Particle spectrum and interactions from correlation functions

Under a basic assumption, from the correlation functions

$$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_s(x_s) \rangle = \langle 0 | T\, \mathcal{O}_1(x_1) \cdots \mathcal{O}_s(x_s) | 0 \rangle$$

we can read off the
- particle spectrum

and
- interactions among particles.

Spectrum from two point functions

Consider a QFT formulated on Minkowski space $\mathbb{R}^d$

$\mathcal{H}$ = space of states.

$\exists$ action of Poincaré group:

$$\begin{cases} \text{translation } \Delta x^\mu P_\mu \\ \text{Lorentz } U(\Lambda) : \Lambda \in SO(d-1,1) \end{cases}$$

$P_\mu = (H, -\mathbb{P})$ commutes with each other

$\rightsquigarrow$ $\exists$ basis of $\mathcal{H}$ in which P_μ are diagonalized.

<u>Assumption</u> There are N particles of masses $m_1, \cdots, m_N$ and $\mathcal{H}$ is spanned by

- A state $|0\rangle$ with $P_\mu = 0$

  Corresponding to the vacuum without any particle.

- A state $|\mathbb{P}_1, i_1; \cdots; \mathbb{P}_\ell, i_\ell\rangle$ $(\ell \geq 1)$ with

  i_1-th particle of momentum $\mathbb{P}_1$,

  i_2-th " $\mathbb{P}_2$,

  $\vdots$

  i_ℓ-th particle of momentum $\mathbb{P}_\ell$.

  It has energy $E = \sum_{a=1}^{\ell} \omega_{\mathbb{P}_a}^{i_a}$; $\omega_{\mathbb{P}}^{i} = \sqrt{\mathbb{P}^2 + m_i^2}$

  and momentum $\mathbb{P} = \sum_{a=1}^{\ell} \mathbb{P}_a$.

<u>Remark</u>

- We have seen that this is the case in free field theories, (up to the ground state energy (see below))

  But the assumption is very non-trivial for a general QFT which is not necessarily free.

- $|0\rangle$ is the unque ground state and all states have non-negative energies $E \geq 0$

(We have seen examples where the ground state has a non-zero energy E_0, but we just "set it off" by taking $H - E_0$ as the new Hamitonian)

Also, the Poincaré relation $U(\Lambda)^{-1} P^\mu U(\Lambda) = \Lambda^\mu{}_\nu P^\nu$ and the uniqueness of the ground state implies that $|0\rangle$ is Lorentz invariant,

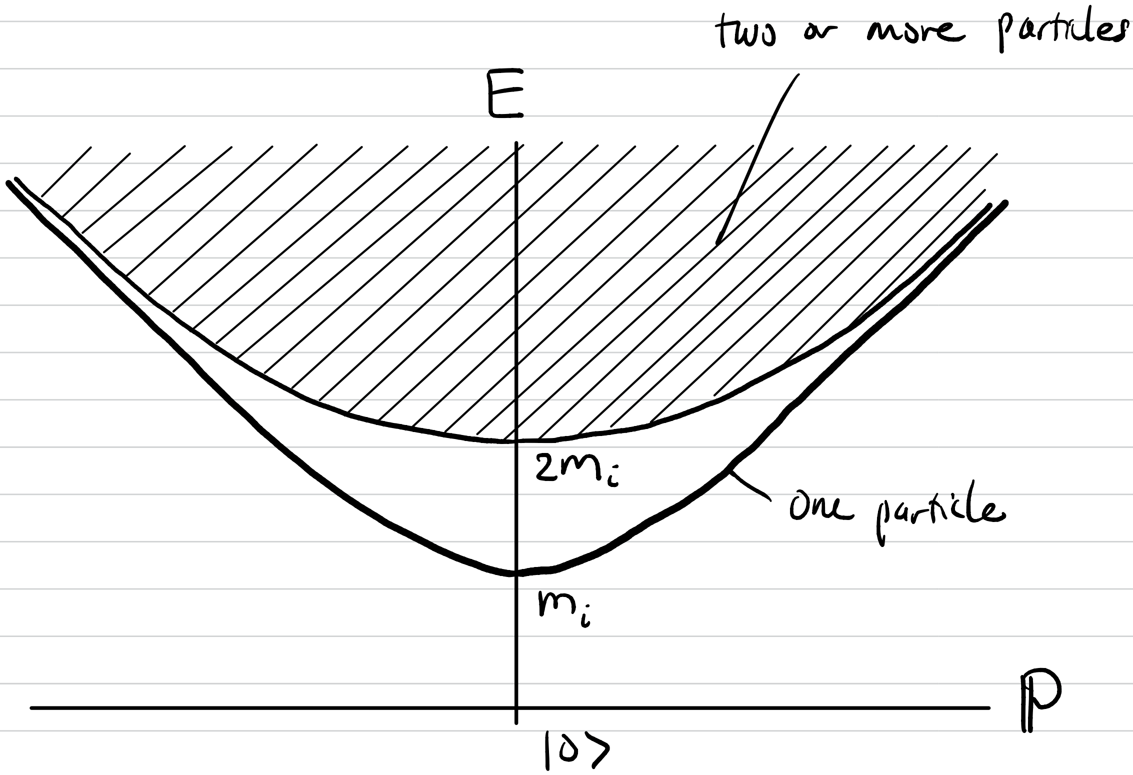
$$U(\Lambda)|0\rangle = |0\rangle.$$

- The particles may have spin and other quantum numbers (such as charges under internal symmetries). Then $(\mathbb{p}, i)$ may be replaced by $(\mathbb{p}, i;\ \underset{\uparrow \atop \text{spin}}{s},\ \underbrace{q_1, \cdots}_{\text{other}})$.

For simplicity, for now, we assume that all the particles are scalar (no spin) and have no additional quantum numbers.

