Scalar propagator

Let us examine the scalar propagator

$$\langle \phi(x)\,\phi(y)\rangle_E = \int \frac{d^dp}{(2\pi)^d}\,\frac{e^{-ip(x-y)}}{p^2+m^2}$$ in Euclidean theory*

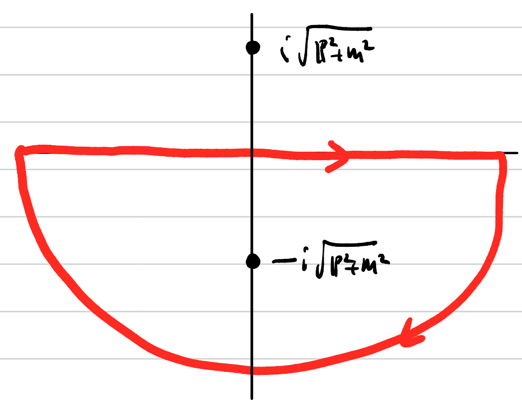
or its Minkowski limit

$$\langle \phi(x)\,\phi(y)\rangle = \int \frac{d^dp}{(2\pi)^d}\,\frac{i\,e^{-ip(x-y)}}{p^2-m^2+i\cdot 0}.$$

Euclidean theory

Take the frame where $x-y=(0,\cdots,0,L)$, $L>0$. Then

$$\langle \phi(x)\,\phi(y)\rangle_E = \int \frac{d^{d-1}\mathbb{p}\,dp_d}{(2\pi)^d}\,\frac{e^{-ip_dL}}{p_d^2+\mathbb{p}^2+m^2}$$



$$= \int \frac{d^{d-1}\mathbb{p}}{(2\pi)^d}\,(-2\pi i)\,\frac{e^{-\sqrt{\mathbb{p}^2+m^2}\,L}}{-2i\sqrt{\mathbb{p}^2+m^2}}$$

$$= \frac{1}{2}\int \frac{d^{d-1}\mathbb{p}}{(2\pi)^{d-1}}\,\frac{e^{-\sqrt{\mathbb{p}^2+m^2}\,L}}{\sqrt{\mathbb{p}^2+m^2}}$$

Note: For $d=1$, this is already an exact result:

$$\langle \phi(x)\,\phi(y)\rangle_E = \frac{1}{2m}\,e^{-m|x-y|}$$

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* Notation has changed a bit from Lecture 6.

We shall find (for $d > 1$),

- exact formula for $m = 0$
- short distance $|x-y| \ll \frac{1}{m}$ behaviour
- long distance $|x-y| \gg \frac{1}{m}$ behaviour

$\left.\begin{matrix} \\ \\ \end{matrix}\right\} m \neq 0$

<u>Massless case</u> $m = 0$

$$\langle \phi(x)\phi(y)\rangle_E = \frac{1}{2}\int \frac{d^{d-1}\mathbb{P}}{(2\pi)^{d-1}} \frac{e^{-|\mathbb{P}|L}}{|\mathbb{P}|}$$

$$= \frac{Vol(S^{d-2})}{2(2\pi)^{d-1}} \int_0^\infty p^{d-2}\,dp\, \frac{e^{-pL}}{p}$$

$Vol(S^{d-2})$ = volume of the sphere S^{d-2} of radius 1

$t = pL$

$$= \frac{Vol(S^{d-2})}{2(2\pi)^{d-1}} \frac{1}{L^{d-2}} \int_0^\infty t^{d-2} \frac{dt}{t} e^{-t} = \Gamma(d-2)$$

Using $Vol(S^{d-2})\Gamma(d-2) = (4\pi)^{\frac{d-2}{2}}\Gamma\left(\frac{d-2}{2}\right)$, we find

$$\boxed{\langle \phi(x)\phi(y)\rangle_E = \frac{\Gamma\left(\frac{d-2}{2}\right)}{4\pi^{d/2}} \frac{1}{|x-y|^{d-2}}}$$

Remarks • For $d=2$, the right hand side does not make sense:

$\Gamma(z)$ has a pole at $z=0$. The problem is already obvious in

$$\langle \phi(x)\phi(y)\rangle_E = \int \frac{d^2p}{(2\pi)^2}\frac{e^{-ip(x-y)}}{p^2}.$$

The integral is logarithmically divergent at $p\to 0$. This is the infra-red singularity of massless scalar in $d=2$.

