Symplectic view on constrained systems

As discussed briefly in the class, some of the constructions (Dirac bracket for 2nd class; reduced phase space for 1st class) can be understood transparently when we view phase spaces as symplectic manifolds. Here I explain that.

phase space = symplectic manifold

A phase space is a manifold M with a Poisson bracket $\{ , \}$ which is a bilinear map from the space of smooth functions on M to itself, satisfying

(i) antisymmetry: $\{f, g\} = -\{g, f\}$,

(ii) derivation: $\{f, gh\} = \{f, g\}h + g\{f, h\}$,

(iii) Jacobi identity: $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$,

(iv) non-degeneracy: for any local coordinate system (x^I) the matrix with entries $\{x^I, x^J\}$ is invertible.

Remarks
- Often, (iv) is dropped in the definition of Poisson bracket. In that definition, a phase space is a manifold with a non-degenerate Poisson bracket.
- Antisymmetric invertible matrix exists only when the size is even.

Thus, the dimension of M is even, $\dim M = 2n$ $(n \in \mathbb{N})$.

- Some of you may consider the phase space as a space with a class of coordinates $(q^1, \dots, q^n, p_1, \dots, p_n)$ by which Poisson bracket is given by $\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} \right)$. Relation to that is given by Darboux's theorem: For $2n$-dimensional manifold M with a Poisson bracket $\{\ ,\ \}$, at any point $x \in M$, one can find a coordinate system $(q^1, \dots, q^n, p_1, \dots, p_n)$ on a neighborhood of x s.t. $\{q^i, p_j\} = \delta^i_j$, $\{q^i, q^j\} = \{p_i, p_j\} = 0$.

A <u>symplectic manifold</u> is a manifold M with a <u>symplectic form</u> ω which is a non-degenerate closed 2-form.

<u>Remarks</u> · "closed" means $d\omega = 0$.

- "non-degenerate" means: for any local coordinate system (x^I), the matrix with entries $\omega\left(\frac{\partial}{\partial x^I}, \frac{\partial}{\partial x^J}\right)$ is invertible.

Again, as antisymmetric matrix can be invertible only when the size is even, the dimension of M must be even, $2n$ $(n \in \mathbb{Z})$.

"non-degeneracy" can also be described as, assuming $\dim M = 2n$, $\omega^n = \underbrace{\omega \wedge \cdots \wedge \omega}_{n} \neq 0$ everywhere.

Poisson bracket $\leftrightarrow$ Symplectic form

A Poisson bracket $\{ , \}$ determines a symplectic form ω, and vice versa: For a local coordinate system $(x^I)_{I=1}^{2n}$,

$$\sum_J \{x^I, x^J\}\, \omega\left(\frac{\partial}{\partial x^J}, \frac{\partial}{\partial x^k}\right) = -\delta^I_k .$$

(i.e. $\{x^I, x^J\}$ = minus the inverse matrix of $\omega\left(\frac{\partial}{\partial x^I}, \frac{\partial}{\partial x^J}\right)$)

In a Darboux coordinate $(q^1, \cdots, q^n, p_1, \cdots, p_n)$,

- $$\{f, g\} = \sum_i \left(\frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p^i}\frac{\partial g}{\partial q_i}\right)$$
- $$\omega = \sum_i dq_i \wedge dp^i .$$

We may write the relationship in a "coordinat-free" way:

For a function f on M, let X_f be the vector field defined by

$$X_f\, g = \{f, g\}$$

for any function g on M. Then, the relationship is

$$\{f, g\} = \omega(X_f, X_g).$$

Indeed, since $X_f g = X_f^I \partial_I g$ & $\{f, g\} = \partial_k f \{x^k, x^I\} \partial_I g$

we find $X_f^I = \partial_k f \{x^k, x^I\}$ and hence

$$-\delta^K_J$$

$$\omega(X_f, X_g) = \omega_{IJ} X_f^I X_g^J = \omega_{IJ}\, \partial_K f \{x^K, x^I\} \partial_L g \{x^L, x^J\}$$

$$= -\partial_K f\, \partial_L g \{x^L, x^K\} = \partial_K f \{x^K, x^L\} \partial_L g = \{f, g\}. \checkmark$$

From the symplectic side, the vector field X_f is characterized by

$$vf = \omega(v, X_f)$$

for any vector field v on M.

