Correlation functions vs vacuum expectation values

Recall $Z(t_f, q_f ; \mathcal{O}_1(t_1)\, \mathcal{O}_2(t_2) ; t_i, q_i)$

$$:= \int_{q(t_f)=q_f,\ q(t_i)=q_i} \mathcal{D}q\, e^{i\int_{t_i}^{t_f} dt\, L(q,\dot{q})}\, \mathcal{O}_1(t_1)\, \mathcal{O}_2(t_2)$$

$$= \begin{cases} \langle q_f |\, e^{-i(t_f - t_1)\hat{H}}\, \hat{\mathcal{O}}_1\, e^{-i(t_1-t_2)\hat{H}}\, \hat{\mathcal{O}}_2\, e^{-i(t_2-t_i)\hat{H}} | q_i \rangle & \text{if } t_1 > t_2 \\ \langle q_f |\, e^{-i(t_f - t_2)\hat{H}}\, \hat{\mathcal{O}}_2\, e^{-i(t_2-t_1)\hat{H}}\, \hat{\mathcal{O}}_1\, e^{-i(t_1-t_i)\hat{H}} | q_i \rangle & \text{if } t_2 > t_1 \end{cases}$$

$$=: \langle q_f |\, e^{-it_f\hat{H}} \cdot \underbrace{T\, \widehat{\mathcal{O}_1(t_1)}\, \widehat{\mathcal{O}_2(t_2)}}_{\text{time ordered product}} \cdot e^{it_i\hat{H}} | q_i \rangle$$

where $\widehat{\mathcal{O}(t)} := e^{it\hat{H}}\, \hat{\mathcal{O}}\, e^{-it\hat{H}}$

Simililary

$$Z(t_f, q_f ; \mathcal{O}_1(t_1) \cdots \mathcal{O}_s(t_s) ; t_i, q_i)$$

$$= \langle q_f |\, e^{-it_f\hat{H}} \cdot T\, \widehat{\mathcal{O}_1(t_1)} \cdots \widehat{\mathcal{O}_s(t_s)} \cdot e^{it_i\hat{H}} | q_i \rangle .$$

Let us take the limit $T \to e^{-i\epsilon}(+\infty)$ in

$$Z(T, q_f ; \mathcal{O}_1(t_1) \cdots \mathcal{O}_s(t_s), -T, q_i)$$

$$= \langle q_f | e^{-iT\hat{H}} \cdot T\, \widehat{\mathcal{O}_1(t_1)} \cdots \widehat{\mathcal{O}_s(t_s)} \cdot e^{-iT\hat{H}} | q_i \rangle$$

$$= \sum_{n,m} \langle q_f | e^{-iTE_n} | n \rangle \langle n | T\, \widehat{\mathcal{O}_1(t_1)} \cdots \widehat{\mathcal{O}_s(t_s)}$$
$$e^{-iTE_m} | m \rangle \langle m | q_i \rangle$$

where $\{ |n\rangle \} \subset \mathcal{H}$ is a basis consisting of Hamiltonian eigenstates $\hat{H}|n\rangle = E_n |n\rangle$.

Let us assume that $|n\rangle$ with label $n=0$ is the unque ground state, the vacuum state $|0\rangle$.

Then, for $n \neq 0$, $E_n > E_0$ and

$$e^{-iT(E_n - E_0)} = e^{-i e^{-i\epsilon} |T| (E_n - E_0)} \searrow 0 \quad \text{as } |T| \to \infty$$

Thus, as long as $\langle q_f | 0 \rangle \neq 0$ and $\langle 0 | q_i \rangle \neq 0$, the term $n = m = 0$ is dominant, and the other terms are exponentially small,

$$Z(T, q_f ; \mathcal{O}_1(t_1) \cdots \mathcal{O}_s(t_s) ; -T, q_i)$$

$$= e^{-2iTE_0} \Big\{ \langle q_f | 0 \rangle \langle 0 | T\, \widehat{\mathcal{O}_1(t_1)} \cdots \widehat{\mathcal{O}_s(t_s)} | 0 \rangle \langle 0 | q_i \rangle$$

$$+ \sum_{m,n \neq 0} \underbrace{e^{-iT(E_n - E_0)} e^{-iT(E_m - E_0)}}_{\to 0 \text{ as } |T| \to \infty} \ast_{n,m} \Big\}$$

$$\langle \mathcal{O}_1(t_1) \cdots \mathcal{O}_s(t_s) \rangle_{T, q_f ; -T, q_i}$$

$$:= \frac{Z(T, q_f ; \mathcal{O}_1(t_1) \cdots \mathcal{O}_s(t_s) ; -T, q_i)}{Z(T, q_f ; -T, q_i)}$$

$$= \frac{e^{-2iE_0 T} \{ \langle q_f | 0 \rangle \langle 0 | T\, \widehat{\mathcal{O}_1(t_1)} \cdots \widehat{\mathcal{O}_s(t_s)} | 0 \rangle \langle 0 | q_i \rangle + \cdots \}}{e^{-2iE_0 T} \{ \langle q_f | 0 \rangle \langle 0 | q_i \rangle + \cdots \}}$$

$$= \langle 0 | T\, \widehat{\mathcal{O}_1(t_1)} \cdots \widehat{\mathcal{O}_s(t_s)} | 0 \rangle + \cdots$$

$$\xrightarrow{T \to e^{-i\epsilon}\infty} \langle 0 | T\, \widehat{\mathcal{O}_1(t_1)} \cdots \widehat{\mathcal{O}_s(t_s)} | 0 \rangle$$

The vacuum expectation value of the time ordered product.

<u>Remarks</u> (i) In a potential theory, $L = \frac{m}{2}\dot{q}^2 - U(q)$, with $U(q) \to \infty$ as $|q| \to \infty$, there is a unique ground state $|0\rangle$ and $\langle q|0\rangle \neq 0$ for any q. Thus we may take any q_i & q_f, say, $q_i = q_f = 0$, or $q_i = q_f =$ a minimum of $U(q)$.