One may introduce an infra-red cut of Λ_{IR}, say

$$\int_{|p|\geq\Lambda_{IR}} \frac{d^2p}{(2\pi)^2}\frac{e^{-ip(x-y)}}{p^2} = -\frac{1}{2\pi}\log\left(\Lambda_{IR}|x-y|\right) + \text{finite } \#$$

where we used

$$\int_\epsilon^\infty \frac{dt}{t}e^{-t} = -\log\epsilon - \gamma + O(\epsilon) \qquad \text{———} (\#)$$

$$\gamma := \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log n\right) = 0.5772\ldots \text{ (Euler's constant)}.$$

• For low values of $d\geq 3$,

d	3	4	5	6
$\langle\phi(x)\phi(0)\rangle_E$	$\frac{1}{4\pi\lvert x\rvert}$	$\frac{1}{4\pi^2\lvert x\rvert^2}$	$\frac{1}{8\pi^2\lvert x\rvert^3}$	$\frac{1}{4\pi^3\lvert x\rvert^4}$

Let us next examine the short & long distance behaviour of $\langle \phi(x)\phi(y)\rangle_E$ in the massive theory $m \neq 0$.

Short distance $m|x-y| \ll 1$

$d>2$ The above result for $m=0$ is the leading short distance singularity:

$$\boxed{\begin{aligned}\langle \phi(x)\phi(y)\rangle_E &= \frac{\Gamma\left(\frac{d-2}{2}\right)}{4\pi^{d/2}} \frac{1}{|x-y|^{d-2}} \\ &+ \frac{1}{|x-y|^{d-2}} O\left((m|x-y|)^2\right) \\ &+ \frac{1}{|x-y|^{d-2}} O\left((m|x-y|)^{2P_d}\right) \log(m|x-y|) \\ &+ \text{regular} \\ \text{as} \quad & m|x-y| \to 0, \\ \text{where} \quad & P_d = \left\lceil \frac{d-3}{2} \right\rceil = \min\left\{ p \in \mathbb{Z} \mid p \geq \frac{d-3}{2} \right\}.\end{aligned}}$$

(·.·) The difference $\langle \phi(x)\phi(y)\rangle_E - \langle \phi(x)\phi(y)\rangle_E\big|_{m=0}$ is

$$\frac{1}{2}\int \frac{d^{d-1}\mathbb{p}}{(2\pi)^{d-1}}\left(\frac{e^{-\sqrt{\mathbb{p}^2+m^2}\,L}}{\sqrt{\mathbb{p}^2+m^2}} - \frac{e^{-|\mathbb{p}|L}}{|\mathbb{p}|}\right)$$

$$= \frac{\mathrm{Vol}(S^{d-2})}{2(2\pi)^{d-1}}\int_0^\infty p^{d-2}dp\left(\frac{e^{-\sqrt{p^2+m^2}\,L}}{\sqrt{p^2+m^2}} - \frac{e^{-pL}}{p}\right)$$

$p=m\ell$

$$= \frac{\mathrm{Vol}(S^{d-2})}{2(2\pi)^{d-1}}\, m^{d-2}\int_0^\infty \ell^{d-2}d\ell\left(\frac{e^{-mL\sqrt{\ell^2+1}}}{\sqrt{\ell^2+1}} - \frac{e^{-mL\ell}}{\ell}\right).$$

In the last expression, the integration region $[0,\infty)$ is decomposed into $[0,1]$ and $[1,\infty)$.

On $[0,1]$, the integrand can be expanded as a power series in mL, yielding a power series $m^{d-2}\sum_{n=0}^\infty b_n(m|x-y|)^n$ that is regular at $m|x-y|\to 0$.

On $[1,\infty)$, $\sqrt{\ell^2+1} = \ell\sqrt{1+1/\ell^2}$ in the integrand can be expanded as a power series in $1/\ell^2$, and that yields the less singular terms. The terms including $\log(m|x-y|)$ are found using (#). //

$\underline{d=2}$ The short distance behaviour is

$$\boxed{\langle \phi(x)\phi(y)\rangle_E = -\frac{1}{2\pi}\log\left(m|x-y|\right) + \text{const}^* + O(m|x-y|) \quad \text{as} \quad m|x-y| \to 0}$$

$$\langle \phi(x)\phi(y)\rangle_E = \frac{1}{2\pi}\int_0^\infty dp\, \frac{e^{-\sqrt{p^2+m^2}\,L}}{\sqrt{p^2+m^2}}$$

$p = m\ell$

$$= \frac{1}{2\pi}\int_0^\infty d\ell\, \frac{e^{-mL\sqrt{\ell^2+1}}}{\sqrt{\ell^2+1}}$$

Decompose $[0,\infty)$ to $[0,1]$ and $[1,\infty)$. The part $[0,1]$ yields a power series in mL. For $[1,\infty)$, $\sqrt{\ell^2+1}$ can be expanded as a power series in $1/\ell^2$. The integration is regular at $mL \to 0$ except the leading term:

$$\frac{1}{2\pi}\int_1^\infty d\ell\, \frac{e^{-mL\ell}}{\ell} = \frac{1}{2\pi}\int_{mL}^\infty \frac{dt}{t}\, e^{-t} \overset{(\#)}{=} -\log(mL) - \gamma + O(mL) \quad //$$

$\underline{\text{Rmks}}$

- The constant const^* is $\frac{1}{2\pi}(\log 2 - \gamma)$.
- Small m effectively acts as Λ_{IR} of the massless theory.