A useful commutation relation

For any pair of functions f and g on M,

$$[X_f, X_g] = X_{\{f,g\}}$$

proof For another function h, Jacobi identity

$$\underbrace{\{f, \{g, h\}\}}_{X_f X_g h} + \underbrace{\{g, \underbrace{\{h, f\}}_{-\{f,h\}}\}}_{-X_g X_f h} + \underbrace{\{h, \{f, g\}\}}_{-\{\{f,g\},h\} = -X_{\{f,g\}} h} = 0$$

reads $[X_f, X_g]h = X_{\{f,g\}}h$. As h is arbitrary, this means the claimed relation. $\square$

The case of second class constraint

Let $(M, \{ , \})$ be a $2n$-dimensional phase space and

let $N \subset M$ a submanifold of 2nd class constraint.

That is, for any point x of N, there is an open neighborhood U

of x in M with a set of functions $\{\varphi^a\}_{a=1}^m$ on U such that

$N \cap U = \{ \varphi^1 = \cdots = \varphi^m = 0 \}$ and $\det\{\varphi^a, \varphi^b\} \neq 0$ on $N \cap U$.

Rmk As $\{\varphi^a, \varphi^b\}$ is an antisymmetric and invertible matrix, its

size must be even. That is, the codimension m of N in M

must be even.

Let ω be the symplectic form corresponding to $\{ , \}$. We

denote by $\omega|_N$ the restriction of ω to N. It is a closed

2-form on N : $d(\omega|_N) = (d\omega)|_N = 0$.

Theorem

(i) $\omega|_N$ is non-degenerate.

In particular, it is a symplectic form on N.

(ii) The Dirac bracket $\{ , \}_N$ is the Poisson bracket

corresponding to $\omega|_N$.

<u>proof</u> It is enough to prove the statements at/near any point p of N.

Choose local coordinates $(x^r)_{r=1}^{2n-m}$ on a neighborhood of p in N.

We can find extensions $\tilde{x}^r$ of x^r defined on a neighborhood U of p in M s.t.

$$\{\tilde{x}^r, \varphi^a\}|_{N\cap U} = 0.$$

Construction: Let $\tilde{x}'^r$ be any extension of x^r and put

$$\tilde{x}^r = \tilde{x}'^r - \{\tilde{x}'^r, \varphi^b\} D_{bc} \varphi^c.$$

Note: $\{\varphi^a, \varphi^b\}$ is invertible at p, hence is invertible in a neighborhood of p in M. D_{bc} here is the inverse.

Thus $\tilde{x}^r$ are defined on a neighborhood U of p in M and

$$\{\tilde{x}^r, \varphi^a\}|_{N\cap U} = \left(\{\tilde{x}'^r, \varphi^a\} - \{\tilde{x}'^r, \varphi^b\} \underbrace{D_{bc}\{\varphi^c, \varphi^a\}}_{\delta_b{}^a}\right)\Big|_{N\cap U}$$

$$= 0. \qquad \underline{\text{OK}}.$$

Narrowing down U if necessary, $(\tilde{x}^1, \ldots, \tilde{x}^{2n-m}, \varphi^1, \ldots, \varphi^m)$ form coordinates on U.