(ii) In general, there can be more than one ground states $|0_I\rangle$ $(I = 1, 2, 3, \cdots)$ and/or an appropriate boundary condition needs to be specified at $t_f = T$ and $t_i = -T$. Showing the dependence on the boundary condition as a superscript, we have

$$\lim_{T \to e^{-i\epsilon}\infty} \langle \mathcal{O}_1(t_1) \cdots \mathcal{O}_s(t_s) \rangle^{I}_{T;-T} = \langle 0_I | T\, \widehat{\mathcal{O}_1(t_1)} \cdots \widehat{\mathcal{O}_s(t_s)} | 0_I \rangle.$$

(iii) We may consider, not just t_i & t_f, but all $t_1, \cdots, t_s$ to lie on $e^{-i\epsilon}\mathbb{R}$ and then take the limit $\epsilon \searrow 0$ after the computation. Then, the LHS (before $\epsilon \searrow 0$) can be regarded as the correlation function of a slightly Wick rotated theory, or equivalently, the real time limit of the reverse Wick rotation of the Euclidean theory

$$\lim_{\epsilon \searrow 0} \langle \mathcal{O}_1(e^{-i\epsilon} t_1) \cdots \mathcal{O}_s(e^{-i\epsilon} t_s) \rangle^I_{e^{-i\epsilon}\mathbb{R}}$$

$$= \lim_{\epsilon \searrow 0} \lim_{T \nearrow \infty} \langle \mathcal{O}_1(e^{-i\epsilon} t_1) \cdots \mathcal{O}_s(e^{-i\epsilon} t_s) \rangle^I_{[-e^{-i\epsilon}T, e^{-i\epsilon}T]}$$

$$= \langle 0_I | T \widehat{\mathcal{O}_1(t_1)} \cdots \widehat{\mathcal{O}_s(t_s)} | 0_I \rangle.$$

(iv) The same holds in QFT in $d > 1$:

$$\lim_{\epsilon \searrow 0} \langle \mathcal{O}_1(x_1^\epsilon) \cdots \mathcal{O}(x_s^\epsilon) \rangle^I_{e^{-i\epsilon}\mathbb{R} \times \mathbb{R}^{d-1}}$$

$$= \langle 0_I | T \widehat{\mathcal{O}_1(x_1)} \cdots \widehat{\mathcal{O}_n(x_s)} | 0_I \rangle$$

where $x^\epsilon = (e^{-i\epsilon} t, \mathbb{x})$ for $x = (t, \mathbb{x})$.

Note that an appropriate boundary condition at spatial infinity $|\mathbb{x}| \to \infty$, that depends on the ground state $|0_I\rangle$, also needs to be specified.

In the Euclidean theory, where there is no time-space distinction, the boundary conditions at space & time ∞'s may be unified, and we may say

$$\langle \mathcal{O}_1(x_1^E) \cdots \mathcal{O}_s(x_s^E) \rangle^{I}_{\mathbb{R}^d_E} \xrightarrow{\text{Minkowski limit}} \langle 0_I | T\, \mathcal{O}_1(x_1) \cdots \mathcal{O}_s(x_s) | 0_I \rangle .$$

We shall mostly consider theories with a unique ground state and suppress the label I of boundary condition.

To avoid cluttering the notation, we may simply write

$$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_s(x_s) \rangle$$

for the Minkowski limit. Then

$$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_s(x_s) \rangle = \langle 0 | T\, \mathcal{O}_1(x_1) \cdots \mathcal{O}_s(x_s) | 0 \rangle$$

Free field theories

A theory is said to be <u>free</u> when the action is quadratic in variables.

e.g. n real variables $\phi = (\phi_1, \cdots, \phi_n)$

$$S_E(\phi) = \frac{1}{2} \sum_{i,j=1}^{n} \phi_i A_{ij} \phi_j \qquad A_{ij} = A_{ji} \text{ symmetric, positive eigenvalues}$$

$$d^n\phi = d\phi_1 \cdots d\phi_n$$

$$Z = \int d^n\phi \, e^{-S_E(\phi)} = \sqrt{\frac{(2\pi)^n}{\det A}}$$

$$\langle \phi_{i_1} \cdots \phi_{i_s} \rangle = \frac{1}{Z} \int d^n\phi \, e^{-S_E(\phi)} \phi_{i_1} \cdots \phi_{i_s} = ?$$

A trick:

$$f(A, J) := \int d^n\phi \, e^{-S_E(\phi) + \sum_{i=1}^{n} J_i \phi_i}$$

$$\frac{\partial}{\partial J_{i_1}} \cdots \frac{\partial}{\partial J_{i_s}} f(A, J) = \int d^n\phi \, e^{-S_E(\phi) + \sum_i J_i \phi_i} \phi_{i_1} \cdots \phi_{i_s}$$

$$\xrightarrow{J \to 0} Z \langle \phi_{i_1} \cdots \phi_{i_s} \rangle$$

But $f(A,J)$ can be computed as

$$f(A,J) = \int d^n\varphi \; e^{-\frac{1}{2}(\varphi - A^{-1}J)\cdot A(\varphi - A^{-1}J) + \frac{1}{2} J\cdot A^{-1} J}$$

$$= \sqrt{\frac{(2\pi)^n}{\det A}} \; e^{\frac{1}{2} J\cdot A^{-1} J} = Z \cdot e^{\frac{1}{2} J\cdot A^{-1} J}$$

$$\therefore \langle \varphi_{i_1} \cdots \varphi_{i_s} \rangle = \frac{1}{Z} \frac{\partial}{\partial J_{i_1}} \cdots \frac{\partial}{\partial J_{i_s}} f(A,J) \Big|_{J=0}$$

$$= \underbrace{\frac{\partial}{\partial J_{i_1}} \cdots \frac{\partial}{\partial J_{i_s}} e^{\frac{1}{2} J\cdot A^{-1} J}} \Big|_{J=0}.$$