- For a one particle state $|\mathbb{p}, i\rangle$, energy is determined by the total momentum $\mathbb{P} = \mathbb{p}$; $E = \sqrt{\mathbb{P}^2 + m_i^2}$.

But for states with two or more particles, for each total momentum $\mathbb{P}$, the possible values of $E = \sum_a \sqrt{\mathbb{p}_a^2 + m_{ia}^2}$ is not bounded above. e.g. for two particles of the same type with $\mathbb{p}_1 + \mathbb{p}_2 = \mathbb{P}$, E can take any value s.t. $2E_{\mathbb{P}/2} \leq E < \infty$.



Organization/normalization of basis:

$$\{|0\rangle\} \cup \{|\mathbb{P},\lambda\rangle\}_{\mathbb{P}}$$

$\mathbb{P}$ = total momentum

λ = all other labels

- For one particle states, $\lambda = i$ the particle species (discrete label).

- For two or more particle states, λ = particle species (discrete) + relative momenta (continuous).

$|\mathbb{P},\lambda\rangle$ has

$$\left.\begin{aligned} \mathbb{P} &= \mathbb{P} \\ E &= \sqrt{\mathbb{P}^2 + m_\lambda^2} =: \omega_{\mathbb{P}}^\lambda \end{aligned}\right\} \quad p^\mu_{\mathbb{P},\lambda} := (\omega^\lambda_{\mathbb{P}}, \mathbb{P})$$

and is related to $|0,\lambda\rangle$ with $\mathbb{P} = 0$ & $E = m_\lambda$ by

$$|\mathbb{P},\lambda\rangle = U(\Lambda_{m_\lambda,\mathbb{P}})|0,\lambda\rangle$$

where $\Lambda_{m,\mathbb{P}}$ is the Lorentz boost in the direction of $\mathbb{P}$ sending $\begin{pmatrix} m \\ 0 \end{pmatrix}$ to $\begin{pmatrix} \sqrt{\mathbb{P}^2+m^2} \\ \mathbb{P} \end{pmatrix}$.

They are normalized so that

$$\langle \mathbb{P},\lambda | \mathbb{P}',\lambda'\rangle = (2\pi)^{d-1}\, 2\,\omega^\lambda_{\mathbb{P}}\, \delta^{d-1}(\mathbb{P}-\mathbb{P}')\, \delta_{\lambda,\lambda'}.$$

e.g. For a particle of mass m, we may use creation/annihilation operators of real scalar $[a(\mathbb{p}), a(\mathbb{p}')^\dagger] = \delta^{d-1}(\mathbb{p}-\mathbb{p}')$, ... to describe the states:

One particle state: $\lambda = \cdot$,

$$|\mathbb{0}, \cdot\rangle = a(\mathbb{0})^\dagger |0\rangle \times \sqrt{(2\pi)^{d-1} 2m}$$

$$|\mathbb{p}, \cdot\rangle = U(\Lambda_{m,\mathbb{p}}) |\mathbb{0}, \cdot\rangle = a(\mathbb{p})^\dagger |0\rangle \times \sqrt{(2\pi)^{d-1} 2\omega_{\mathbb{p}}}$$

two particle state: $\lambda = \mathbb{q}$,

$$|\mathbb{0}; \mathbb{q}\rangle = a(\mathbb{q})^\dagger a(-\mathbb{q})^\dagger |0\rangle \times \underbrace{C_{\mathbb{q},m}}_{\text{some constant}}$$

with $\mathbb{P} = 0$ & $E = 2\omega_{\mathbb{q}}$

$$|\mathbb{p}; \mathbb{q}\rangle = U(\Lambda_{2\omega_{\mathbb{q}}, \mathbb{p}}) |\mathbb{0}; \mathbb{q}\rangle$$

$\vdots$

l-particle state: $\lambda = (\mathbb{q}_1, \cdots, \mathbb{q}_l)$; $\mathbb{q}_1 + \cdots + \mathbb{q}_l = 0$,

$$|\mathbb{0}; \mathbb{q}_1, \cdots, \mathbb{q}_l\rangle = a(\mathbb{q}_1)^\dagger \cdots a(\mathbb{q}_l)^\dagger |0\rangle \times \underbrace{C_{\mathbb{q}_1, \cdots, \mathbb{q}_l, m}}_{\text{some constant}}$$

with $\mathbb{P} = 0$, $E = \omega_{\mathbb{q}_1} + \cdots + \omega_{\mathbb{q}_l} =: m_{\mathbb{q}_1, \cdots, \mathbb{q}_l}$

$$|\mathbb{p}; \mathbb{q}_1, \cdots, \mathbb{q}_l\rangle = U(\Lambda_{m_{\mathbb{q}_1, \cdots, \mathbb{q}_l}, \mathbb{p}}) |\mathbb{0}; \mathbb{q}_1, \cdots, \mathbb{q}_l\rangle$$

Exercise: For $\Lambda \in SO(d-1,1)$ and $\omega_{\mathbb{P}} = \sqrt{\mathbb{P}^2 + m^2}$

define $\Lambda(\mathbb{P})$ by $\Lambda \begin{pmatrix} \omega_{\mathbb{P}} \\ \mathbb{P} \end{pmatrix} = \begin{pmatrix} \omega_{\Lambda(\mathbb{P})} \\ \Lambda(\mathbb{P}) \end{pmatrix}$. Then

① $U(\Lambda)\, a(\mathbb{P})\, U(\Lambda)^{-1} = \sqrt{\dfrac{\omega_{\Lambda(\mathbb{P})}}{\omega_{\mathbb{P}}}}\, a(\Lambda(\mathbb{P}))$

② $d^{d-1}\mathbb{P}/\omega_{\mathbb{P}}$ is invariant under $\mathbb{P} \to \Lambda(\mathbb{P})$.