(i) The inverse $\begin{pmatrix} \omega_{rs} & \omega_{rb} \\ \omega_{as} & \omega_{ab} \end{pmatrix}$ of $-\begin{pmatrix} \{\tilde{x}^r, \tilde{x}^s\} & \{\tilde{x}^r, \varphi^b\} \\ \{\varphi^a, \tilde{x}^s\} & \{\varphi^a, \varphi^b\} \end{pmatrix}$

enters in to the symplectic form on U:

$$\omega|_U = \frac{1}{2}\omega_{rs}\, d\tilde{x}^r \wedge d\tilde{x}^s + \omega_{rb}\, d\tilde{x}^r \wedge d\varphi^b + \frac{1}{2}\omega_{ab}\, d\varphi^a \wedge d\varphi^b.$$

As $\{\tilde{x}^r, \varphi^a\}|_{N\cap U} = 0$, this inverse is of the form $\begin{pmatrix} \omega_{rs} & 0 \\ 0 & \omega_{ab} \end{pmatrix}$ on $N\cap U$. In particular, (ω_{rs}) is invertible on $N\cap U$.

Thus, $\omega|_{N\cap U} = \frac{1}{2}\omega_{rs}\, dx^r \wedge dx^s$ is non-degenerate.

(ii) Let $f(x)$ & $g(x)$ be functions of $x \in N\cap U$. As their extensions to U, take $\tilde{f}(\tilde{x},\varphi) = f(\tilde{x})$ & $\tilde{g}(\tilde{x},\varphi) = g(\tilde{x})$. Then

$$\{f,g\}_N|_{N\cap U} = \left(\{\tilde{f},\tilde{g}\} - \{\tilde{f},\varphi^a\}D_{ab}\{\varphi^b,\tilde{g}\}\right)\Big|_{N\cap U}$$

$$\left[\ \{\tilde{f},\varphi^a\}|_{N\cap U} = \frac{\partial f(\tilde{x})}{\partial \tilde{x}^r}\{\tilde{x}^r,\varphi^a\}\Big|_{N\cap U} = 0\right.$$

$$= \{\tilde{f},\tilde{g}\}|_{N\cap U} = \frac{\partial f}{\partial \tilde{x}^r}(\tilde{x})\frac{\partial g}{\partial \tilde{x}^s}(\tilde{x})\{\tilde{x}^r,\tilde{x}^s\}\Big|_{N\cap U}$$

$$= \frac{\partial f}{\partial x^r}(x)\frac{\partial g}{\partial x^s}(x)\{\tilde{x}^r,\tilde{x}^s\}|_{N\cap U} = -(\omega^{-1})^{rs}$$

= Poisson bracket corresponding to $\omega|_{N\cap U}$ of f & g.

Q.E.D.

The case of first class constraint

Let $(M, \{\cdot,\cdot\})$ be a phase space and let $\{\varphi^a = 0\}_{a=1}^m$ be a first class constraint. It must obey

$$\{\varphi^a, \varphi^b\} = C^{ab}_c \varphi^c$$

for some function C^{ab}_c. Here we <u>assume that C^{ab}_c is a constant.</u>

By Jacobi identity of Poisson bracket, C^{ab}_c may be regarded as the structure constants of a Lie algebra $\mathfrak{g}$:

$$[e^a, e^b] = C^{ab}_c e^c \quad \text{for a basis } \{e^a\}_{a=1}^m \subset \mathfrak{g}.$$

Recall that we have vector fields X_{φ^a} on M s.t. $X_{\varphi^a} f = \{\varphi^a, f\}$.

They satisfy $[X_{\varphi^a}, X_{\varphi^b}] = X_{\{\varphi^a, \varphi^b\}} = C^{ab}_c X_{\varphi^c}$. In other words, if we write $X_\xi = \xi_a X_{\varphi^a}$ for $\xi = \xi_a e^a \in \mathfrak{g}$,

$$[X_\xi, X_\eta] = X_{[\xi, \eta]}. \qquad \text{———} (*)$$

This means that $\mathfrak{g}$ acts on M. Also, if we write

$$\mu_\xi = \xi_a \varphi^a \quad \text{for } \xi = \xi_a e^a \in \mathfrak{g},$$

we have

$$X_\xi \mu_\eta = \mu_{[\xi, \eta]}. \qquad \text{———} (**)$$

Indeed $X_\xi \mu_\eta = \xi_a \eta_b X_{\varphi^a} \varphi^b = \xi_a \eta_b \{\varphi^a, \varphi^b\} = \underbrace{\xi_a \eta_b C^{ab}_c}_{[\xi, \eta]_c} \varphi^c.$