→ Terms where $\frac{\partial}{\partial J}$ hits only one of the two J's in $\frac{1}{2} J A^{-1} J$ vanish as $J=0$. Terms that survive are those where both J's in $\frac{1}{2} J A^{-1} J$ are hit by $\frac{\partial}{\partial J}$'s. Thus, the result is the sum of terms where the derivatives $\frac{\partial}{\partial J_{i_1}}, \cdots, \frac{\partial}{\partial J_{i_s}}$ form pairs, which is possible only when s is even, each pair $\left\{\frac{\partial}{\partial J_{i_a}}, \frac{\partial}{\partial J_{i_b}}\right\}$ producing

$\frac{\partial}{\partial J_{i_a}} \frac{\partial}{\partial J_{i_b}} \left(\frac{1}{2} J \cdot A^{-1} J \right) = A^{-1}_{i_a i_b}$. It is the sum of pairwise contractions, called <u>Wick contractions</u>:

$$\langle \phi_i \rangle = 0 ,$$

$$\langle \phi_i \phi_j \rangle = \phi_i \phi_j = A^{-1}_{ij} ,$$

$$\langle \phi_i \phi_j \phi_k \rangle = 0 ,$$

$$\langle \phi_i \phi_j \phi_k \phi_l \rangle = \phi_i \phi_j \phi_k \phi_l + \phi_i \phi_j \phi_k \phi_l + \phi_i \phi_j \phi_k \phi_l$$

$$= A^{-1}_{ij} A^{-1}_{kl} + A^{-1}_{ik} A^{-1}_{jl} + A^{-1}_{il} A^{-1}_{jk} ,$$

$$\vdots$$

e.g. real scalar field in d dimensions

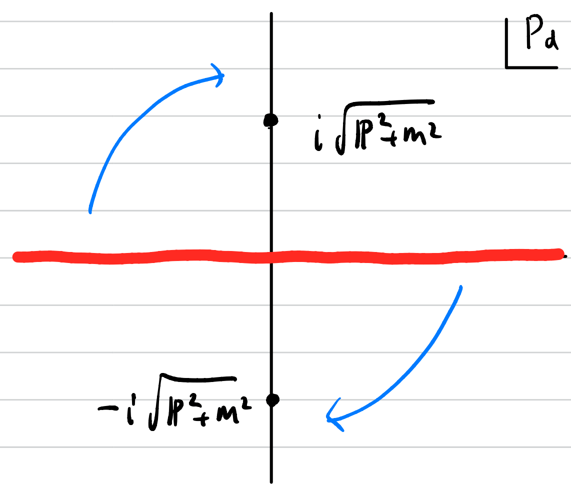
variable: $\phi(x)$ a function of $x \in \mathbb{R}^d_E$

$$S_E[\phi] = \int d^d x_E \left(\frac{1}{2} \partial\phi \cdot \partial\phi + \frac{m^2}{2}\phi^2 \right)$$

$$= \frac{1}{2} \int d^d x_E \; \phi(x_E)\left(-\partial^2 + m^2\right)\phi(x_E)$$

$$\langle \phi(x_E)\,\phi(y_E) \rangle_{\mathbb{R}^d_E} = \left(-\partial^2 + m^2\right)^{-1}_{x_E y_E} = \int \frac{d^d p_E}{(2\pi)^d} \frac{e^{-i p_E \cdot (x_E - y_E)}}{p_E^2 + m^2}$$

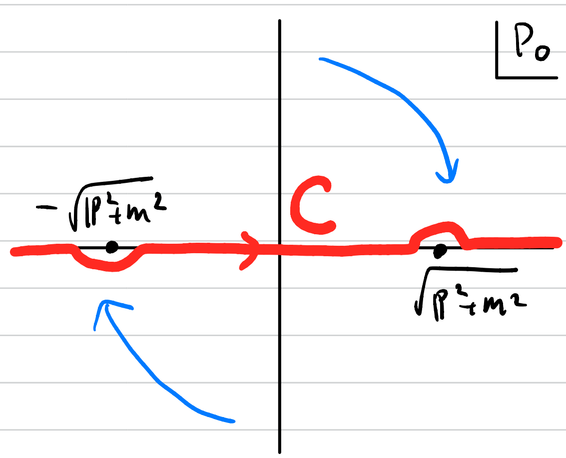
$$= \int \frac{d^{d-1}\mathbb{P}\, dp_d}{(2\pi)^d} \frac{e^{-i\mathbb{P}\cdot(\mathbb{x}-\mathbb{y}) - i p_d (x^d - y^d)}}{\mathbb{P}^2 + p_d^2 + m^2}$$



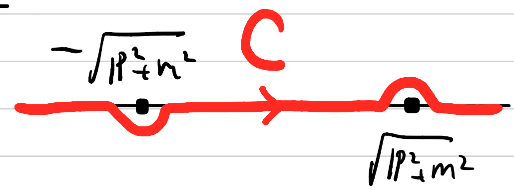
reverse Wick rotation

$x^d \to i x^0 , \; y^d \to i y^0 \; ; \quad p_d \to -i p_0$

$$\to \int_{\mathbb{R}^{d-1}\times C} \frac{d^{d-1}\mathbb{P}\,(-i\,dp_0)}{(2\pi)^d} \frac{e^{-i\mathbb{P}\cdot(\mathbb{x}-\mathbb{y}) - i p_0 (x^0 - y^0)}}{\mathbb{P}^2 - p_0^2 + m^2}$$



$$\langle \phi(x)\phi(y)\rangle = \int_{\mathbb{R}^{d-1}\times C} \frac{d^{d-1}\mathbb{p}\, dp_0}{(2\pi)^d} \frac{i\, e^{-ip\cdot(x-y)}}{p_0^2 - \mathbb{p}^2 - m^2}$$



