Introduction to perturbation theory

Consider the theory of a single real variable ϕ with measure $d\phi$ and action

$$S_E(\phi) = \frac{1}{2} a \phi^2 + \frac{\lambda}{4!} \phi^4$$

partition function

$$Z = \int_{\mathbb{R}} d\phi \; e^{-S_E(\phi)} = \int_{\mathbb{R}} d\phi \; e^{-\frac{1}{2} a \phi^2} \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{\lambda}{4!}\right)^n \phi^{4n}$$

Its <u>perturbative expansion</u> is

$$Z_{pert} := \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{\lambda}{4!}\right)^n \int_{\mathbb{R}} d\phi \; e^{-\frac{1}{2} a \phi^2} \phi^{4n}$$

It is <u>not convergent</u>.

(If it had a convergence radius $R > 0$, $Z_{pert}(\lambda) = Z(\lambda)$ for $|\lambda| < R$, but $Z(\lambda)$ is not obviously analytic at $\lambda = 0$: the integral is badly divergent for real negative λ.

Instead, it is an <u>asymptotic expansion</u> of $Z(\lambda)$:

$$\lambda^{-N} \left(Z(\lambda) - Z_{pert}^{\leq N}(\lambda) \right) \to 0 \quad \text{as} \quad \lambda \searrow 0$$

truncation at order $n \leq N$

Each term of Z_{pert} is a correlation function of the free theory with action $S_{E,free}(\phi) = \frac{1}{2} a \phi^2$:

$$Z_{pert} = Z_{free} \cdot \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{\lambda}{4!}\right)^n \langle \phi^{4n} \rangle_{free}$$

Similarly for correlation functions

$$\langle f(\phi) \rangle = \frac{1}{Z} \int_{\mathbb{R}} d\phi \, e^{-S_E(\phi)} f(\phi) \quad \rightsquigarrow$$

$$\langle f(\phi) \rangle_{pert} := \frac{1}{Z_{pert}} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{-\lambda}{4!}\right)^n \int_{\mathbb{R}} d\phi \, e^{-S_{E,free}(\phi)} f(\phi) \, \phi^{4n}$$

$$= \frac{\cancel{Z}_{free} \cdot \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{\lambda}{4!}\right)^n \langle f(\phi) \, \phi^{4n} \rangle_{free}}{\cancel{Z}_{free} \cdot \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{\lambda}{4!}\right)^n \langle \phi^{4n} \rangle_{free}}$$

<u>Recall</u>: free correlators can be computed as the sum of Wick contractions

$$\langle \phi^4 \rangle_{free} = \phi\phi\phi\phi + \phi\phi\phi\phi + \phi\phi\phi\phi$$

$$= 3 \cdot a^{-1}$$

$$Z_{pert}/Z_{free} = 1 + \left\langle -\frac{\lambda}{4!}\phi^4 \right\rangle_{free} + \frac{1}{2!}\left\langle \left(-\frac{\lambda}{4!}\phi^4\right)^2 \right\rangle_{free} + \cdots$$

$$\left\langle -\frac{\lambda}{4!}\phi^4 \right\rangle_{free} = -\frac{\lambda}{4!}\,\phi\phi\,\phi\phi \times 3 \qquad \text{A}$$

$$= \frac{-\lambda}{4\cdot 2}\left(a^{-1}\right)^2$$

$$\frac{1}{2!}\left\langle \left(-\frac{\lambda}{4!}\phi^4\right)^2 \right\rangle_{free}$$

$$= \frac{1}{2!}\left(-\frac{\lambda}{4!}\right)^2 \Bigg\{ \phi\phi\phi\phi \times 3 \times \phi\phi\phi\phi \times 3 \qquad \text{B}$$

$$+ \phi\phi\phi\phi\,\phi\phi\phi\phi \times \binom{4}{2}^2 \times 2 \qquad \text{C}$$

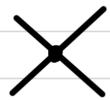
$$+ \phi\phi\phi\phi\,\phi\phi\phi\phi \times 4! \Bigg\} \qquad \text{D}$$

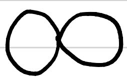
$$= \left\{ \frac{1}{2(4\cdot 2)^2} + \frac{1}{4^2} + \frac{1}{2\cdot 4!} \right\} (-\lambda)^2 \left(a^{-1}\right)^4$$

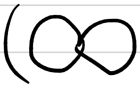
$$\vdots$$

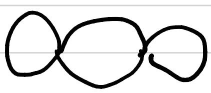
Diagramatic presentation :

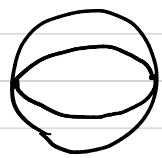
propagator $\overbrace{\phi\phi} = a^{-1}$ $\rightsquigarrow$ line —

interaction $-\frac{\lambda}{4!}\phi^4$ $\rightsquigarrow$ vertex 

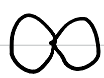
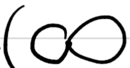
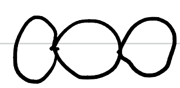
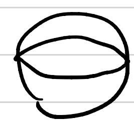
A : 

B : $\frac{1}{2!}\Big($  $\Big)^2$

C : 

D : 

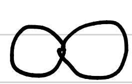
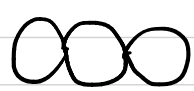
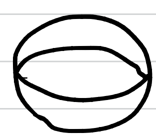
. . . .

$Z_{pert}/Z_{free} = 1 +$  $+ \frac{1}{2!}\Big($  $\Big)^2 +$  $+$  $+ \cdots$

sum of vacuum diagrams

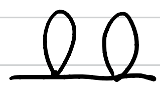
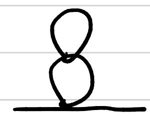
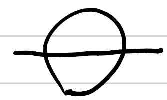
↑

no external line

$\overset{!}{=} \exp\Big($  $+$  $+$  $+ \cdots \Big)$

sum of connected vacuum diagrams

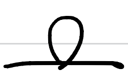
$$\langle\phi\phi\rangle_{pert} = \frac{\langle\phi\phi\rangle_{free} + \left\langle\phi\phi\left(-\frac{\lambda}{4!}\phi^4\right)\right\rangle_{free} + \cdots}{1 + \left\langle -\frac{\lambda}{4!}\phi^4\right\rangle_{free} + \cdots}$$

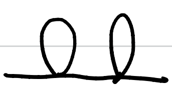
= +  +  +  +  + ⋯

sum of diagrams with two external lines

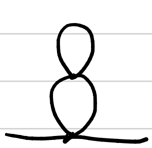
without vacuum diagram ‼ ← division by Z_{pert}