③ $\langle \mathbb{P}; \vec{q} \mid \mathbb{P}'; \vec{q}' \rangle \propto \sqrt{\mathbb{P}^2 + m_{\vec{q}}^2}\; \delta^{d-1}(\mathbb{P} - \mathbb{P}')$

The identity operator on $\mathcal{H}$ may be expanded as

$$\mathrm{id}_{\mathcal{H}} = |0\rangle\langle 0| + \sum_{i=1}^{N} \int \frac{d^{d-1}\mathbb{P}}{(2\pi)^{d-1}\, 2\omega^i_{\mathbb{P}}} |\mathbb{P}, i\rangle\langle \mathbb{P}, i|$$

$$+ \text{ projection to multiparticle states}$$

$$= |0\rangle\langle 0| + \sum\!\!\!\!\!\!\int_{\lambda} \int \frac{d^{d-1}\mathbb{P}}{(2\pi)^{d-1}\, 2\omega^{\lambda}_{\mathbb{P}}} |\mathbb{P}, \lambda\rangle\langle \mathbb{P}, \lambda| .$$

symbol for mixed discrete-continuous sum

Let us consider the two point function

$$\langle \mathcal{O}(x)\,\mathcal{O}(y)\rangle = \langle 0|\,T\,\mathcal{O}(x)\,\mathcal{O}(y)\,|0\rangle$$

of a hermitian scalar operator $\mathcal{O} = \mathcal{O}^\dagger$

$$\text{s.t.}\quad \langle \mathcal{O}(x)\rangle = \langle 0|\,\mathcal{O}(x)\,|0\rangle = 0.$$

Inserting $\mathrm{id}_{\mathcal{H}}$ = above in the middle, we find for $x^0 > y^0$

$$\langle 0|\,\mathcal{O}(x)\,\mathcal{O}(y)\,|0\rangle = \sum_\lambda \int \frac{d^{d-1}\mathbb{P}}{(2\pi)^{d-1}\,2\omega^\lambda_{\mathbb{P}}}\,\langle 0|\,\mathcal{O}(x)\,|\mathbb{P},\lambda\rangle\langle \mathbb{P},\lambda|\,\mathcal{O}(y)\,|0\rangle$$

$$\langle 0|\,\mathcal{O}(x)\,|\mathbb{P},\lambda\rangle = \underbrace{\langle 0|\,e^{iPx}}_{\langle 0|}\,\mathcal{O}(0)\,\underbrace{e^{-iPx}\,|\mathbb{P},\lambda\rangle}_{e^{-iP_{\mathbb{P},\lambda}x}\,|\mathbb{P},\lambda\rangle}$$

$$= e^{-iP_{\mathbb{P},\lambda}x}\,\langle 0|\,\mathcal{O}(0)\,\underbrace{|\mathbb{P},\lambda\rangle}_{U(\Lambda_{m_\lambda,\mathbb{P}})|0,\lambda\rangle}$$

$$= e^{-iP_{\mathbb{P},\lambda}x}\,\langle 0|\,\mathcal{O}(0)\,U(\Lambda_{m_\lambda,\mathbb{P}})\,|0,\lambda\rangle$$

$\mathcal{O}$ is a scalar

$$= e^{-iP_{\mathbb{P},\lambda}x}\,\langle 0|\,\mathcal{O}(0)\,|0,\lambda\rangle$$

Similarly, $\langle \mathbb{P},\lambda|\,\mathcal{O}(y)\,|0\rangle = e^{iP_{\mathbb{P},\lambda}y}\,\langle 0,\lambda|\,\mathcal{O}(0)\,|0\rangle$

$$\therefore \langle 0|\mathcal{O}(x)\mathcal{O}(y)|0\rangle = \sum\!\!\!\!\!\!\int_\lambda \int \frac{d^{d-1}\mathbb{P}}{(2\pi)^{d-1}2\omega^\lambda_{\mathbb{P}}} e^{-iP_{\mathbb{P},\lambda}(x-y)} \left|\langle \mathbb{O},\lambda|\mathcal{O}(0)|0\rangle\right|^2$$

similarly for $\langle 0|\mathcal{O}(y)\mathcal{O}(x)|0\rangle$ for $y^0 > x^0$

$\langle 0|T\,\mathcal{O}(x)\mathcal{O}(y)|0\rangle$

$$= \sum\!\!\!\!\!\!\int_\lambda \underbrace{\int \frac{d^{d-1}\mathbb{P}}{(2\pi)^{d-1}2\omega^\lambda_{\mathbb{P}}} e^{-i\omega^\lambda_{\mathbb{P}}|x^0-y^0| + i\mathbb{P}\cdot(\mathbb{x}-\mathbb{y})}}_{\displaystyle\int \frac{d^d p}{(2\pi)^d}\frac{i\,e^{-ip(x-y)}}{p^2-m_\lambda^2+i0} =: D_F(x-y)_{m_\lambda}} \left|\langle \mathbb{O},\lambda|\mathcal{O}(0)|0\rangle\right|^2$$

$$= \int_0^\infty \frac{dM^2}{2\pi}\rho(M^2)\,D_F(x-y)_M$$

where $\displaystyle \rho(M^2) := \sum\!\!\!\!\!\!\int_\lambda 2\pi\,\delta(M^2-m_\lambda^2)\left|\langle \mathbb{O},\lambda|\mathcal{O}(0)|0\rangle\right|^2$

$$= \sum_{i=1}^N 2\pi\,\delta(M^2-m_i^2)\,Z_i + \rho_{\text{multi}}(M^2)$$

$Z_i := \left|\langle \mathbb{O},i|\mathcal{O}(0)|0\rangle\right|^2$

multiparticle continuum supported on $M^2 \geq 4m_{\min}^2$

Fourier transform of the two point function :