Equivalently, slightly moving the poles,

$$\langle \phi(x)\phi(y)\rangle = \int_{\mathbb{R}^d} \frac{d^d p}{(2\pi)^d} \frac{i\, e^{-ip\cdot(x-y)}}{p^2 - m^2 + i\cdot 0}$$

$$p^2 := \eta^{\mu\nu} p_\mu p_\nu = p_0^2 - \mathbb{p}^2 \qquad \left(\begin{array}{c} \text{Our convention:} \\ \eta_{00} = 1, \quad \eta_{ij} = -\delta_{ij} \end{array} \right)$$

This is the inverse of the kinetic operator in the Minkowski action

$$S[\phi] = \int d^d x \left(\frac{1}{2} \eta^{\mu\nu} \partial_\mu \phi\, \partial_\nu \phi - \frac{m^2}{2} \phi^2 \right)$$

$$= \frac{1}{2} \int d^d x\, \phi(x) \left(-\eta^{\mu\nu} \partial_\mu \partial_\nu - m^2 \right) \phi(x)$$

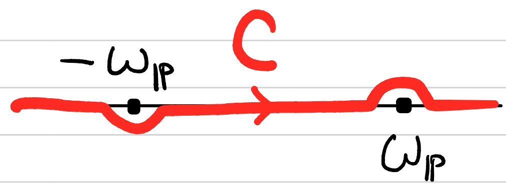
with the prescription $\frac{1}{p^2 - m^2} \to \frac{1}{p^2 - m^2 + i\cdot 0}$ to avoid poles.

Let us continue the computation. Defining

$$\omega_{\mathbb{p}} := \sqrt{\mathbb{p}^2 + m^2},$$

$$\langle \phi(x)\,\phi(y) \rangle$$

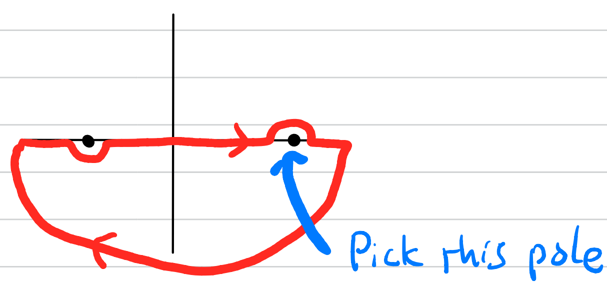
$$= \int_{\mathbb{R}^{d-1}} \frac{d^{d-1}\mathbb{p}}{(2\pi)^{d-1}} \int_C \frac{dp_0}{2\pi} \, \frac{i\, e^{-ip_0(x^0-y^0) - i\mathbb{p}\cdot(\mathbb{x}-\mathbb{y})}}{(p_0-\omega_{\mathbb{p}})(p_0+\omega_{\mathbb{p}})}$$



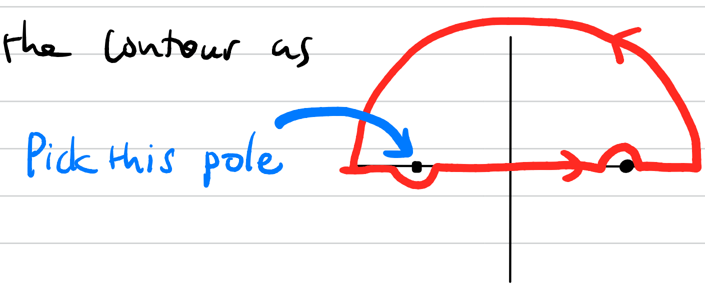
We may perform the p_0-integration

... Look at $e^{-ip_0(x^0-y^0)}$

$x^0 - y^0 > 0$: close the contour as



$x^0 - y^0 < 0$: close the contour as



The result

$$\langle \phi(x)\,\phi(y) \rangle = \int \frac{d^{d-1}\mathbb{p}}{(2\pi)^{d-1}\, 2\omega_{\mathbb{p}}} \, e^{-i\omega_{\mathbb{p}}|x^0-y^0| - i\mathbb{p}\cdot(\mathbb{x}-\mathbb{y})}$$

Comparison with operator result

$$\mathcal{L} = \frac{1}{2}\partial^\mu \phi\, \partial_\mu \phi - \frac{m^2}{2}\phi^2$$

$$\phi(\mathbb{x}) = \int \frac{d^{d-1}\mathbb{p}}{(2\pi)^{d-1}}\, e^{i\mathbb{p}\cdot\mathbb{x}}\, \phi(\mathbb{p}), \qquad \phi(\mathbb{p})^* = \phi(-\mathbb{p})$$

$$L = \int d^{d-1}\mathbb{x}\, \mathcal{L}$$

$$= \int \frac{d^{d-1}\mathbb{p}}{(2\pi)^{d-1}} \left\{ \frac{1}{2}\dot{\phi}(-\mathbb{p})\dot{\phi}(\mathbb{p}) - \frac{1}{2}\underbrace{(\mathbb{p}^2+m^2)}_{\omega_{\mathbb{p}}^2}\phi(-\mathbb{p})\phi(\mathbb{p}) \right\}$$

$$\pi(\mathbb{p}) = \frac{\delta L}{\delta \dot{\phi}(\mathbb{p})} = \frac{1}{(2\pi)^{d-1}}\dot{\phi}(-\mathbb{p}), \qquad \pi(\mathbb{p})^* = \pi(-\mathbb{p})$$

$$H = \int d^{d-1}\mathbb{p}\; \pi(\mathbb{p})\dot{\phi}(\mathbb{p}) - L$$

$$= \int d^{d-1}\mathbb{p} \left\{ \frac{(2\pi)^{d-1}}{2}\pi(\mathbb{p})\pi(-\mathbb{p}) + \frac{\omega_{\mathbb{p}}^2}{2(2\pi)^{d-1}}\phi(-\mathbb{p})\phi(\mathbb{p}) \right\}$$

Let us quantize the system.