Another assumption the $\mathfrak{g}$ action on M integrates to an action of a connected Lie group G which has $\mathfrak{g}$ as its Lie algebra, $\mathfrak{g} = \mathrm{Lie}(G)$. By the sign of $(*)$, it must be a right action (See exercise for Lec 4):

$$X_\xi f(x) = \frac{d}{dt} f(x e^{t\xi}) \Big|_{t=0} .$$

Then $(**)$ integrates to

$$\mu_\eta(xg) = \mu_{g\eta g^{-1}}(x) \qquad \widetilde{(**)}.$$

When we view the phase space as a symplectic manifold, the structure we are having is said succinctly as

"(M, ω) has a Hamiltonian G-action"

or more specifically as

"G acts on M preserving a symplectic form ω with a moment map $\mu : M \to \mathfrak{g}^*$."

"<u>G preserves ω</u>" means $g^*\omega = \omega$ for $\forall g \in G$.

"a moment map $\mu : M \to \mathfrak{g}^*$ is a map satisfying

(i) For $\forall \xi \in \mathfrak{g}$, $d\langle \mu, \xi \rangle = -i_{X_\xi}\omega$.

(ii) It is G-equivariant.

Note $\mathfrak{g}^* := \mathrm{Hom}_{\mathbb{R}}(\mathfrak{g}, \mathbb{R})$, the space of linear forms on $\mathfrak{g}$.
There is a right G-action on $\mathfrak{g}^*$: for $f \in \mathfrak{g}^*$ and $g \in G$,
$fg \in \mathfrak{g}^*$ is given by $\langle fg, \xi \rangle = \langle f, g\xi g^{-1} \rangle$ for $\xi \in \mathfrak{g}$.

For $\xi \in \mathfrak{g}$, $\langle \mu, \xi \rangle$ is a $\mathbb{R}$-valued function on M.

Then (i) and (ii) means the following:

(i) For any vector field v on M,

$$v\langle \mu, \xi \rangle = -\omega(X_\xi, v) = \omega(v, X_\xi).$$

(ii) For $x \in M$ and $g \in G$, $\mu(xg) = \mu(x)g$. In other words,

$$\langle \mu(xg), \xi \rangle = \langle \mu(x), g\xi g^{-1} \rangle \quad \text{for } \xi \in \mathfrak{g}.$$

If we denote $\mu_\xi = \langle \mu, \xi \rangle$, these read

(i) $\quad v\mu_\xi = \omega(v, X_\xi)$.

(ii) $\quad \mu_\xi(xg) = \mu_{g\xi g^{-1}}(x)$.

Let us check that is indeed the case.

- G action preserves ω, $g^*\omega = \omega \quad \forall g$.

  As G is connected, this is equivalent to its infinitesimal form:

$$\mathcal{L}_{X_\xi}\omega = 0 \qquad \forall \xi \in \mathfrak{g}$$

  We test it for vector fields X_f, X_g:

$$(\mathcal{L}_{X_\xi}\omega)(X_f, X_g)$$

$$= \underbrace{X_\xi}_{X_{\mu_\xi}} \underbrace{\omega(X_f, X_g)}_{\{f,g\}} - \omega(\underbrace{[X_\xi, X_f]}_{[X_{\mu_\xi}, X_f] = X_{\{\mu_\xi, f\}}}, X_g) - \omega(X_f, \underbrace{[X_\xi, X_g]}_{[X_{\mu_\xi}, X_g] = X_{\{\mu_\xi, g\}}})$$

$$= \{\mu_\xi, \{f, g\}\} - \{\{\mu_\xi, f\}, g\} - \{f, \{\mu_\xi, g\}\}$$

$$= 0 \quad \text{by Jacobi identity.} \quad \checkmark$$

- $v\mu_\xi = \omega(v, X_\xi)$ follows from $X_\xi = X_{\mu_\xi}$ and $vf = \omega(v, X_f)$.

