_f} dt\,\left(p_i \dot{q}^i - H(p,q) + \lambda_a \varphi^a(q,p)\right)}$$

Yang-Mills theory $\quad \mathcal{M} = \{ (A_{ia}(\mathbf{x}), E_{ia}(\mathbf{x})) \}$

$$H = \int d^{d-1}\mathbf{x} \left(\frac{e^2}{2} \sum_{i,a} E_{ia}(\mathbf{x})^2 + \frac{1}{2e^2} \sum_{\substack{i<j \\ a}} F_{ija}(\mathbf{x})^2 \right)$$

$$\varphi(\mathbb{A},\mathbb{E})^a(\mathbf{x}) = (\mathbb{D}\cdot\mathbb{E})_a(\mathbf{x})$$

$$\chi(\mathbb{A},\mathbb{E})_a(\mathbf{x}) = (\nabla\cdot\mathbb{A})_a(\mathbf{x})$$

$$\{ \varphi^a(\mathbf{x}), \chi_b(\mathbf{y}) \} = (\nabla\cdot\mathbb{D}\,\delta(\mathbf{x}-\mathbf{y}))^a{}_b$$

$$\{ \chi_a(\mathbf{x}), \chi_b(\mathbf{y}) \} = 0 \qquad \checkmark.$$

$$Z = \int \mathcal{D}\mathbb{A}\,\mathcal{D}\mathbb{E}\,dA_0 \prod_t \left(\prod_{\mathbf{x}} \delta(\nabla\cdot\mathbb{A}(t,\mathbf{x})) \cdot \det(\nabla\cdot\mathbb{D}\,\delta(\mathbf{x}-\mathbf{y})(t)) \right)$$
$$\exp\left(\frac{i}{\hbar} \int d^d x \left(\mathbb{E}\cdot\dot{\mathbb{A}} - \frac{e^2}{2}\mathbb{E}^2 - \frac{1}{2e^2}\sum_{ij} F_{ij}^2 + A_0\,\mathbb{D}\cdot\mathbb{E} \right) \right)$$

$$\overset{\substack{\text{integrate out} \\ \mathbb{E}}}{=} \int \mathcal{D}\mathbb{A} \prod_x \delta(\nabla\cdot\mathbb{A}(x)) \det(\nabla\cdot\mathbb{D}\,\delta(x-y))\, e^{\frac{i}{\hbar}S[\mathbb{A}]}$$

...agrees with the earlier result with

gauge fixing condition $\chi = \nabla\cdot\mathbb{A}$.