(We omit hat ^ for operators.)

By the reality of variables, $\phi(\mathbb{p})^\dagger = \phi(-\mathbb{p})$, $\pi(\mathbb{p})^\dagger = \pi(-\mathbb{p})$.

Canonical commutation relation is

$$[\phi(\mathbb{P}_1), \pi(\mathbb{P}_2)] = i\delta^{d-1}(\mathbb{P}_1 - \mathbb{P}_2)$$

$$[\phi(\mathbb{P}_1), \phi(\mathbb{P}_2)] = [\pi(\mathbb{P}_1), \pi(\mathbb{P}_2)] = 0$$

Looking at H, we see that the system is just the sum of copies of harmonic oscillators. This motivates us to take

$$a(\mathbb{P}) := \sqrt{\frac{\omega_{\mathbb{P}}}{2(2\pi)^{d-1}}}\,\phi(\mathbb{P}) + i\sqrt{\frac{(2\pi)^{d-1}}{2\omega_{\mathbb{P}}}}\,\pi(-\mathbb{P})$$

$$a(\mathbb{P})^{\dagger} = \sqrt{\frac{\omega_{\mathbb{P}}}{2(2\pi)^{d-1}}}\,\phi(-\mathbb{P}) - i\sqrt{\frac{(2\pi)^{d-1}}{2\omega_{\mathbb{P}}}}\,\pi(\mathbb{P})$$

Then,

$$[a(\mathbb{P}_1), a(\mathbb{P}_2)^{\dagger}] = \delta^{d-1}(\mathbb{P}_1 - \mathbb{P}_2)$$

$$[a(\mathbb{P}_1), a(\mathbb{P}_2)] = [a(\mathbb{P}_1)^{\dagger}, a(\mathbb{P}_2)^{\dagger}] = 0$$

$$H = \int d^{d-1}\mathbb{P}\ \omega_{\mathbb{P}} \left(\frac{1}{2} a(\mathbb{P})^{\dagger} a(\mathbb{P}) + \frac{1}{2} a(-\mathbb{P}) a(-\mathbb{P})^{\dagger} \right)$$

$$= \int d^{d-1}\mathbb{P}\ \omega_{\mathbb{P}} \left(a(\mathbb{P})^{\dagger} a(\mathbb{P}) + \frac{1}{2} \delta^{d-1}(0) \right)$$

$$[H, a(\mathbb{p})] = -\omega_{\mathbb{p}}\, a(\mathbb{p}), \quad [H, a(\mathbb{p})^\dagger] = \omega_{\mathbb{p}}\, a(\mathbb{p})^\dagger.$$

Thus, $a(\mathbb{p})^\dagger / a(\mathbb{p})$ are indeed creation/annihilation operators.

The state $|0\rangle$ annihilated by all $a(\mathbb{p})$ is the unique ground state with energy $E_0 = \int d^{d-1}\mathbb{p}\, \frac{1}{2}\omega_{\mathbb{p}}\, \underbrace{\delta^{d-1}(0)}$.

best understood by putting the system in a finite volume

Other states are obtained from $|0\rangle$ by operating $a(\mathbb{p})^\dagger$'s. Each operation increases the energy by $\omega_{\mathbb{p}}$.

$$\phi(\mathbb{x}) = \int \frac{d^{d-1}\mathbb{p}}{(2\pi)^{d-1}}\, e^{i\mathbb{p}\cdot\mathbb{x}} \sqrt{\frac{(2\pi)^{d-1}}{2\omega_{\mathbb{p}}}} \left(a(\mathbb{p}) + a(-\mathbb{p})^\dagger \right)$$

$$= \int \frac{d^{d-1}\mathbb{p}}{\sqrt{(2\pi)^{d-1} 2\omega_{\mathbb{p}}}} \left(e^{i\mathbb{p}\cdot\mathbb{x}} a(\mathbb{p}) + e^{-i\mathbb{p}\cdot\mathbb{x}} a(\mathbb{p})^\dagger \right)$$

$$\phi(t, \mathbb{x}) = e^{itH}\, \phi(\mathbb{x})\, e^{-itH}$$

$$= \int \frac{d^{d-1}\mathbb{p}}{\sqrt{(2\pi)^{d-1} 2\omega_{\mathbb{p}}}} \left(e^{i\mathbb{p}\cdot\mathbb{x}} e^{-i\omega_{\mathbb{p}} t} a(\mathbb{p}) + e^{-i\mathbb{p}\cdot\mathbb{x} + i\omega_{\mathbb{p}} t} a(\mathbb{p})^\dagger \right)$$

$$\langle 0 | T \phi(x) \phi(y) | 0 \rangle$$

$$\overset{x^0 > y^0}{=} \langle 0 | \phi(x) \phi(y) | 0 \rangle$$

$$= \int \frac{d^{d-1}\mathbb{p}_1 \, d^{d-1}\mathbb{p}_2}{(2\pi)^{d-1} 2\sqrt{\omega_{\mathbb{p}_1}\omega_{\mathbb{p}_2}}} \underbrace{\langle 0 | e^{i\mathbb{p}_1\cdot\mathbb{x} - i\omega_{\mathbb{p}_1} x^0} a(\mathbb{p}_1) \, e^{-i\mathbb{p}_2\cdot\mathbb{y} + i\omega_{\mathbb{p}_2} y^0} a(\mathbb{p}_2)^\dagger | 0 \rangle}_{e^{-i\omega_{\mathbb{p}_1}x^0 + i\omega_{\mathbb{p}_2}y^0 + i\mathbb{p}_1\cdot\mathbb{x} - i\mathbb{p}_2\cdot\mathbb{y}} \, \delta^{d-1}(\mathbb{p}_1 - \mathbb{p}_2)}$$

$$= \int \frac{d^{d-1}\mathbb{p}}{(2\pi)^{d-1} 2\omega_{\mathbb{p}}} e^{-i\omega_{\mathbb{p}}(x^0 - y^0) + i\mathbb{p}\cdot(\mathbb{x} - \mathbb{y})}$$

$$\overset{y^0 > x^0}{=} \int \frac{d^{d-1}\mathbb{p}}{(2\pi)^{d-1} 2\omega_{\mathbb{p}}} e^{-i\omega_{\mathbb{p}}(y^0 - x^0) + i\mathbb{p}\cdot(\mathbb{y} - \mathbb{x})}$$