- $\mu_\xi(xg) = \mu_{g\xi g^{-1}}(x)$ is nothing but (**).

Thus, indeed, G preserves the symplectic form ω and

$\mu_\xi = \xi_a \varphi^a$ defines a moment map $\mu: M \to \mathfrak{g}^*$.

Conversely, if there is a G-action on M preserving ω with a moment map $\mu: M \to \mathfrak{g}^*$, it defines a 1st class constraint on the corresponding phase space: $\varphi^a := \mu e^a$ for a basis $\{e^a\}_{a=1}^m$ of $\mathfrak{g} = \mathrm{Lie}(G)$.

Symplectic quotients

Let (M, ω) be a symplectic manifold and suppose a Lie group G acts on M preserving ω with a moment map $\mu: M \to \mathfrak{g}^*$.

Under some assumption, another symplectic manifold called the symplectic quotient is associated to this.

First, we consider the subspace $\mu^{-1}(0) = \{x \in M \mid \mu(x) = 0\} \subset M$.

By G-equivariance of μ, if $\mu(x) = 0$ then $\mu(xg) = 0$.

Thus G acts on the subspace $\mu^{-1}(0)$.

<u>Assumption: the G action on $\mu^{-1}(0)$ is free</u>

Then at any $x \in \mu^{-1}(0)$, $d\mu_x : T_x M \to \mathfrak{g}^*$ is surjective.

$\because$ For $v \in T_x M$ and $\xi \in \mathfrak{g}$, $\langle d\mu_x(v), \xi \rangle = \omega(v, X_\xi(x))$.

As the G-action is free, $\xi \in \mathfrak{g} \mapsto X_\xi(x) \in T_x M$ is injective.

By non-degeneracy of ω, $\{d\mu_x(v) \mid v \in T_x M\}$ spans $\mathfrak{g}^*$. //

Thus the condition $\mu = 0$ cuts down d.o.f. just by $\dim \mathfrak{g}$.

$\mu^{-1}(0) \subset M$ is a submanifold of dimension $\dim M - \dim G$.

We make an additional technical assumption that

$\mu^{-1}(0)/G$ has a structure of manifold and the quotient map

$\pi : \mu^{-1}(0) \longrightarrow \mu^{-1}(0)/G$ defines a principal G-bundle.

(This is automatic if G is compact.)

The task is to construct a symplectic form on $\mu^{-1}(0)/G$.

<u>Preparation</u> For $x \in \mu^{-1}(0)$, $\mathfrak{g}_x := \{ X_\xi(x) \mid \xi \in \mathfrak{g} \} \subset T_x \mu^{-1}(0)$,

$$\omega_x \big|_{T_x \mu^{-1}(0) \times \mathfrak{g}_x} = 0$$

[☺ $\forall v \in T_x \mu^{-1}(0)$, $\forall \xi \in \mathfrak{g}$, $\omega(v, X_\xi(x)) = \langle d\mu_x(v), \xi \rangle = v\mu_\xi$.

But μ_ξ is constantly 0 in $\mu^{-1}(0)$ and hence $v\mu_\xi = 0$ $\square$

<u>Construction</u> For $p \in \mu^{-1}(0)/G$ and $v, w \in T_p(\mu^{-1}(0)/G)$, choose

arbitrary lifts $x \in \mu^{-1}(0)$ and $\tilde{v}, \tilde{w} \in T_x \mu^{-1}(0)$ ("lift"

means $\pi(x) = p$ and $\pi_* \tilde{v} = v$, $\pi_* \tilde{w} = w$) and put

$$\tilde{\omega}_p(v, w) = \omega_x(\tilde{v}, \tilde{w}).$$

<u>Need to check</u> (i) independent of the choice of lifts.