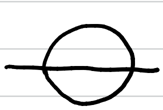
(= sum of connected diagrams $\langle\phi\phi\rangle_{conn}$)

 $= -\frac{\lambda}{4!}\phi(\phi\phi\phi\phi)\phi \times 4\cdot 3 = -\frac{\lambda}{2}(a^{-1})^3$

 $= \frac{1}{2!}\left(-\frac{\lambda}{4!}\right)^2 \phi(\phi\phi\phi\phi)(\phi\phi\phi\phi)\phi \times 2\cdot 4^2\cdot 3^2 = \frac{(-\lambda)^2}{4}(a^{-1})^5$

$(\phi\phi\phi\phi)$

 $= \frac{1}{2!}\left(-\frac{\lambda}{4!}\right)^2 \phi(\phi\phi\phi\phi)\phi \times 2\cdot 4\cdot 3\cdot\binom{4}{2}\cdot 2 = \frac{(-\lambda)^2}{4}(a^{-1})^5$

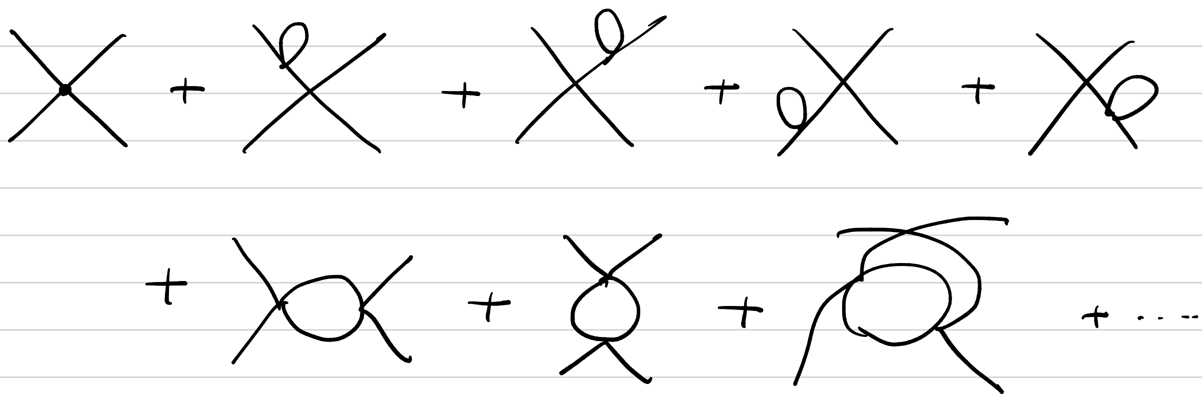
 $= \frac{1}{2!}\left(-\frac{\lambda}{4!}\right)^2 \phi(\phi\phi\phi\phi)(\phi\phi\phi\phi)\phi \cdot 2\cdot 4^2\cdot 3! = \frac{(-\lambda)^2}{6}(a^{-1})^5$

⋮

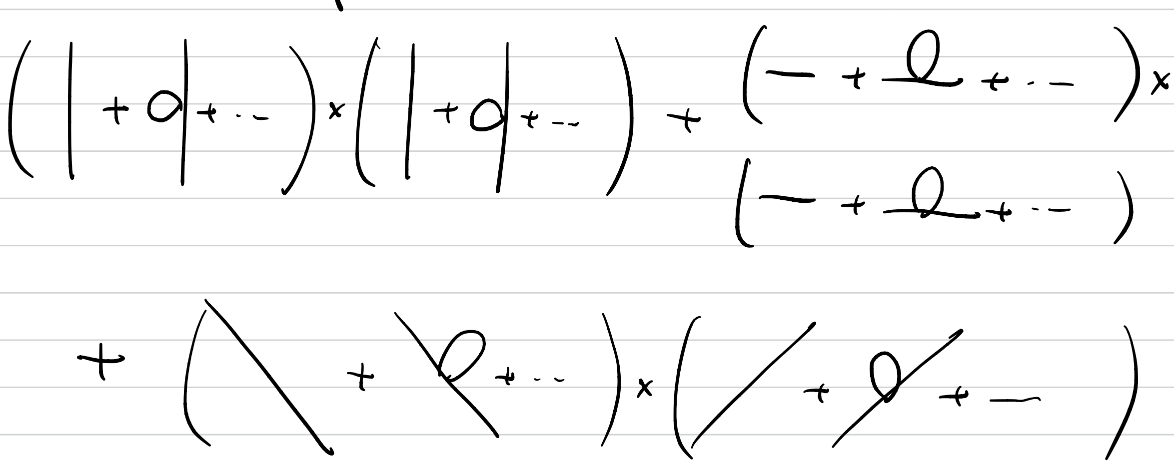
$\langle\phi\phi\phi\phi\rangle_{pert}$ = sum of diagrams with 4 external lines
<u>without vacuum diagrams</u> !!

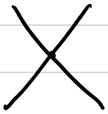
$$\overset{!!}{=} \underbrace{\langle\phi\phi\phi\phi\rangle_{conn}}_{\alpha} + \underbrace{\langle\phi\phi\rangle_{conn}\cdot\langle\phi\phi\rangle_{conn} \times 3}_{\beta}$$

α = connected part

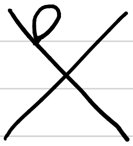


β = disconnected part



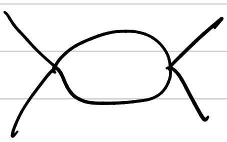


$$= -\frac{\lambda}{4!}\,\phi\phi\,\phi\phi\phi\phi\,\phi\phi \times 4! = -\lambda(a^{-1})^4,$$



$$= \frac{1}{2!}\left(-\frac{\lambda}{4!}\right)^2 \phi\phi\,\phi\phi\phi\phi\,\phi\phi\phi\phi\,\phi\phi \times 2\cdot 4\cdot(3\cdot 4)\cdot 3!$$

$$= \frac{(-\lambda)^2}{2}(a^{-1})^6,$$



$$= \frac{1}{2!}\left(-\frac{\lambda}{4!}\right)^2 \phi\phi\,\phi\phi\phi\phi\,\phi\phi\phi\phi\,\phi\phi \times 2\cdot(4\cdot 3)^2\cdot 2$$