In eigther case

$$\langle 0 | T \phi(x) \phi(y) | 0 \rangle$$

$$= \int \frac{d^{d-1}\mathbb{p}}{(2\pi)^{d-1} 2\omega_{\mathbb{p}}} e^{-i\omega_{\mathbb{p}}|x^0 - y^0| - i\mathbb{p}\cdot(\mathbb{x} - \mathbb{y})}$$

It matches with $\langle \phi(x) \phi(y) \rangle$.

e.g. A part of gauge fixed Maxwell theory

$$\tilde{\mathcal{L}}_E = \frac{1}{4e^2}\sum_{\mu,\nu} F_{\mu\nu}^2 + \frac{e^2\xi}{2}B^2 - iB\cdot\partial^\mu A_\mu + \bar{c}\cdot\partial^2 c$$

(Eliminate B

$$\tilde{\mathcal{L}}_E = \underbrace{\frac{1}{4e^2}\sum_{\mu,\nu} F_{\mu\nu}^2 + \frac{1}{2e^2\xi}(\partial\cdot A)^2}_{\text{Consider this part}} + \bar{c}\cdot\partial^2 c$$

$$S_E[A] = \int d^d x_E \left(\frac{1}{4e^2}\sum_{\mu,\nu} F_{\mu\nu}^2 + \frac{1}{2e^2}(\partial\cdot A)^2 \right)$$

$$= \int d^d x_E \frac{1}{2e^2}\sum_{\mu,\nu} A_\mu(x_E)\left(-\delta_{\mu\nu}\partial^2 + \partial_\mu\partial_\nu - \frac{1}{\xi}\partial_\mu\partial_\nu\right)A_\nu(x_E)$$

$$=: \frac{1}{2}A\cdot\Delta A$$

$$\langle A_\mu(x_E)A_\nu(y_E)\rangle_{\mathbb{R}^d_E} = \Delta^{-1}_{(\mu,x_E),(\nu,y_E)}$$

$$= e^2\int\frac{d^d p_E}{(2\pi)^d}\frac{e^{-ip_E(x-y)}}{p_E^2}\left(\delta_{\mu\nu} + (\xi-1)\frac{p^E_\mu p^E_\nu}{p_E^2}\right)$$

reverse Wick rotation $x^d \to ix^0$; $p_d \to -ip_0$

$$A_d(x_E) \longrightarrow -i A_0(x)$$

$$\left.\begin{array}{lcl} \delta_{00} & \longrightarrow & -1 \\ \delta_{ij} & \longrightarrow & \delta_{ij} \end{array}\right\} \quad \delta_{\mu\nu} \longrightarrow -\eta_{\mu\nu}$$

$$\langle A_\mu(x) A_\nu(y) \rangle$$

$$= e^2 \int \frac{d^d p}{(2\pi)^d} \frac{i \, e^{-ip(x-y)}}{p^2 + i\cdot 0} \left(-\eta_{\mu\nu} - (\xi - 1) \frac{p_\mu p_\nu}{p^2 + i\cdot 0} \right)$$

The result depends on ξ $\longleftrightarrow$ A_μ is not physical,

$$\delta_B A_\mu = \partial_\mu C \neq 0.$$

$$\langle F_{\mu\nu}(x) F_{\rho\lambda}(y) \rangle = ?$$

$$\langle \partial_\mu A_\nu(x) \partial_\rho A_\lambda(y) \rangle$$

$$= e^2 \int \frac{d^d p}{(2\pi)^d} \frac{i \, e^{-ip(x-y)}}{p^2 + i\cdot 0} \left(-p_\mu p_\rho \eta_{\nu\lambda} - (\xi - 1) \underbrace{\frac{p_\mu p_\nu p_\rho p_\lambda}{p^2 + i\cdot 0}} \right)$$

Symmetric in $(\mu\nu)$, $(\rho\lambda)$

$\rightarrow$ vanishes in $[\mu\nu]$, $[\rho\lambda]$ alternating sum

$$\langle F_{\mu\nu}(x) F_{\rho\lambda}(y) \rangle$$

$$= e^2 \int \frac{d^d p}{(2\pi)^d} \frac{i\, e^{-ip(x-y)}}{p^2 + i0}$$

$$\left(- p_\mu p_\rho \eta_{\nu\lambda} + p_\nu p_\rho \eta_{\mu\lambda} + p_\mu p_\lambda \eta_{\nu\rho} - p_\nu p_\lambda \eta_{\mu\rho} \right)$$

This is the full correlation function of the gauge fixed theory:

$$\langle F_{\mu\nu}(x) F_{\rho\lambda}(y) \rangle_{\text{full}} = \text{the same},$$

Since the ghost path-integral simply produces

$$\frac{\int \mathcal{D}\bar{c}\,\mathcal{D}c\; e^{\int d^dx\, \bar{c}\, \partial^2 c}}{\int \mathcal{D}\bar{c}\,\mathcal{D}c\; e^{\int d^dx\, \bar{c}\, \partial^2 c}} = 1.$$

Since $F_{\mu\nu}$ is physical, $\delta_B F_{\mu\nu} = 0$, this is a physically meaningful result. Indeed, there is no ξ-dependence.