Long distance $m|x-y| \gg 1$

For $d \geq 2$, the long distance behaviour is

$$\boxed{\langle \phi(x)\phi(y)\rangle_E = \frac{m^{d-2}}{2(2\pi m|x-y|)^{\frac{d-1}{2}}}\, e^{-m|x-y|}\left(1+O\left(\frac{1}{m|x-y|}\right)\right)}$$

$$\langle \phi(x)\phi(y)\rangle_E = \frac{\mathrm{Vol}(S^{d-2})}{2(2\pi)^{d-1}}\int_0^\infty p^{d-2}\,dp\,\frac{e^{-\sqrt{p^2+m^2}\,L}}{\sqrt{p^2+m^2}}$$

$p = m\ell$

$$= \frac{\mathrm{Vol}(S^{d-2})}{2(2\pi)^{d-1}}\, m^{d-2}\int_0^\infty \ell^{d-2}\,d\ell\,\frac{e^{-mL\sqrt{\ell^2+1}}}{\sqrt{\ell^2+1}} =: I$$

$$\sqrt{\ell^2+1} = s+1 \;\Rightarrow\; \frac{\ell\,d\ell}{\sqrt{\ell^2+1}} = ds \;\Rightarrow\; \frac{\ell^{d-2}\,d\ell}{\sqrt{\ell^2+1}} = (s^2+2s)^{\frac{d-3}{2}}\,ds$$

$$\therefore\; I = \int_0^\infty \underbrace{(s^2+2s)^{\frac{d-3}{2}}}_{2^{\frac{d-3}{2}}\, s^{\frac{d-3}{2}}\,(1+s/2)^{\frac{d-3}{2}}}\,ds\; e^{-mL(s+1)}$$

$mLs = u$

$$= 2^{\frac{d-3}{2}}(mL)^{-\frac{d-1}{2}}\, e^{-mL}\int_0^\infty u^{\frac{d-3}{2}}\underbrace{\left(1+\frac{u}{2mL}\right)^{\frac{d-3}{2}}}_{1+\frac{d-3}{2}\frac{u}{2mL}+\cdots} e^{-u}\,du$$

$$= 2^{\frac{d-3}{2}} (mL)^{-\frac{d-1}{2}} e^{-mL} \left[\Gamma\left(\frac{d-1}{2}\right) + \frac{d-3}{4mL} \Gamma\left(\frac{d+1}{2}\right) + \cdots \right]$$

$$= 2^{\frac{d-3}{2}} (mL)^{-\frac{d-1}{2}} e^{-mL} \Gamma\left(\frac{d-1}{2}\right) \left(1 + \frac{(d-3)(d-1)}{8mL} + \cdots \right)$$

$$\therefore \langle \phi(x) \phi(y) \rangle_E = \frac{\mathrm{Vol}(S^{d-2})}{2(2\pi)^{d-1}} m^{d-2} \cdot I$$

$$\left[\mathrm{Vol}(S^{d-2}) \Gamma\left(\frac{d-1}{2}\right) = 2 \cdot \pi^{\frac{d-1}{2}} \right.$$

$$= \frac{2 \cdot \pi^{\frac{d-1}{2}} \cdot 2^{\frac{d-3}{2}}}{2(2\pi)^{d-1}} m^{d-2} (mL)^{-\frac{d-1}{2}} e^{-mL} \left(1 + \frac{(d-3)(d-1)}{8mL} + \cdots \right)$$

$$= \frac{m^{d-2}}{2(2\pi mL)^{\frac{d-1}{2}}} e^{-mL} \left(1 + \frac{(d-3)(d-1)}{8mL} + \cdots \right) \quad //$$

<u>Remarks</u>

- The formula applies also for $d = 1$
- For $d = 3$, subleading terms vanish. Thus

$$\langle \phi(x) \phi(y) \rangle_E = \frac{1}{4\pi |x-y|} e^{-m|x-y|} \qquad \text{exact.}$$

- The coefficients of subleading terms can be determined:

$$\langle \phi(x)\phi(y)\rangle_E$$

$$= \frac{m^{d-2}}{2(2\pi m|x-y|)^{\frac{d-1}{2}}} e^{-m|x-y|} \left(1+\sum_{n=1}^{\infty} \frac{c_n}{(m|x-y|)^n}\right)$$

$$c_n = \frac{1}{2^n n!}\left(\frac{d-1}{2}+n-1\right)\cdots\left(\frac{d-1}{2}-n\right) = \frac{1}{2^n n!}\frac{\Gamma\left(\frac{d-1}{2}+n\right)}{\Gamma\left(\frac{d-1}{2}-n\right)}$$

When d is odd $c_n = 0$ for $n \geq \frac{d-1}{2}$.