$$= \frac{(-\lambda)^2}{2}(a^{-1})^6,$$

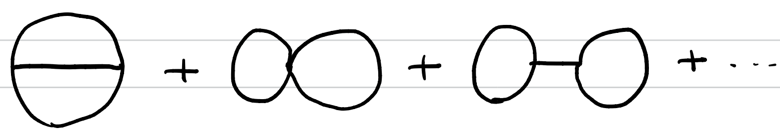
$\vdots$

Generalizations

more interactions

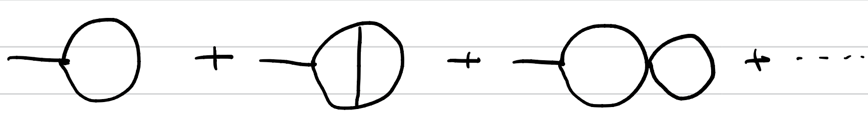
$$S_E(\phi) = \underbrace{\frac{1}{2} a \phi^2}_{\text{free}} + \underbrace{\frac{\lambda_3}{3!} \phi^3 + \frac{\lambda_4}{4!} \phi^4 \quad \left(+ \cdots \right)}_{\text{interactions}}$$

$$Z_{pert} / Z_{free} = \exp\Big($$



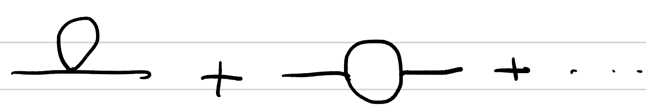
$$\Big)$$

$$\langle \phi \rangle_{pert} =$$

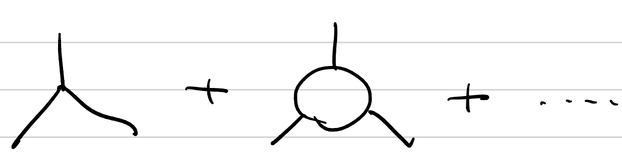


$$\left(= \langle \phi \rangle_{conn} \right)$$

$$\langle \phi \phi \rangle_{pert} = \langle \phi \phi \rangle_{conn} + \langle \phi \rangle_{conn} \cdot \langle \phi \rangle_{conn}$$



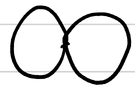
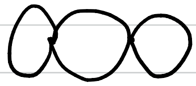
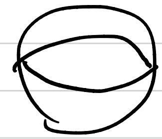
$$\langle \phi \phi \phi \rangle_{pert} = \langle \phi \phi \phi \rangle_{conn} + \langle \phi \rangle_{conn} \langle \phi \phi \rangle_{conn} \times 3 + \langle \phi \rangle_{conn}^3$$

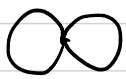


⋮

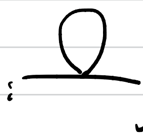
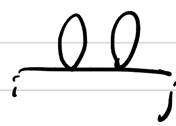
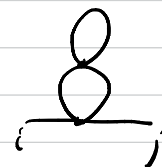
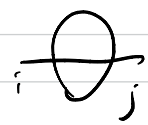
n variables

$$S_E(\varphi) = \underbrace{\frac{1}{2}\sum_{ij}\varphi_i A_{ij}\varphi_j}_{\text{free}} + \underbrace{\frac{\lambda}{4!}\sum_i \varphi_i^4}_{\text{interaction}}$$

$Z_{pert}/Z_{free} = \exp\Big($  $+$  $+$  $+ \cdots \Big)$

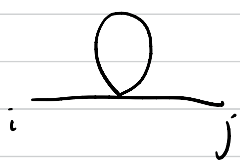
 $= -\frac{\lambda}{4!}\sum_i \varphi_i\varphi_i\varphi_i\varphi_i \times 3 = \frac{-\lambda}{4\cdot 2}\sum_i (A^{-1})_{ii}^2$

$\vdots$

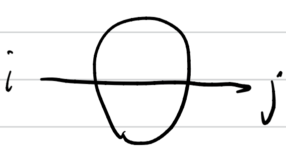
$\langle \varphi_i\varphi_j\rangle_{pert} =$  $+$  $+$  $+$  $+$ 

$= \langle \varphi_i\varphi_j\rangle_{conn}$

 $= A^{-1}_{ij}$

 $= -\frac{\lambda}{4!}\sum_k \varphi_i(\varphi_k\varphi_k\varphi_k\varphi_k)\varphi_j \times 4\cdot 3$

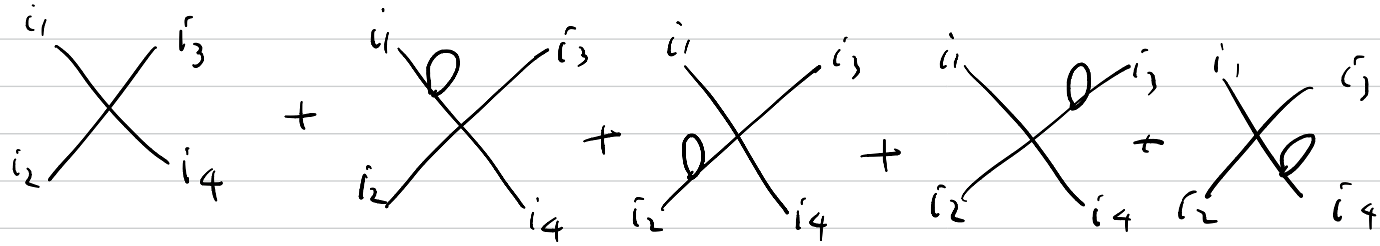
$$= \frac{-\lambda}{2}\sum_k A^{-1}_{ik}A^{-1}_{kk}A^{-1}_{kj},$$

 $= \frac{1}{2!}\left(\frac{-\lambda}{4!}\right)^2 \sum_{k,l} \varphi_i\varphi_k\varphi_k\varphi_k\varphi_k\varphi_l\varphi_l\varphi_l\varphi_l\varphi_j \times 2\cdot 4^2\cdot 3!$