Exercise Compute the same in the canonical quantization of Maxwell theory, and compare.

<u>Remark</u> The expression for $\langle A_\mu(x) A_\nu(y)\rangle$ simplifies at $\xi = 1$ called <u>Feynman gauge</u>:

$$\langle A_\mu(x) A_\nu(y)\rangle = e^2 \int \frac{d^d p}{(2\pi)^d} \frac{i e^{-ip(x-y)}}{p^2 + i0} (-\eta_{\mu\nu})$$

(Simplification at $\xi = 1$ is obvious in $S[A]$.)

Of course, the physics (of the full gauge fixed system, with "physical = BRST cohomology" taken into account) should not depend on ξ. In other words, $\xi = 1$ is a convenient choice.

Let us continue with the computation of $\langle A_\mu(x) A_\nu(y)\rangle$. As it simplifies at $\xi = 1$, we just use it. Then, we can borrow the result for real scalar and find

$$\langle A_\mu(x) A_\nu(y)\rangle$$

$$= - e^2 \eta_{\mu\nu} \int \frac{d^{d-1} \mathbb{P}}{(2\pi)^{d-1} 2|\mathbb{P}|} e^{-i|\mathbb{P}|\cdot|x^0 - y^0| - i\mathbb{P}\cdot(\mathbf{x}-\mathbf{y})}.$$

Let us compare this with operator result (continuing with $\xi=1$).

$$S[A] = \int d^d x \left(-\frac{1}{4e^2} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2e^2} (\partial^\mu A_\mu)^2 \right)$$

$$= \int d^d x \, \frac{1}{2e^2} A_\mu(x) \, \eta^{\mu\nu} \partial^2 A_\nu(x)$$

$$L = \int d^{d-1} \mathbb{x} \left[-\frac{1}{2e^2} \dot{A}_0^2 + \frac{1}{2e^2} (\nabla A_0)^2 + \frac{1}{2e^2} \sum_i \left(\dot{A}_i^2 - (\nabla A_i)^2 \right) \right]$$

$$A_\mu(\mathbb{x}) = \int \frac{d^{d-1} \mathbb{p}}{(2\pi)^{d-1}} e^{i \mathbb{p}\cdot\mathbb{x}} A_\mu(\mathbb{p}) \quad , \qquad A_\mu(\mathbb{p})^* = A_\mu(-\mathbb{p})$$

$$L = \int \frac{d^{d-1}\mathbb{p}}{(2\pi)^{d-1}} \left[-\frac{1}{2e^2} \dot{A}_0(-\mathbb{p}) A_0(\mathbb{p}) + \frac{\mathbb{p}^2}{2e^2} A_0(-\mathbb{p}) A_0(\mathbb{p}) \right.$$
$$\left. + \frac{1}{2e^2} \sum_i \left(\dot{A}_i(-\mathbb{p}) A_i(\mathbb{p}) - \mathbb{p}^2 A_i(-\mathbb{p}) A_i(\mathbb{p}) \right) \right]$$

$$\Pi^0(\mathbb{p}) = -\frac{1}{(2\pi)^{d-1} e^2} \dot{A}_0(-\mathbb{p}) , \quad \Pi^i(\mathbb{p}) = \frac{1}{(2\pi)^{d-1} e^2} \dot{A}_i(-\mathbb{p}) ,$$

$$\Pi^\mu(\mathbb{p})^* = \Pi^\mu(-\mathbb{p}),$$

$$H = \int d^{d-1}\mathbb{p} \left[-\frac{(2\pi)^{d-1} e^2}{2} \Pi^0(\mathbb{p}) \Pi^0(-\mathbb{p}) - \frac{\mathbb{p}^2}{2(2\pi)^{d-1} e^2} A_0(-\mathbb{p}) A_0(\mathbb{p}) \right.$$
$$\left. + \sum_i \left(\frac{(2\pi)^{d-1} e^2}{2} \Pi^i(\mathbb{p}) \Pi^i(-\mathbb{p}) + \frac{\mathbb{p}^2}{2(2\pi)^{d-1} e^2} A_i(-\mathbb{p}) A_i(\mathbb{p}) \right) \right]$$

Quantization

$$A_\mu(\mathbb{P})^\dagger = A_\mu(-\mathbb{P}), \quad \Pi^\mu(\mathbb{P})^\dagger = \Pi^\mu(-\mathbb{P})$$

$$[A_\mu(\mathbb{P}_1), \Pi^\nu(\mathbb{P}_2)] = i\delta_\mu^\nu \, \delta^{d-1}(\mathbb{P}_1 - \mathbb{P}_2)$$

$$[A_\mu(\mathbb{P}_1), A_\nu(\mathbb{P}_2)] = [\Pi^\mu(\mathbb{P}_1), \Pi^\nu(\mathbb{P}_2)] = 0$$

If we put

$$a_\mu(\mathbb{P}) := \sqrt{\frac{|\mathbb{P}|}{2(2\pi)^{d-1}e^2}} A_\mu(\mathbb{P}) + i\sqrt{\frac{(2\pi)^{d-1}e^2}{2|\mathbb{P}|}} \Pi^\mu(-\mathbb{P})$$

$$a_\mu(\mathbb{P})^\dagger = \sqrt{\frac{|\mathbb{P}|}{2(2\pi)^{d-1}e^2}} A_\mu(-\mathbb{P}) - i\sqrt{\frac{(2\pi)^{d-1}e^2}{2|\mathbb{P}|}} \Pi^\mu(\mathbb{P})$$