So the sum terminates at $n = \frac{d-3}{2}$ $\rightsquigarrow$ exact formula.

When d is even, it is an infinite sum.

$$\frac{c_{n+1}}{c_n} = \frac{2^n n!}{2^{n+1}(n+1)!}\left(\frac{d-1}{2}+n\right)\left(\frac{d-1}{2}-n-1\right)$$

$$= \frac{-1}{2(n+1)}\left(n+\frac{d-1}{2}\right)\left(n+1-\frac{d-1}{2}\right) \underset{n\to\infty}{\sim} -\frac{n}{2}$$

The convergence radius in $\frac{1}{m|x-y|}$ is zero.

So, it is at best an asymptotic expansion.

Minkowski limit

From the above results on $\langle \phi(x)\phi(y)\rangle_E$, what can we learn about the behaviour of the Minkowski limit $\langle \phi(x)\phi(y)\rangle$?

The reverse Wick rotation is

$$|x-y| \to \sqrt{(\mathbf{x}-\mathbf{y})^2 + e^{2i\epsilon}(x^0-y^0)^2}$$

and the limit is $\epsilon \nearrow \pi/2$.

- Space-like separation : $(\mathbf{x}-\mathbf{y})^2 > (x^0-y^0)^2$

  The results on $\langle \phi(x)\phi(y)\rangle_E$ holds on $\langle \phi(x)\phi(y)\rangle$ under the replacement $|x-y| \to \sqrt{(\mathbf{x}-\mathbf{y})^2-(x^0-y^0)^2}$

- Time-like separation : $(\mathbf{x}-\mathbf{y})^2 < (x^0-y^0)^2$

  The results on $\langle \phi(x)\phi(y)\rangle_E$ holds on $\langle \phi(x)\phi(y)\rangle$ under the replacement $|x-y| \to i\sqrt{(x^0-y^0)^2-(\mathbf{x}-\mathbf{y})^2}$

- Light-like separation : $(\mathbf{x}-\mathbf{y})^2 = (x^0-y^0)^2$

  The limit $\epsilon \nearrow \pi/2$ is singular. Nothing can be said on $\langle \phi(x)\phi(y)\rangle$ from $\langle \phi(x)\phi(y)\rangle_E$.

$\underline{d=1}$ $\langle \phi(t)\,\phi(0)\rangle = \frac{1}{2m}\, e^{-im|t|}$ exact.

$\underline{d\geq 2}$ If $x^2 := (x^0)^2 - \mathbf{x}^2 \neq 0$, we have the following behaviour using

$$|x| := \begin{cases} \sqrt{-x^2} & x^2<0 \quad \text{spacelike} \\ i\sqrt{x^2} & x^2>0 \quad \text{timelike} \end{cases}$$

$\underline{m=0\ (d>2)}$

$$\langle \phi(x)\,\phi(0)\rangle = \frac{\Gamma\left(\frac{d-2}{2}\right)}{4\pi^{d/2}}\frac{1}{|x|^{d-2}} \qquad \text{exact.}$$

$\underline{m\neq 0,\ 0\neq m|x|\ll 1}$

$$\langle \phi(x)\,\phi(0)\rangle = \begin{cases} \dfrac{\Gamma\left(\frac{d-2}{2}\right)}{4\pi^{d/2}}\dfrac{1}{|x|^{d-2}} + \dfrac{O((m|x|)^2)}{|x|^{d-2}} + \dfrac{O((m|x|)^{2p_d})}{|x|^{d-2}}\log(m|x|) + \text{regular} & d>2 \\ -\dfrac{1}{2\pi}\log(m|x|) + \text{regular} & d=2 \end{cases}$$

where $p_d = \left\lfloor \frac{d-2}{2} \right\rfloor$

$m \neq 0, \ m|x| \gg 1$

$$\langle \phi(x)\phi(0)\rangle = \frac{m^{d-2}}{2(2\pi m|x|)^{\frac{d-1}{2}}} e^{-m|x|}\left(1+O\left(\frac{1}{m|x|}\right)\right)$$

$d \geq 1$

For odd d, there is an exact formula, which is valid for any x with $x^2 \neq 0$:

$d=1$: already written

$d=3$: $\langle \phi(x)\phi(0)\rangle = \frac{1}{4\pi|x|} e^{-m|x|}$ exact.