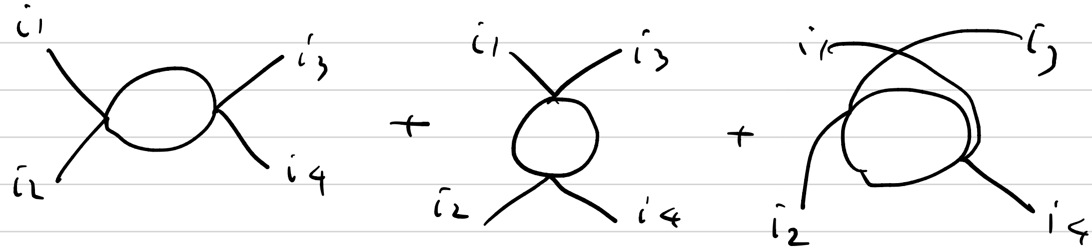
$$= \frac{(-\lambda)^2}{6}\sum_{kl} A^{-1}_{ik}(A^{-1}_{kl})^3 A^{-1}_{lj},$$

$$\langle \phi_{i_1}\phi_{i_2}\phi_{i_3}\phi_{i_4}\rangle_{pert} = \langle \phi_{i_1}\phi_{i_2}\phi_{i_3}\phi_{4}\rangle_{conn} + \langle\phi_{i_1}\phi_{i_2}\rangle_{conn}\langle\phi_{i_3}\phi_{i_4}\rangle_{conn}$$

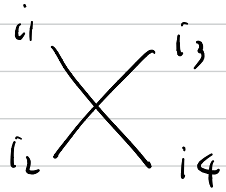
$$+ (i_1 i_3)(i_2 i_4) + (i_1 i_4)(i_2 i_3)$$



$+$

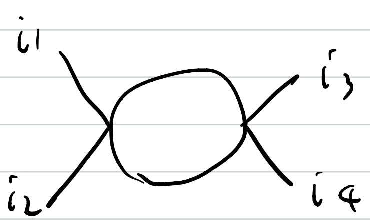


$+ \cdots$



$$= -\frac{\lambda}{4!}\sum_j \overset{\phi_{i_1}}{\phi_j}\,\underset{\phi_{i_2}}{\phi_j}\,\underset{\phi_{i_4}}{\phi_j}\,\overset{\phi_{i_3}}{\phi_j} \times 4!$$

$$= -\lambda \sum_j A^{-1}_{i_1 j} A^{-1}_{i_2 j} A^{-1}_{i_3 j} A^{-1}_{i_4 j}$$



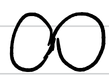
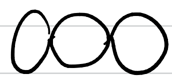
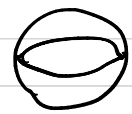
$$= \frac{1}{2!}\left(-\frac{\lambda}{4!}\right)^2 \sum_j \overset{\phi_{i_1}}{\phi_j}\,\underset{\phi_{i_2}}{\phi_j}\,\phi_j\,\phi_j \sum_k \phi_k\,\phi_k\,\underset{\phi_{i_4}}{\phi_k}\,\overset{\phi_{i_3}}{\phi_k} \times 2\cdot(4\cdot 3)^2\cdot 2$$

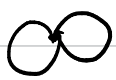
$$= \frac{(-\lambda)^2}{2}\sum_{kl} A^{-1}_{i_1 j} A^{-1}_{i_2 j} \left(A^{-1}_{jk}\right)^2 A^{-1}_{k i_3} A^{-1}_{k i_4}$$

Scalar field in d-dimensions

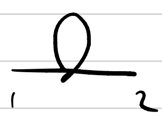
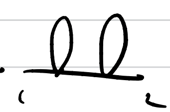
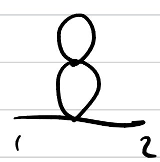
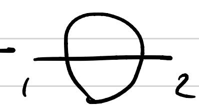
$$S_E[\phi] = \int d^dx \left(\underbrace{\frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2}_{\text{free}} + \underbrace{\frac{\lambda}{4!}\phi^4}_{\text{interaction}} \right)$$

"ϕ^4-theory"

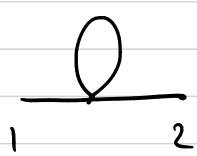
$Z_{pert}/Z_{free} = \exp\Big($  $+$  $+$  $+ \cdots \Big)$

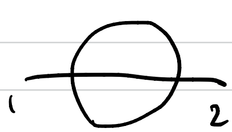
 $= \dfrac{-\lambda}{4\cdot 2}\displaystyle\int d^dx \left(\overline{\phi(x)\,\phi(x)}\right)^2$

⋮

$\langle \phi(x_1)\,\phi(x_2)\rangle_{pert} =$  $+$  $+$  $+$  $+$ 

 $= \overline{\phi(x_1)\,\phi(x_2)} = \displaystyle\int \frac{d^dp}{(2\pi)^d}\,\frac{e^{-ip(x_1-x_2)}}{p^2+m^2}$

 $= \dfrac{-\lambda}{2}\displaystyle\int d^dx\; \overline{\phi(x_1)\,\phi(x)}\;\overline{\phi(x)\phi(x)}\;\overline{\phi(x)\,\phi(x_2)}$

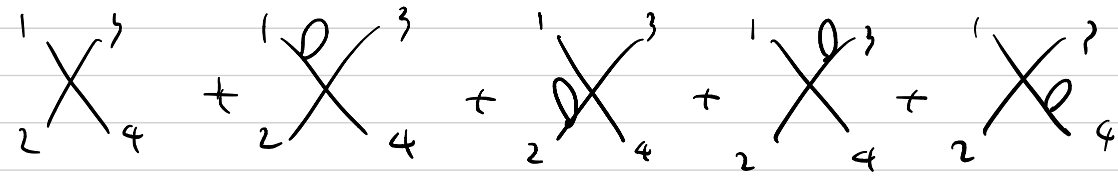
 $= \dfrac{(-\lambda)^2}{6}\displaystyle\int d^dx\, d^dy\; \overline{\phi(x_1)\phi(x)}\left(\overline{\phi(x)\,\phi(y)}\right)^3\overline{\phi(y)\,\phi(x_2)}$

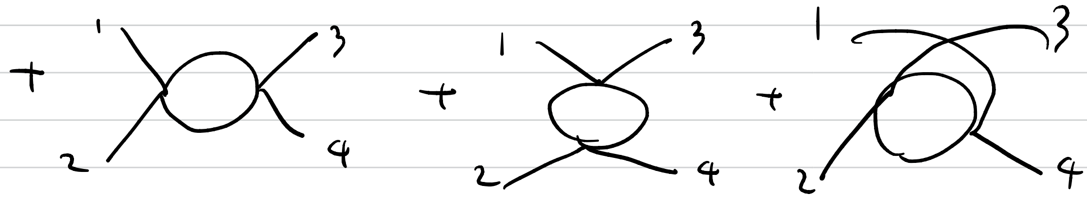
⋮

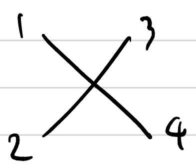
$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle_{pert} = \langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle_{conn}$$

$$+ \langle \phi(x_1)\phi(x_2)\rangle_{conn} \langle \phi(x_3)\phi(x_4)\rangle_{conn}$$