Then, $$[a_\mu(\mathbb{P}_1), a_\nu(\mathbb{P}_2)^\dagger] = \delta_{\mu,\nu}\, \delta^{d-1}(\mathbb{P}_1 - \mathbb{P}_2)$$

$$[a_\mu(\mathbb{P}_1), a_\nu(\mathbb{P}_2)] = [a_\mu(\mathbb{P}_1)^\dagger, a_\nu(\mathbb{P}_2)^\dagger] = 0$$

$$H = \int d^{d-1}\mathbb{P} \Big[-|\mathbb{P}| \big(a_0(\mathbb{P})^\dagger a_0(\mathbb{P}) + \tfrac{1}{2}\delta^{d-1}(0) \big) + \sum_i |\mathbb{P}| \big(a_i(\mathbb{P})^\dagger a_i(\mathbb{P}) + \tfrac{1}{2}\delta^{d-1}(0) \big) \Big]$$

$$[H, a_0(\mathbb{P})] = |\mathbb{P}|\, a_0(\mathbb{P}), \quad [H, a_0(\mathbb{P})^\dagger] = -|\mathbb{P}|\, a_0(\mathbb{P})^\dagger$$

$$[H, a_i(\mathbb{P})] = -|\mathbb{P}|\, a_i(\mathbb{P}), \quad [H, a_i(\mathbb{P})^\dagger] = |\mathbb{P}|\, a_i(\mathbb{P})^\dagger$$

$a_0(\mathbb{P})$, $a_i(\mathbb{P})^\dagger$: creation operators

$a_0(\mathbb{P})^\dagger$, $a_i(\mathbb{P})$: annihilation operators

The state $|0\rangle$ annihilated by $a_0(\mathbb{P})^\dagger$ and $a_i(\mathbb{P})$ is the unique ground state, with energy

$$E_0 = \int d^{d-1}\mathbb{P}\, \frac{d}{2}|\mathbb{P}|\, \delta^{d-1}(0).$$

Other states are obtained from $|0\rangle$ by operating $a_0(\mathbb{P})$ & $a_i(\mathbb{P})^\dagger$ each operation increasing energy by $|\mathbb{P}|$.

e.g. The 1st excitate states:

$$|\mathbb{P};0\rangle = a_0(\mathbb{P})|0\rangle, \quad |\mathbb{P};i\rangle = a_i(\mathbb{P})^\dagger|0\rangle.$$

Note: assuming $\langle 0|0\rangle = 1$,

$$\langle \mathbb{P}_1; i|\mathbb{P}_2; j\rangle = \langle 0|a_i(\mathbb{P}_1)\, a_j(\mathbb{P}_2)^\dagger|0\rangle$$

$$= \delta_{ij}\, \delta^{d-1}(\mathbb{P}_1 - \mathbb{P}_2), \quad \text{this is normal.}$$

$$\langle \mathbb{P}_1; 0|\mathbb{P}_2; 0\rangle = \langle 0|\, a_0(\mathbb{P}_1)^\dagger a_0(\mathbb{P}_2)\,|0\rangle$$

$$= -\delta_{ij}\, \delta^{d-1}(\mathbb{P}_1 - \mathbb{P}_2), \quad \text{negative norm states!}$$

$$A_\mu(\mathbb{x}) = e \int \frac{d^{d-1}\mathbb{P}}{\sqrt{(2\pi)^{d-1} 2|\mathbb{P}|}} e^{i\mathbb{P}\cdot\mathbb{x}} \left(a_\mu(\mathbb{P}) + a_\mu(-\mathbb{P})^+ \right)$$

$$= e \int \frac{d^{d-1}\mathbb{P}}{\sqrt{(2\pi)^{d-1} 2|\mathbb{P}|}} \left(e^{i\mathbb{P}\cdot\mathbb{x}} a_\mu(\mathbb{P}) + e^{-i\mathbb{P}\cdot\mathbb{x}} a_\mu(\mathbb{P})^+ \right)$$

$$A_\mu(t,\mathbb{x}) = e^{itH} A_\mu(\mathbb{x}) e^{-itH}$$

$$= \begin{cases} e \int \frac{d^{d-1}\mathbb{P}}{\sqrt{(2\pi)^{d-1} 2|\mathbb{P}|}} \left(e^{-i|\mathbb{P}|t + i\mathbb{P}\cdot\mathbb{x}} a_i(\mathbb{P}) + e^{i|\mathbb{P}|t - i\mathbb{P}\cdot\mathbb{x}} a_i(\mathbb{P})^+ \right) & \mu = i \\ e \int \frac{d^{d-1}\mathbb{P}}{\sqrt{(2\pi)^{d-1} 2|\mathbb{P}|}} \left(e^{i|\mathbb{P}|t + i\mathbb{P}\cdot\mathbb{x}} a_0(\mathbb{P}) + e^{-i|\mathbb{P}|t - i\mathbb{P}\cdot\mathbb{x}} a_0(\mathbb{P})^+ \right) & \mu = 0 \end{cases}$$

$$\langle 0 | T A_i(x) A_j(y) | 0 \rangle$$

$$= e^2 \delta_{ij} \int \frac{d^{d-1}\mathbb{P}}{(2\pi)^{d-1} 2|\mathbb{P}|} e^{-i|\mathbb{P}||x^0 - y^0| - i\mathbb{P}\cdot(\mathbb{x}-\mathbb{y})}$$

$$\langle 0 | T A_0(x) A_0(y) | 0 \rangle$$

$$= -e^2 \int \frac{d^{d-1}\mathbb{P}}{(2\pi)^{d-1} 2|\mathbb{P}|} e^{-i|\mathbb{P}||x^0 - y^0| - i\mathbb{P}\cdot(\mathbb{x}-\mathbb{y})}$$

... match with

$$\langle A_\mu(x) A_\nu(y) \rangle = -e^2 \eta_{\mu\nu} \int \frac{d^{d-1}\mathbb{P}}{(2\pi)^{d-1} 2|\mathbb{P}|} e^{-i|\mathbb{P}||x^0 - y^0| - i\mathbb{P}\cdot(\mathbb{x}-\mathbb{y})} .$$