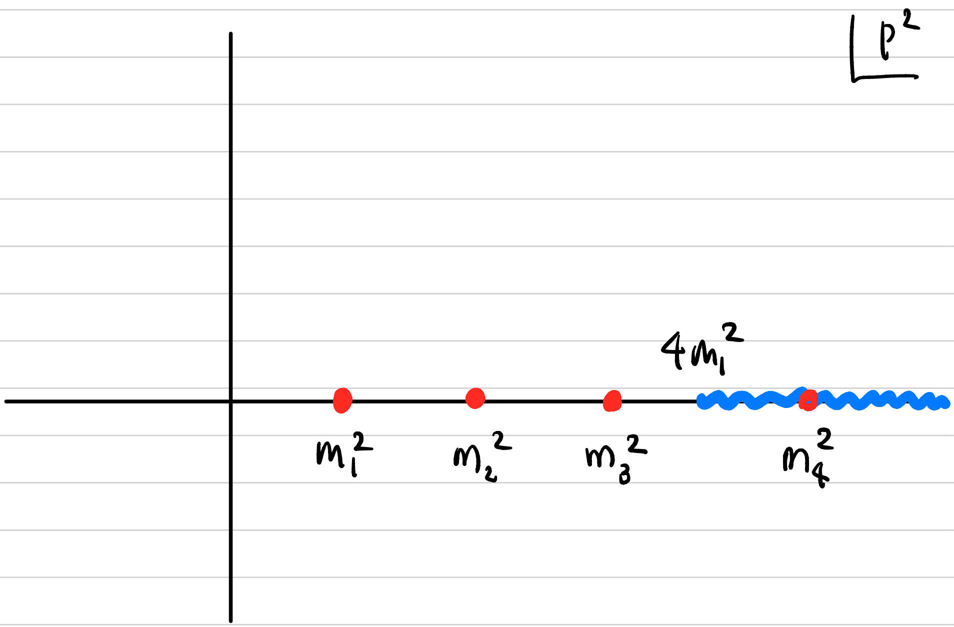
$$\int d^d x \, e^{ipx} \langle 0 | T \mathcal{O}(x) \mathcal{O}(0) | 0 \rangle$$

$$= \int_0^\infty \frac{dM^2}{2\pi} \rho(M^2) \frac{i}{p^2 - M^2 + i0}$$

$$= \sum_{i=1}^{N} \underbrace{\frac{i Z_i}{p^2 - m_i^2 + i0}} + \int_{4m_{min}^2}^{\infty} \frac{dM^2}{2\pi} \rho_{multi}(M^2) \frac{i}{p^2 - M^2 + i0}$$

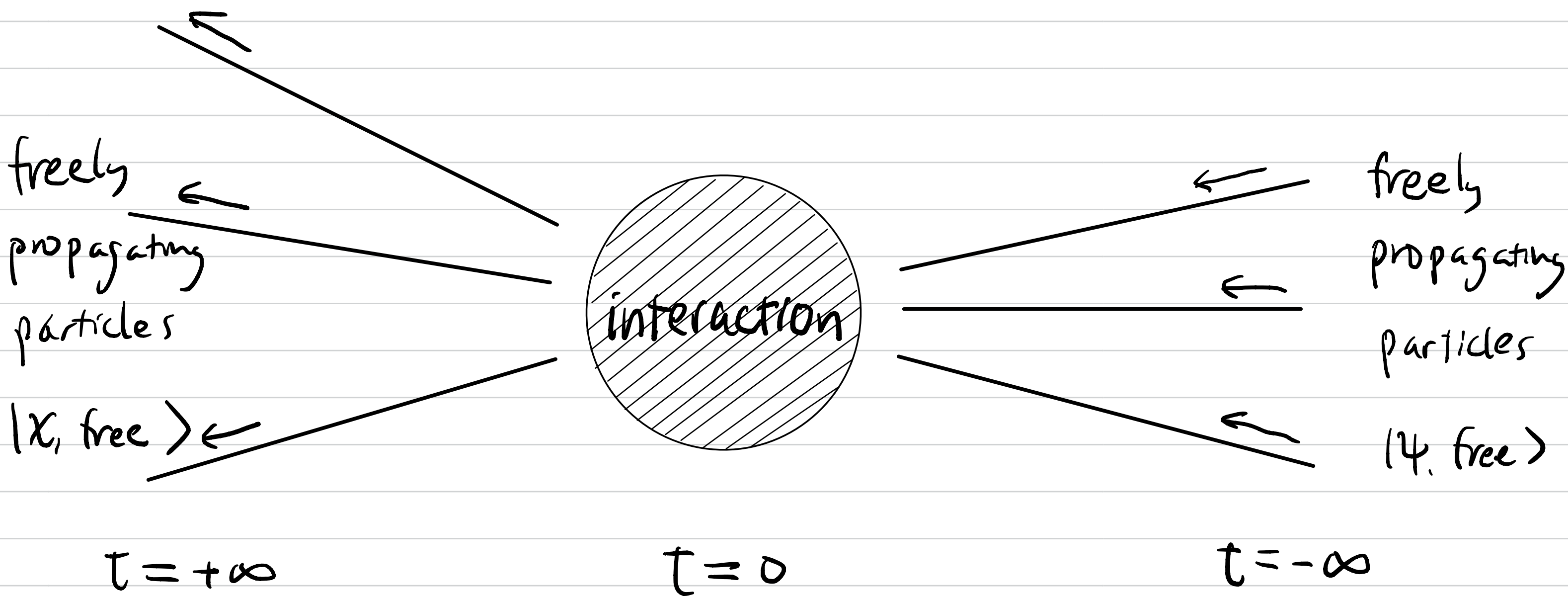
pole for each particle with $Z_i \neq 0$

branch cut on $[4m_{min}^2, \infty)$



Asymptotic states

Scattering process



$$|\psi\rangle_{in} \xleftarrow{\text{time evolution}} |\psi, \text{free}\rangle$$

$$|\chi, \text{free}\rangle \xleftarrow{\text{time evolution}} |\chi\rangle_{out}$$

S-matrix

$$\langle \chi, \text{free} | S | \psi, \text{free} \rangle := {}_{out}\langle \chi | \psi \rangle_{in}$$

We need $|\psi\rangle_{in}$ and $|\chi\rangle_{out}$.