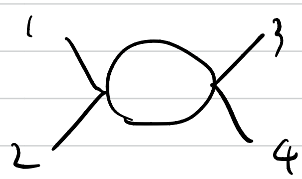
$$+ (13)(24) + (14)(23)$$







$$= -\lambda \int d^dx \prod_{i=1}^{4} \phi(x_i)\phi(x)$$



$$= \frac{(-\lambda)^3}{2} \int d^dx\, d^dy\ \phi(x_1)\phi(x)\ \phi(x_2)\phi(x)\ \phi(x)\phi(y)$$

$$\times\ \phi(y)\phi(x_3)\ \phi(y)\phi(x_4)$$

$\vdots$

Minkowski limit

$$S[\phi] = \int d^dx \, \Big(\underbrace{\tfrac{1}{2}\partial^\mu \phi \, \partial_\mu \phi - \tfrac{m^2}{2}\phi^2}_{\text{free}} \underbrace{- \tfrac{\lambda}{4!}\phi^4}_{\text{interaction}} \Big)$$

Z_{pert}, $\langle \phi(x_1) \cdots \phi(x_s) \rangle_{pert}$ is obtained from the result of Euclidean theory by the replacement

$$\phi(x)\phi(y) \longrightarrow \int \frac{d^dp}{(2\pi)^d} \frac{e^{-ip(x-y)}}{p^2 - m^2 + i \cdot 0}$$

$$-\lambda \int d^dx_E \longrightarrow -i\lambda \int d^dx$$

Gauge theory (e.g. QCD)

variables · A_μ gauge potential of gauge group G

· ψ Dirac fermion in a representation R of G

$$\mathcal{L} = -\frac{1}{4e^2} F^{\mu\nu} \cdot F_{\mu\nu} + i\bar{\psi} \not{D}_A \psi - m\bar{\psi}\psi$$

gauge fixing by $\partial^\mu A_\mu = 0$

$$\rightsquigarrow \tilde{\mathcal{L}} = \mathcal{L} - \frac{1}{2e^2\xi}(\partial^\mu A_\mu)^2 + i\partial^\mu \bar{c} D_\mu c \; ; \; c, \bar{c} : \text{F.P. ghosts}$$

$$\tilde{\mathcal{L}} = \mathcal{L}_{\text{free}} + \mathcal{L}_{\text{int}} \qquad (\text{rescale } A_\mu \to eA_\mu)$$

$$\mathcal{L}_{\text{free}} = -\frac{1}{4}(\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu) - \frac{1}{2\xi}(\partial^\mu A_\mu)^2$$
$$+ i\bar{\psi}\not{\partial}\psi - m\bar{\psi}\psi + i\partial^\mu \bar{c}\,\partial_\mu c$$

$$\mathcal{L}_{\text{int}} = -\frac{e}{2}(\partial^\mu A^\nu - \partial^\nu A^\mu)\cdot[A_\mu, A_\nu] - \frac{e^2}{4}[A^\mu, A^\nu]\cdot[A_\mu, A_\nu]$$
$$+ ie\bar{\psi}\not{A}\psi + ie\,\partial^\mu \bar{c}[A_\mu, c]$$

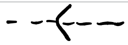
Perturbative expansion of partition/correlation functions can be computed using

$$A_\mu = e^a A_{\mu a}\,, \quad C = e^a C_a\,, \quad \bar{C} = e^a \bar{C}_a$$

propagators

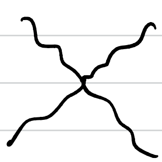
 $$= \overbrace{A_{\mu a}(x)\, A_{\nu b}(y)} = \delta_{ab} \int \frac{d^d p}{(2\pi)^d} \frac{-i\, e^{-ip(x-y)}}{p^2 + i\cdot 0} \left(\eta_{\mu\nu} - (1-\xi) \frac{p_\mu p_\nu}{p^2 + i\cdot 0} \right),$$

 $$= \overbrace{\psi(x)\, \bar{\psi}(y)} = \int \frac{d^d p}{(2\pi)^d} \frac{i\, e^{-ip(x-y)}}{p^2 - m^2 + i\cdot 0} (\not{p} + m),$$

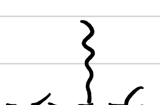
 $$= \overbrace{C_a(x)\, \bar{C}_b(y)} = \delta_{ab} \int \frac{d^d p}{(2\pi)^2} \frac{e^{-ip(x-y)}}{p^2 + i\cdot 0}\,, \quad \text{and}$$

vertices

 $$= -\frac{ie}{2} \int d^d x\, (\partial^\mu A^\nu - \partial^\nu A^\mu) \cdot [A_\mu, A_\nu]\,,$$

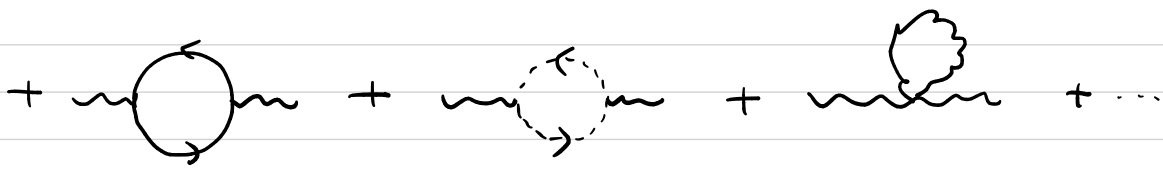
 $$= -\frac{ie^2}{4} \int d^d x\, [A^\mu, A^\nu] \cdot [A_\mu, A_\nu]\,,$$

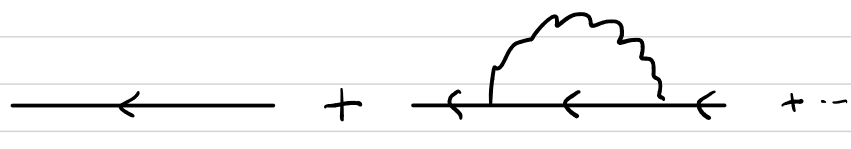
 $$= ie \int d^d x\, i \bar{\psi} \not{A} \psi\,,$$