(ii) closed, $d\bar{\omega} = 0$

(iii) nondegenerate.

(i). Choice of $\tilde{v}$ & $\tilde{w}$ (same $x \in \pi^{-1}(p)$):

Another choice of $\tilde{v}$ is $\tilde{v} + X_\xi(x)$ for some $\xi \in \mathfrak{g}$.

But this does not change $\omega_x(v, w)$ by <u>preparation</u>.

Same for $\tilde{w}$.

· Choice of $x \in \pi^{-1}(p)$:

Another choice is xg for some $g \in G$.

As the lifts of v & w, we may take $\tilde{v}g$, $\tilde{w}g$.

Then $\omega_{g(x)}(\tilde{v}g, \tilde{w}g) = (g^*\omega)_x(\tilde{v}, \tilde{w}) = \omega_x(\tilde{v}, \tilde{w})$. OK

(ii) $d\bar{\omega} = 0$?

Take $v_1, v_2, v_3 \in T_p(\mu^{-1}(0)/G)$. Extend them to vector fields on a neighborhood U of p with lifts to vector fields $\tilde{v}_1, \tilde{v}_2, \tilde{v}_3$ on $\pi^{-1}(U)$ s.t. $[\tilde{v}_i, \tilde{v}_j]$ are lifts of $[v_i, v_j]$. This is possible by taking U s.t. $\pi^{-1}(U) \to U$ is trivializable as a G-bundle, $\pi^{-1}(U) \cong U \times G \xrightarrow{\text{left}} U$.

$$d\bar{\omega}(v_1, v_2, v_3) = v_1\,\bar{\omega}(v_2, v_3) + \text{cyclic} - \bar{\omega}([v_1, v_2], v_3) - \text{cyclic}$$

$$= \tilde{v}_1\,\omega(\tilde{v}_2, \tilde{v}_3) + \text{cyclic} - \omega([\tilde{v}_1, \tilde{v}_2], \tilde{v}_3) - \text{cyclic}$$

$$= d\omega(\tilde{v}_1, \tilde{v}_2, \tilde{v}_3) = 0.$$

$\therefore\ d\bar{\omega} = 0.$ OK.

(iii) $\bar{\omega}$ non-degenerate ?

Take $x \in \mu^{-1}(0)$.

As ω_x is non-degenerate on $T_xM \times T_xM$ and 0 on $\mathfrak{g}_x \times T_x\mu^{-1}(0)$

and since $N_x\mu^{-1}(0) = T_xM / T_x\mu^{-1}(0)$ has the same dimension as $\mathfrak{g}_x$,

ω_x induces a non-degenerate bilinear form on $N_x\mu^{-1}(0) \times \mathfrak{g}_x$.

Thus, for the decomposition

$$T_xM \cong T_x\mu^{-1}(0) \oplus N_x\mu^{-1}(0) \cong T_x\mu^{-1}(0)/\mathfrak{g}_x \oplus \mathfrak{g}_x \oplus N_x\mu^{-1}(0)$$

ω_x is represented by a matrix

$$\left(\begin{array}{c|c|c} * & 0 & * \\ \hline 0 & 0 & * \\ \hline * & * & * \end{array}\right).$$

maximal rank

This means that ω_x is non-degenerate on $T_x\mu^{-1}(0)/\mathfrak{g}_x \times T_x\mu^{-1}(0)/\mathfrak{g}_x$.

That is, $\bar{\omega}_p : T_p\mu^{-1}(0)/G \times T_p\mu^{-1}(0)/G \to \mathbb{R}$ is non-degenerate.

OK.

(i), (ii), (iii) are all checked.