For simplicity, consider a theory with a single species of scalar particle of mass m.

Suppose $\exists$ a hermitian scalar operator $\mathcal{O}$ s.t.

$$\langle 0 | \mathcal{O}(x) | 0 \rangle = 0, \quad \langle \mathbf{p} | \mathcal{O}(x) | 0 \rangle = \sqrt{Z} \neq 0 .$$

Then $\langle \mathbf{p} | \mathcal{O}(x) | 0 \rangle = \sqrt{Z}\, e^{i P_{\mathbf{p}} x} = \sqrt{Z}\, e^{i\omega_{\mathbf{p}} t - i \mathbf{p}\cdot\mathbf{x}}$.

For a positive energy wave packet $f(x)$

$$f(x) = \int \frac{d^{d-1}\mathbf{k}}{(2\pi)^{d-1} 2\omega_{\mathbf{k}}} \tilde{f}(\mathbf{k})\, e^{-i P_{\mathbf{k}} x},$$

define

$$\mathcal{O}_f(t) := \frac{-i}{\sqrt{Z}} \int d^{d-1}\mathbf{x}\, f(t,\mathbf{x}) \overleftrightarrow{\partial_t} \mathcal{O}(t,\mathbf{x})$$

$$\overleftrightarrow{\partial_t} := \overrightarrow{\partial_t} - \overleftarrow{\partial_t}$$

<u>Study $\mathcal{O}_f(t)|0\rangle$:</u>

$$\langle 0 | \mathcal{O}_f(t) | 0 \rangle = 0$$

$$\langle \mathbf{p} | \mathcal{O}_f(t) | 0 \rangle = -i \int d^{d-1}\mathbf{x} \int \frac{d^{d-1}\mathbf{k}}{(2\pi)^{d-1} 2\omega_{\mathbf{k}}} \tilde{f}(\mathbf{k})\, e^{-i(P_{\mathbf{k}} - P_{\mathbf{p}})x}\, i(\omega_{\mathbf{k}} + \omega_{\mathbf{p}})$$

$$= \tilde{f}(\mathbf{p}) \qquad (t\text{-independent})$$

c.f. $$\langle 0 | \mathcal{O}_f(t) | \mathbf{p} \rangle = -i \int d^{d-1}\mathbf{x} \int \frac{d^{d-1}\mathbf{k}}{(2\pi)^{d-1} 2\omega_{\mathbf{k}}} \tilde{f}(\mathbf{k})\, e^{-i(P_{\mathbf{k}} + P_{\mathbf{p}})x}\, i(\omega_{\mathbf{k}} - \omega_{\mathbf{p}})$$

$$= 0$$

multiparticle state $\langle \mathbb{p},\lambda | \mathcal{O}_f(t) | 0 \rangle =$

$$-i\int d^{d-1}\mathbf{x} \int \frac{d^{d-1}\mathbb{k}}{(2\pi)^{d-1}2\omega_{\mathbb{k}}} \tilde{f}(\mathbb{k})\, e^{-i(P_{\mathbb{k}}-P_{\mathbb{p},\lambda})x}\, i(\omega_{\mathbb{k}}+\omega_{\mathbb{p}}) \langle \mathbb{0},\lambda | \mathcal{O}(0) | 0 \rangle$$

$$= \frac{\omega_{\mathbb{p}}+\omega_{\mathbb{p}}^{\lambda}}{2\omega_{\mathbb{p}}} \tilde{f}(\mathbb{p}) \langle \mathbb{0},\lambda | \mathcal{O}(0) | 0 \rangle\, e^{i(\omega_{\mathbb{p}}^{\lambda}-\omega_{\mathbb{p}})t}$$

For a test state $|\psi\rangle$,

$$\langle \psi | \mathcal{O}_f(t) | 0 \rangle = \langle \psi | 0 \rangle \cancel{\langle 0 | \mathcal{O}_f(t) | 0 \rangle}$$

$$+ \int \frac{d^{d-1}\mathbb{p}}{(2\pi)^{d-1}2\omega_{\mathbb{p}}} \langle \psi | \mathbb{p} \rangle \langle \mathbb{p} | \mathcal{O}_f(t) | 0 \rangle \;= \tilde{f}(\mathbb{p})$$

$$+ \int_{\lambda} \int \frac{d^{d-1}\mathbb{p}}{(2\pi)^{d-1}2\omega_{\mathbb{p}}} \langle \psi | \mathbb{p},\lambda \rangle \underbrace{\langle \mathbb{p},\lambda | \mathcal{O}_f(t) | 0 \rangle}_{\cdots\, e^{i\omega_{\mathbb{p}}^{\lambda} t}}$$

$|t| \to \infty$ $\downarrow$ 0

Wild oscillation of $e^{i\omega_{\mathbb{p}}^{\lambda} t}$ as a function of λ

$$\therefore \langle \psi | \mathcal{O}_f(t) | 0 \rangle \xrightarrow{|t|\to\infty} \int \frac{d^{d-1}\mathbb{p}}{(2\pi)^{d-1}2\omega_{\mathbb{p}}} \langle \psi | \mathbb{p} \rangle \tilde{f}(\mathbb{p})$$

i.e. $$\mathcal{O}_f(t)|0\rangle \xrightarrow{t\to\pm\infty} \int \frac{d^{d-1}\mathbb{p}}{(2\pi)^{d-1}2\omega_{\mathbb{p}}} |\mathbb{p}\rangle \tilde{f}(\mathbb{p}) =: |f\rangle$$