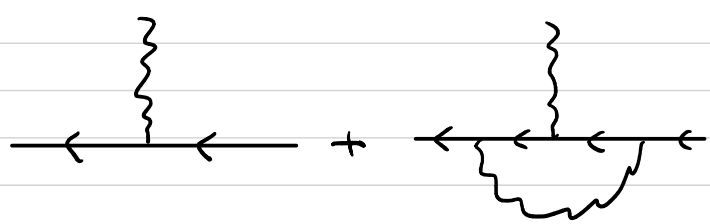
 $$= ie \int d^d x\, i\, \partial^\mu \bar{C} \cdot [A_\mu, C]\,.$$

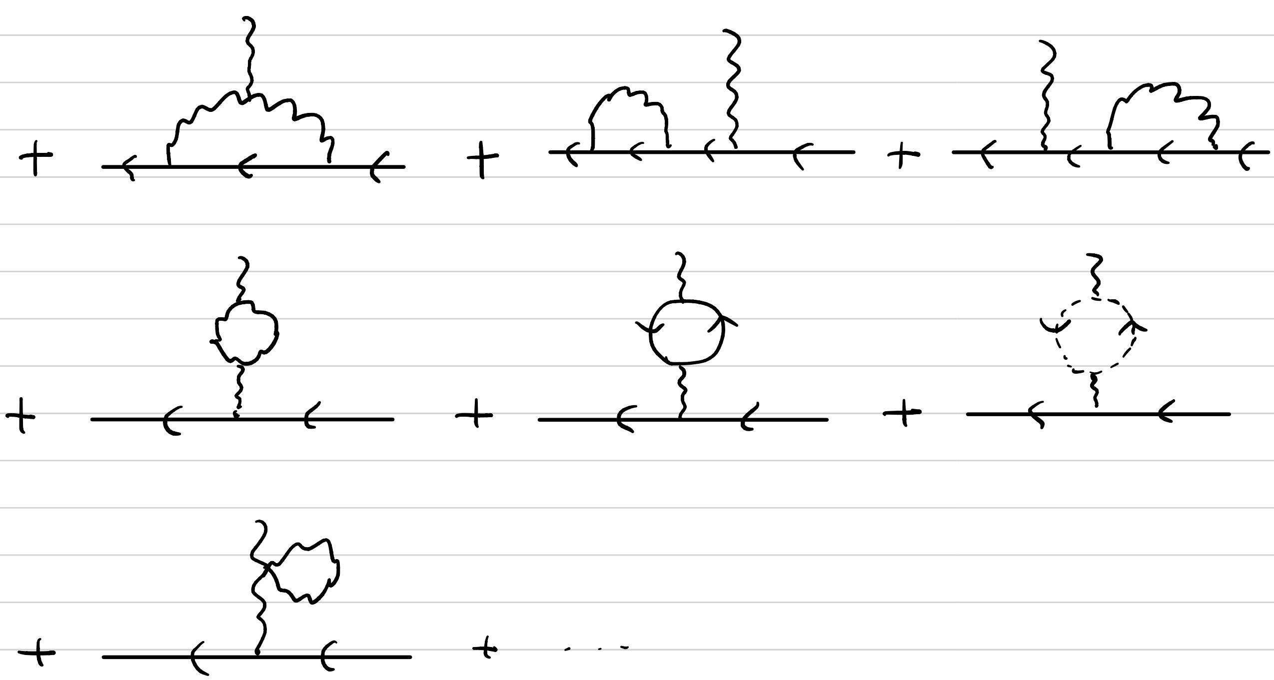
eg,

$\langle A_\mu(x)\, A_\nu(y) \rangle_{pert} =$ 



$\langle \psi(x)\, \bar{\psi}(y) \rangle_{pert} =$ 

$\langle \psi(x)\, \bar{\psi}(y)\, A_\mu(z) \rangle_{pert} =$ 



$\vdots$

In general,

- Z_{pert}/Z_{free} = sum of vacuum diagrams

$\overset{!}{=}$ exp (sum of connected vacuum diagrams)

- For $\Phi_1, \cdots, \Phi_s$ = elementary fields inserted at points

e.g. $\phi(x_1) \cdots \phi(x_s)$ in ϕ^4 theory

e.g. $A_{\mu_1}(x_1) \cdots, \psi(y_1), \cdots, \bar{\psi}(z_1), \cdots$ in QCD

$$\langle \Phi_1 \cdots \Phi_s \rangle_{pert} \overset{!!}{=} \sum_{\{1,\cdots,s\} = I_1 \cup \cdots \cup I_\ell} \underbrace{\pm}_{} \left\langle \prod_{i \in I_1} \Phi_{i_1} \right\rangle_{conn} \cdots \left\langle \prod_{i_\ell \in I_\ell} \Phi_{i_s} \right\rangle_{conn}$$

permutation of fermionic Φ_i's

Sum over decompositions of $\{1, \cdots, s\}$ to non-empty subsets $I_1, \cdots, I_\ell \subset \{1, \cdots, s\}$

* In particular, no term has a vacuum diagram factor.

Decomposition to connected parts

Here we explain

(i) $Z_{\text{pert}}/Z_{\text{free}} \overset{!}{=} \exp\left(\text{sum of connected vacuum diagrams}\right)$

(ii) $\langle \Phi_1 \cdots \Phi_s \rangle_{\text{pert}}$

$$\overset{!!}{=} \sum_{\substack{\{1,\dots,s\} \\ = I_1 \cup \dots \cup I_\ell}} \underbrace{\pm}_{} \left\langle \prod_{i \in I_1} \Phi_{i_1} \right\rangle_{\text{conn}} \cdots \left\langle \prod_{i_\ell \in I_\ell} \Phi_{i_s} \right\rangle_{\text{conn}}$$

permutation of fermionic Φ_i's

Sum over decompositions of $\{1,\dots,s\}$ to non-empty subsets $I_1,\dots,I_\ell \subset \{1,\dots,s\}$

* In particular, no term has a vacuum diagram factor.