Thus, $\bar{\omega}$ is a symplectic form on $\mu^{-1}(0)/G$.

The symplectic manifold $(\mu^{-1}(0)/G, \bar{\omega})$ is called the symplectic quotien of (M, ω) by G action with moment map μ.

Reduced phase space = symplectic quotient

Finally, we would like to show that, under the assumptions made,

the reduced phase space $(M^*, \{,\}_{M^*})$ corresponds to the symplectic quotient $(\mu^{-1}(0)/G, \bar{\omega})$.

Note that $N = \{\varphi^1 = \cdots = \varphi^m = 0\} = \{\mu = 0\} = \mu^{-1}(0)$ and the equivalence relation $\sim$ in N is equal to G-equivalence.

Thus $M^* = N/\sim \; = \mu^{-1}(0)/G$.

Therefore, the question is whether the Poisson bracket $\{,\}_{M^*}$ corresponds to the symplectic form $\bar{\omega}$. This is judged by examining, say, for functions f and g on M^*, whether

$$\{f, g\}_{M^*} \overset{?}{=} \bar{\omega}(X_f, X_g).$$

Recall that, if $\tilde{f}$ and $\tilde{g}$ are extension of f and g to a neighborhood of N in M s.t. $\{\tilde{f}, \varphi^a\} = f^a_b \varphi^b$, $\{\tilde{g}, \varphi^a\} = g^a_b \varphi^b$, then

$$\{f, g\}_{M^*} = \{\tilde{f}, \tilde{g}\} = \omega(X_{\tilde{f}}, X_{\tilde{g}}).$$

Recall also that, if $\widetilde{X}_f$ and $\widetilde{X}_g$ are lifts of X_f and X_g to $\mu^{-1}(0)$,

$$\bar{\omega}(X_f, X_g) = \omega(\widetilde{X}_f, \widetilde{X}_g).$$

Thus the question is whether $X_{\tilde{f}}$ and $X_{\tilde{g}}$ are lifts of X_f and X_g.

We first consider $X_{\tilde{f}}$.

Is $X_{\tilde{f}}$ tangent to $\mu^{-1}(0) = N$?

The answer is YES as $X_{\tilde{f}} \varphi^a = \{\tilde{f}, \varphi^a\} = f^a_b \varphi^b = 0$ on N.

Then, $\pi_* X_{\tilde{f}} = X_f$?

X_f is characterized by $v f = \bar{\omega}(v, X_f)$ $\forall v \in T_p(\mu^{-1}(0)/G)$,

and $X_{\tilde{f}}$ is characterized by $\tilde{v} \tilde{f} = \omega(\tilde{v}, X_{\tilde{f}})$ $\forall \tilde{v} \in T_x M$.

For any $v \in T_p(\mu^{-1}(0)/G)$ take a lift $\tilde{v} \in T_x \mu^{-1}(0)$, $x \in \pi^{-1}(p)$.

Then, as $\tilde{f} = f \circ \pi$, $\tilde{v}\tilde{f} = \tilde{v}(f \circ \pi) = \pi_* \tilde{v} f = vf$.

$\therefore vf = \tilde{v}\tilde{f} = \omega(\tilde{v}, X_{\tilde{f}}) = \bar{\omega}(\pi_* \tilde{v}, \pi_* X_{\tilde{f}}) = \bar{\omega}(v, \pi_* X_{\tilde{f}})$.

This shows $\pi_* X_{\tilde{f}} = X_f$ indeed, hence $X_{\tilde{f}}$ is a lift of X_f.

Similarly, $X_{\tilde{g}}$ is a lift of X_g.

$\therefore \bar{\omega}(X_f, X_g) = \omega(X_{\tilde{f}}, X_{\tilde{g}}) = \{\tilde{f}, \tilde{g}\} = \{f, g\}_{M^*}$.

The Poisson bracket $\{\ ,\ \}_{M^*}$ corresponds to the symplectic form $\bar{\omega}$.

$\square$