$U(t) = e^{-itH}$

Note: $|f\rangle = U(T)\,U(-T)\,|f\rangle$

$$= U(T) \underbrace{\underbrace{\int \frac{d^{d-1}p}{(2\pi)^{d-1} 2\omega_p} |p\rangle \tilde{f}(p)\, e^{-i\omega_p(-T)}}_{\text{free propagation of one particle with wave packet } f \text{ at } t=-T}}_{\text{its time evolution to } t=0}$$

$T \to \infty$: $|f\rangle = |f\rangle_{in}$

$T \to -\infty$: $|f\rangle = |f\rangle_{out}$

Thus

$$\mathcal{O}_f(t)\,|0\rangle \xrightarrow{t \to \pm\infty} |f\rangle = |f\rangle_{in} = |f\rangle_{out}.$$

adjoint

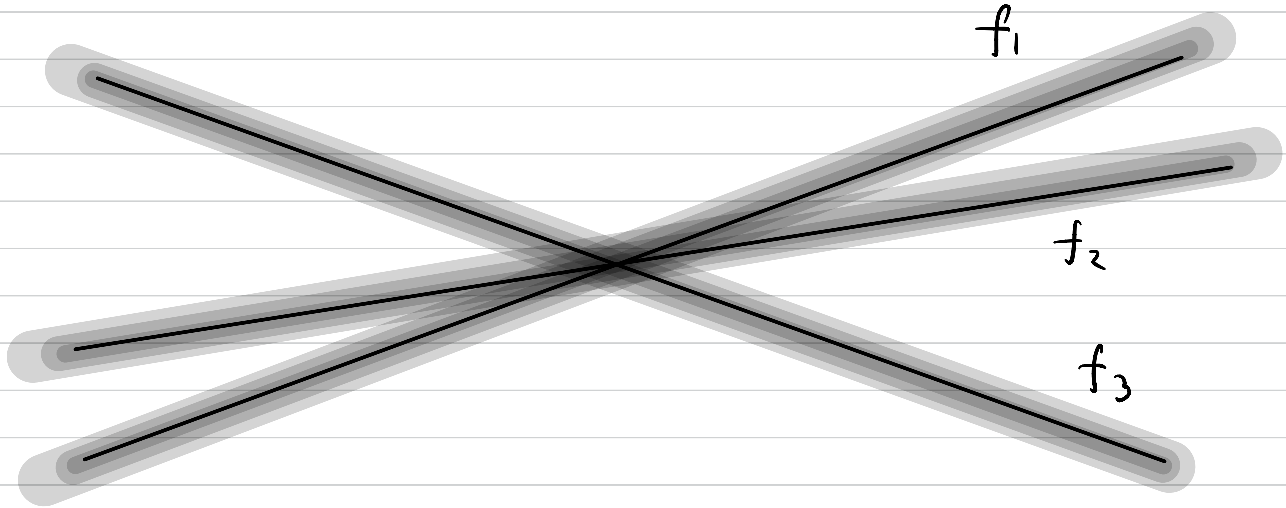
$$\langle 0|\,\mathcal{O}_f(t)^\dagger \xrightarrow{t \to \pm\infty} \langle f| = {}_{in}\langle f| = {}_{out}\langle f|$$

By a similar computation, we find $\langle 0|\mathcal{O}_f(t)|\psi\rangle \xrightarrow{|t| \to \infty} 0$ for any test state $|\psi\rangle$.

$$\langle 0|\,\mathcal{O}_f(t) \longrightarrow 0 \quad \text{as } t \to \pm\infty.$$

$$\mathcal{O}_f(t)^\dagger\,|0\rangle \longrightarrow 0 \quad \text{as } t \to \pm\infty$$

$f_1(x), \cdots, f_n(x)$: wave packets with no overlap at $|t| \to \infty$



$$\mathcal{O}_{f_1}(-T) \cdots \mathcal{O}_{f_n}(-T) |0\rangle$$

$$= U(T)\, U(T)^{-1} \prod_{i=1}^{n} \frac{-i}{\sqrt{Z}} \int d^{d-1}\mathbf{x}_i \, f_i(-T, \mathbf{x}_i) \overleftrightarrow{\partial_t} \mathcal{O}(-T, \mathbf{x}_i) |0\rangle$$

$$\underbrace{= U(T) \underbrace{\prod_{i=1}^{n} \frac{-i}{\sqrt{Z}} \int d^{d-1}\mathbf{x}_i \, f_i(-T, \mathbf{x}_i) \overleftrightarrow{\partial_t} \mathcal{O}(0, \mathbf{x}_i) |0\rangle}_{\substack{\text{free propagation of } n \text{ particles} \\ \text{with wave packets } f_1, \cdots, f_n \text{ at } t=-T}}}_{\text{its time evolution to } t=0.}$$

$$\therefore \; |f_1, \cdots, f_n\rangle_{in} = \lim_{T \to \infty} \mathcal{O}_{f_1}(-T) \cdots \mathcal{O}_{f_n}(-T) |0\rangle$$

$$= \lim_{T_1 \to \infty} \cdots \lim_{T_n \to \infty} \mathcal{O}_{f_1}(-T_1) \cdots \mathcal{O}_{f_n}(-T_n) |0\rangle$$

The ordering does not matter since $f_1, \cdots, f_n$ has no overlap at $t \to -\infty$.

$$\mathcal{O}_{f_1}(T) \cdots \mathcal{O}_{f_n}(T) |0\rangle$$

$$= U(T)^{-1} U(T) \prod_{i=1}^{n} \frac{-i}{\sqrt{Z}} \int d^{d-1}\mathbf{x}_i \, f_i(T, \mathbf{x}_i) \overleftrightarrow{\partial_t} \mathcal{O}(T, \mathbf{x}_i) |0\rangle$$

$$= U(T)^{-1} \underbrace{\prod_{i=0}^{n} \frac{-i}{\sqrt{Z}} \int d^{d-1}\mathbf{x}_i \, f_i(T, \mathbf{x}_i) \overleftrightarrow{\partial_t} \mathcal{O}(0, \mathbf{x}_i) |0\rangle}_{\text{free propagation of } n \text{ particles with wave packets } f_1, \cdots, f_n \text{ at } t = T}$$

its time reversal to $t = 0$

$$\therefore |f_1, \cdots, f_n\rangle_{out} = \lim_{T \to \infty} \mathcal{O}_{f_1}(T) \cdots \mathcal{O}_{f_n}(T) |0\rangle$$

$$= \lim_{T_1 \to \infty} \cdots \lim_{T_n \to \infty} \mathcal{O}_{f_1}(T_1) \cdots \mathcal{O}_{f_n}(T_n) |0\rangle$$

The ordering does not matter since $f_1, \cdots, f_n$ has no overlap at $t \to +\infty$.