(ii) follows from (i) Define

$$Z_{\text{pert}}(J) := \left[\int \mathcal{D}\Phi \; e^{-S_{\text{free}}(\Phi) - S_{\text{int}}(\Phi) + J\cdot\Phi}\right]_{\text{pert}}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \int \mathcal{D}\Phi \; e^{-S_{\text{free}}(\Phi)} \left(-S_{\text{int}}(\Phi) + J\cdot\Phi\right)^n.$$

a part of interaction

Then,

$$\langle \Phi_1 \cdots \Phi_s \rangle_{pert}$$

$$= \frac{1}{Z_{pert}(J)} \frac{\partial}{\partial J_1} \cdots \frac{\partial}{\partial J_s} Z_{pert}(J) \Big|_{J=0} .$$

On the other hand, (i) implies

$$Z_{pert}(J) = Z_{free} \cdot \exp\left(Z_{conn}(J) \right), \text{ where}$$

$$Z_{conn}(J) = \text{Sum of Connected vacuum diagrams}$$

($J \cdot \Phi$ is a part of interaction and corresponds to a vertex of the form J——.)

Thus,

$$\langle \Phi_1 \cdots \Phi_s \rangle_{pert} = e^{-Z_{conn}(J)} \frac{\partial}{\partial J_1} \cdots \frac{\partial}{\partial J_s} e^{Z_{conn}(J)} \Big|_{J=0}$$

$$= \sum_{\{1,\cdots,s\} = I_1 \sqcup \cdots \sqcup I_\ell} \pm \prod_{i_1 \in I_1} \frac{\partial}{\partial J_{i_1}} Z_{conn}(J) \cdots \prod_{i_\ell \in I_\ell} \frac{\partial}{\partial J_{i_\ell}} Z_{conn}(J) \Big|_{J=0}$$

$$= \sum_{\{1,\cdots,s\} = I_1 \sqcup \cdots \sqcup I_\ell} \pm \Big\langle \prod_{i_1 \in I_1} \Phi_{i_1} \Big\rangle_{conn} \cdots \Big\langle \prod_{i_\ell \in I_\ell} \Phi_{i_\ell} \Big\rangle_{conn}$$

//

Thus, it remains to show (i)

Notation in this discussion: for a diagram D, we write $[D]$ for the contribution of D to Z_{pert}/Z_{free}.

Thus $$Z_{pert}/Z_{free} = \sum_D [D]$$

Case 1 $-S_{int} = V$, a single type of vertex

e.g. ϕ^4 theory without source term.

Then $$Z_{pert}/Z_{free} = \sum_{n=0}^{\infty} \frac{1}{n!} \langle V^n \rangle_{free}$$

Suppose a connected diagram C has v_C vertices.

Then $[C]$ is a term in $\frac{1}{v_C!} \langle V^{v_C} \rangle_{free}$.

Also, $[\underbrace{C \cdots C}_{m}]$ is a term in $\frac{1}{(mv_C)!} \langle V^{mv_C} \rangle_{free}$,

and is included in its part

$$\frac{1}{(mv_C)!} \underbrace{\langle V^{v_C} \rangle_{free} \cdots \langle V^{v_C} \rangle_{free}}_{m} \times \begin{pmatrix} \text{number of ways to} \\ \text{decompose } mv_C \text{ elements} \\ \text{to } m \text{ groups of } v_C \text{ elements} \end{pmatrix}$$

$$\begin{pmatrix}\text{number of ways to} \\ \text{decompose } m\nu_c \text{ elements} \\ \text{to } m \text{ groups of } \nu_c \text{ elements}\end{pmatrix}$$

$$= \underbrace{\binom{m\nu_c}{\nu_c}\binom{m\nu_c-\nu_c}{\nu_c}\cdots\binom{2\nu_c}{\nu_c}\binom{\nu_c}{\nu_c}}_{\text{number of ways to put } m\nu_c \text{ elements to } m \text{ labeled boxes}} \times \underbrace{\frac{1}{m!}}_{\text{forget the labels of the boxes}}$$

$$= \frac{(m\nu_c)!}{(\nu_c!)^m\, m!}$$

$\therefore [\underbrace{C \cdots C}_{m}]$ is a term in

$$\frac{1}{\cancel{(m\nu_c)!}} \langle V^{\nu_c}\rangle_{\text{free}} \cdots \langle V^{\nu_c}\rangle_{\text{free}} \times \frac{\cancel{(m\nu_c)!}}{(\nu_c!)^m\, m!}$$

$$= \frac{1}{m!}\Big(\underbrace{\frac{1}{\nu_c!}\langle V^{\nu_c}\rangle_{\text{free}}}_{[C] + \text{others}}\Big)^m$$

$$\therefore [\underbrace{C \cdots C}_{m}] = \frac{1}{m!}[C]^m$$

If $C_1, \cdots, C_k$ are connected diagrams of $v_{C_1}, \cdots, v_{C_k}$ vertices,

$$[\underbrace{C_1 \cdots C_1}_{m_1} \underbrace{C_2 \cdots C_2}_{m_2} \cdots \underbrace{C_k \cdots C_k}_{m_k}]$$

is a term in $\displaystyle \frac{1}{(m_1 v_{C_1} + \cdots + m v_{C_k})!} \left\langle V^{m_1 v_{C_1} + \cdots + m v_{C_k}} \right\rangle_{free}$

and is included in

$$\frac{1}{(m_1 v_{C_1} + \cdots + m v_{C_k})!} \left\langle V^{v_{C_1}} \right\rangle_{free}^{m_1} \cdots \left\langle V^{v_{C_k}} \right\rangle_{free}^{m_k}$$

$$\times \left(\underbrace{\begin{array}{l} \text{number of ways to decompose } m_1 v_{C_1} + \cdots + m v_{C_k} \\ \text{elements to } m_1 \text{ groups of } v_{C_1} \text{ elements}, \cdots, \\ \cdots, m_k \text{ groups of } v_{C_k} \text{ elements} \end{array}}_{\frac{(m_1 v_{C_1} + \cdots + m v_{C_k})!}{(v_{C_1}!)^{m_1} \cdots (v_{C_k}!)^{m_k}} \frac{1}{m_1! \cdots m_k!}} \right)$$

$$= \frac{1}{m_1!} \left(\frac{1}{v_{C_1}!} \left\langle V^{v_{C_1}} \right\rangle_{free} \right)^{m_1} \cdots \frac{1}{m_k!} \left(\frac{1}{v_{C_k}!} \left\langle V^{v_{C_k}} \right\rangle_{free} \right)^{m_k}$$

$$\therefore [\underbrace{C_1 \cdots C_1}_{m_1} \cdots \underbrace{C_k \cdots C_k}_{m_k}] = \frac{1}{m_1!} [C_1]^{m_1} \cdots \frac{1}{m_k!} [C_k]^{m_k}$$