LSZ reduction formula

$$\langle g_1, \cdots, g_n \text{ free} | S | f_1, f_2, \text{ free} \rangle = {}_{out}\langle g_1, \cdots, g_n | f_1, f_2 \rangle_{in} = ?$$

$$\mathcal{O}_f(-T) - \mathcal{O}_f(T) = -\int_{-T}^{T} dt \frac{\partial}{\partial t} \mathcal{O}_f(t)$$

$$= \frac{i}{\sqrt{Z}} \int_{-T}^{T} dt \int d^{d-1}\mathbf{x}\, \underbrace{\partial_t (f \partial_t \mathcal{O} - \partial_t f \mathcal{O})}_{f \partial_t^2 \mathcal{O} - \underbrace{\partial_t^2 f}_{=(\nabla^2 - m^2) f} \mathcal{O}}$$

[as $f(t, \mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$,
spatial partial integration is allowed.

$$= \frac{i}{\sqrt{Z}} \int_{-T}^{T} dt \int d^{d-1}\mathbf{x}\, f (\partial_t^2 - \nabla^2 + m^2) \mathcal{O}$$

$$= \frac{i}{\sqrt{Z}} \int_{[-T,T]\times \mathbb{R}^{d-1}} d^d x\, f (\partial^2 + m^2) \mathcal{O}$$

Taking its adjoint

$$\mathcal{O}_g(T)^\dagger - \mathcal{O}_g(-T)^\dagger = \frac{i}{\sqrt{Z}} \int_{[-T,T]\times \mathbb{R}^{d-1}} d^d x\, g^* (\partial^2 + m^2) \mathcal{O}$$

Consider $X_{T_1 \cdots T_n, T_1', T_2'} :=$

$$\prod_{i=1}^{n} \int_{[-T_i, T_i] \times \mathbb{R}^{d-1}} d^d y_i \, \frac{i}{\sqrt{Z}} g_i(y_i)^* \prod_{j=1}^{2} \int_{[-T_j', T_j'] \times \mathbb{R}^{d-1}} d^d x_j \, \frac{i}{\sqrt{Z}} f_j(x_j)$$

$$\times \left(\partial_{y_1}^2 + m^2\right) \cdots \left(\partial_{y_n}^2 + m^2\right)\left(\partial_{x_1}^2 + m^2\right)\left(\partial_{x_2}^2 + m^2\right)$$

$$\langle 0 | T \, \mathcal{O}(y_1) \cdots \mathcal{O}(y_n) \mathcal{O}(x_1) \mathcal{O}(x_2) | 0 \rangle$$

$$\int_{[-T_1, T_1] \times \mathbb{R}^{d-1}} d^d y_1 \, \frac{i}{\sqrt{Z}} g_1(y_1)^* \left(\partial_{y_1}^2 + m^2\right) \langle 0 | T \, \mathcal{O}(y_1) \cdots \mathcal{O}(x_2) | 0 \rangle$$

$$\xrightarrow[T_1 \to \infty]{} \langle 0 | \mathcal{O}_{g_1}(\infty)^\dagger \, T(\mathcal{O}(y_2) \cdots \mathcal{O}(x_2)) | 0 \rangle$$

$$- \langle 0 | T(\mathcal{O}(y_2) \cdots \mathcal{O}(x_2)) \mathcal{O}_{g_1}(-\infty)^\dagger | 0 \rangle = 0$$

Thus

$$X_{\vec{T}, \vec{T}'} \xrightarrow{T_1 \to \infty} \prod_{i=2}^{n} \cdots \prod_{j=1}^{2} \cdots \quad \cdots \langle 0 | \mathcal{O}_{g_1}(\infty)^\dagger \, T(\mathcal{O}(y_2) \cdots \mathcal{O}(x_2)) | 0 \rangle$$

Repeating this for $T_2, \ldots, T_n$, we find

$$X_{\vec{T}, \vec{T}'} \xrightarrow{T_1, T_2, \ldots T_n \to \infty} \prod_{j=1}^{2} \cdots \quad \cdots \underbrace{\langle 0 | \mathcal{O}_{g_1}(\infty)^\dagger \cdots \mathcal{O}_{g_n}(\infty)^\dagger}_{{}_{\text{out}}\langle g_1, \ldots, g_n |} \, T(\mathcal{O}(x_1) \mathcal{O}(x_2)) | 0 \rangle$$

Further limits :

$$\xrightarrow{T_1'\to\infty} \int_{[-T_2',T_2']\times\mathbb{R}^{d-1}} d^d x_2 \frac{i}{\sqrt{Z}} f_2(x_2)\left(\partial_{x_2}^2+m^2\right)$$