Thus

$$Z_{pert}/Z_{free} = \sum_{D} [D]$$

$$= \sum_{\substack{C_1, \dots, C_k \\ \text{connected}}} [C_1 \cdots C_1 \cdots C_k \cdots C_k]$$

$$[\underbrace{C_1 \cdots C_1}_{m_1} \cdots \underbrace{C_k \cdots C_k}_{m_k}] = \frac{1}{m_1!}[C_1]^{m_1} \cdots \frac{1}{m_k!}[C_k]^{m_k}$$

$$= \prod_{C \text{ connected}} \underbrace{\sum_{m_c=0}^{\infty} \frac{1}{m_c!}[C]^{m_c}}_{\exp([C])}$$

$$= \exp\left(\sum_{C \text{ connected}} [C]\right) \qquad //$$

<u>Case 2</u> $-S_{int} = V_1 + \cdots + V_N$: multiple types of vertices

e.g. ϕ^4 theory with a source $J \cdot \phi$

e.g. QCD with or without a source.

Then, $$Z_{pert}/Z_{free} = \sum_{n_1,\cdots,n_N} \frac{1}{n_1!\cdots n_N!} \langle V_1^{n_1}\cdots V_N^{n_N}\rangle_{free}$$

Suppose a connected diagram C has V_C^1 vertices of type V_1, V_C^2 vertices of type V_2, ---, V_C^N vertices of type V_N.

Then $[C]$ is a term in $$\frac{1}{V_C^1!\cdots V_C^N!} \langle V_1^{V_C^1}\cdots V_N^{V_C^N}\rangle_{free}.$$

Also $[\underbrace{C\cdots C}_{m}]$ is a term in $$\frac{1}{(mV_C^1)!\cdots(mV_C^N)!} \langle V_1^{mV_C^1}\cdots V_N^{mV_C^N}\rangle_{free}$$

and is included in its part

$$\frac{1}{(mV_C^1)!\cdots(mV_C^N)!} \left(\langle V_1^{V_C^1}\cdots V_N^{V_C^N}\rangle_{free}\right)^m$$

$$\times \underbrace{\begin{pmatrix}\text{number of ways to distribute } mV_C^1 \text{ elements of type } 1, \\ mV_C^2 \text{ elements to type } 2, \cdots, mV_C^N \text{ elements of type } N \\ \text{to } m \text{ unlabeled boxes, where each box admit} \\ V_C^1 \text{ elements of type } 1, \cdots, V_C^N \text{ elements of type } N\end{pmatrix}}_{\frac{(mV_C^1)!}{(V_C^1!)^m}\cdots\frac{(mV_C^N)!}{(V_C^N!)^m}\cdot\frac{1}{m!}}$$

$$= \frac{1}{m!}\left(\frac{1}{V_c^1!\cdots V_c^N!}\left\langle V_1^{V_c^1}\cdots V_N^{V_c^N}\right\rangle_{\text{free}}\right)^m$$

$$\therefore\ [\underbrace{C\cdots C}_{m}] = \frac{1}{m!}[C]^m$$

For $i=1,\cdots,k$, let C_i be a connected diagram with $V_{C_i}^j$ vertices of type V_j $(j=1,\cdots,N)$. Then

$[\underbrace{C_1\cdots C_1}_{m_1}\cdots\underbrace{C_k\cdots C_k}_{m_k}]$ is a term in

$$\prod_{j=1}^{N}\frac{1}{\left(\sum_{i=1}^{k} m_i V_{C_i}^j\right)!}\left\langle V_1^{\sum_{i_1=1}^{k} m_{i_1}V_{C_{i_1}}^1}\cdots V_N^{\sum_{i_N=1}^{k} m_{i_N}V_{C_{i_N}}^N}\right\rangle_{\text{free}}$$

and is included in its part

$$\prod_{j=1}^{N}\frac{1}{\left(\sum_{i=1}^{k} m_i V_{C_i}^j\right)!}\prod_{i=1}^{k}\left\langle V_1^{V_{C_i}^1}\cdots V_N^{V_{C_i}^N}\right\rangle_{\text{free}}^{m_i}$$

$\times$ (number of ways to distribute $\sum_{i=1}^{k} m_i V_{C_i}^j$ elements of type j $(j=1,\cdots,N)$ to m_i unlabeled boxes, where each box admit $V_{C_i}^1$ elements of type $1,\cdots,$ $V_{C_i}^N$ elements of type N $(i=1,\cdots,k)$.)

$$= \prod_{j=1}^{N} \frac{1}{\left(\sum_{i=1}^{k} m_i V_{c_i}^j\right)!} \prod_{i=1}^{k} \left\langle V_1^{V_{c_i}^1} \cdots V_N^{V_{c_i}^N} \right\rangle_{\text{free}}^{m_i}$$

$$\times \prod_{j=1}^{N} \left(\frac{\left(\sum_{i=1}^{k} m_i V_{c_i}^j\right)!}{\prod_{i=1}^{k} (m_i V_{c_i}^j)!} \cdot \prod_{i=1}^{k} \frac{(m_i V_{c_i}^j)!}{(V_{c_i}^j !)^{m_i}} \right) \frac{1}{m_1! \cdots m_k!}$$

$$= \prod_{i=1}^{k} \frac{1}{m_i!} \left(\frac{1}{V_{c_i}^1! \cdots V_{c_i}^N!} \left\langle V_1^{V_{c_i}^1} \cdots V_N^{V_{c_i}^N} \right\rangle_{\text{free}} \right)^{m_i}$$

Thus,

$$[\underbrace{C_1 \cdots C_1}_{m_1} \cdots \underbrace{C_k \cdots C_k}_{m_k}] = \frac{1}{m_1!}[C_1]^{m_1} \cdots \frac{1}{m_k!}[C_k]^{m_k} \longrightarrow *$$

$$Z_{\text{pert}}/Z_{\text{free}} = \sum_{D} [D] = \sum_{C_1, \cdots, C_k \text{ conn}} [C_1 \cdots C_1 \cdots C_k \cdots C_k]$$

$$\overset{*}{=} \prod_{C \text{ conn}} \sum_{m_C=0}^{\infty} \frac{1}{m_C!} [C]^{m_C}$$

$$= \prod_{C \text{ conn}} \exp[C] = \exp\left(\sum_{C \text{ conn}} [C]\right)$$

//.