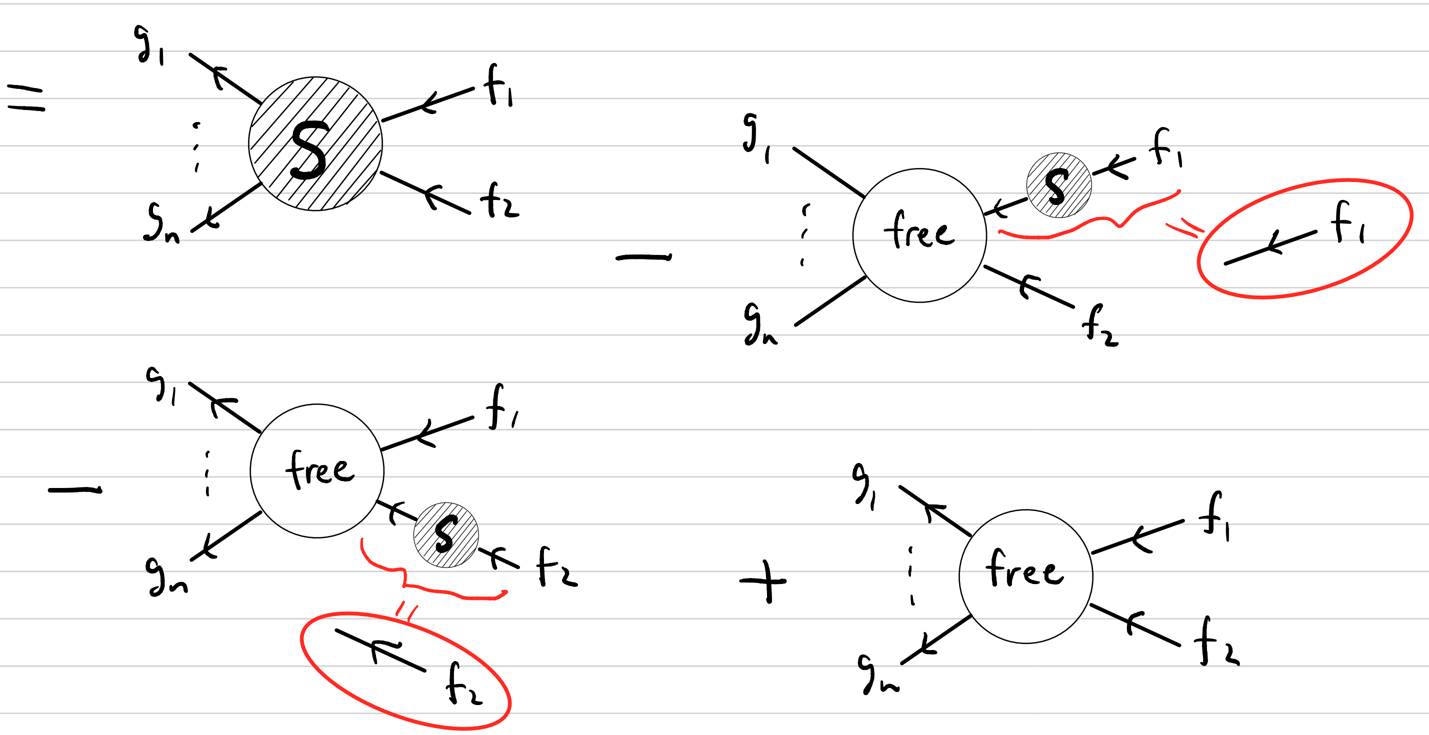
$$\left({}_{out}\langle g_1,\dots,g_n|\mathcal{O}(x_2)\,\mathcal{O}_{f_1}(-\infty)|0\rangle - {}_{out}\langle g_1,\dots,g_n|\mathcal{O}_{f_1}(\infty)\,\mathcal{O}(x_2)|0\rangle \right)$$

$$\xrightarrow{T_2'\to\infty} {}_{out}\langle g_1,\dots g_n|\underbrace{\mathcal{O}_{f_2}(-\infty)\,\mathcal{O}_{f_1}(-\infty)|0\rangle}_{|f_1,f_2\rangle_{in}}$$

$$- {}_{out}\langle g_1,\dots g_n|\underbrace{\mathcal{O}_{f_2}(\infty)\underbrace{\mathcal{O}_{f_1}(-\infty)|0\rangle}_{\mathcal{O}_{f_1}(\infty)|0\rangle}}_{|f_1,f_2\rangle_{out}}$$

$$- {}_{out}\langle g_1,\dots,g_n|\underbrace{\mathcal{O}_{f_1}(\infty)\underbrace{\mathcal{O}_{f_2}(-\infty)}_{\mathcal{O}_{f_1}(\infty)|0\rangle}|0\rangle}_{|f_1,f_2\rangle_{out}}$$

$$+ {}_{out}\langle g_1,\dots,g_n|\underbrace{\mathcal{O}_{f_1}(\infty)\,\mathcal{O}_{f_2}(\infty)|0\rangle}_{|f_1,f_2\rangle_{out}}$$



$$= {}_{\text{out}}\langle g_1, \cdots, g_n | f_1, f_2 \rangle_{\text{in}} - {}_{\text{out}}\langle g_1, \cdots, g_n | f_1, f_2 \rangle_{\text{out}}$$

$$= \langle g_1, \cdots, g_n, \text{free} | S | f_1, f_2, \text{free} \rangle - \langle g_1, \cdots, g_n, \text{free} | f_1, f_2, \text{free} \rangle$$

We obtained a formula

$$\boxed{\begin{aligned} &\langle g_1, \cdots, g_n, \text{free} | S | f_1, f_2, \text{free} \rangle \\ &= \langle g_1, \cdots, g_n, \text{free} | f_1, f_2, \text{free} \rangle \\ &+ \prod_{i=1}^{n} \int d^d y_i \frac{i}{\sqrt{Z}} g_i(y_i)^* (\partial_{y_i}^2 + m^2) \prod_{j=1}^{2} \int d^d x_j \frac{i}{\sqrt{Z}} f_j(x_j) (\partial_{x_j}^2 + m^2) \\ &\qquad \langle 0 | T \mathcal{O}(y_1) \cdots \mathcal{O}(y_n) \mathcal{O}(x_1) \mathcal{O}(x_2) | 0 \rangle \end{aligned}}$$

describing S-matrix by correlation functions.

This is the LSZ reduction formula.

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