Symmetry twisted partition function

Consider a mechanics with a set of variables

$q = (q_1, \cdots, q_n)$ and Lagrangian $L(q, \dot{q})$.

Suppose there is a 1-parameter group of symmetries g_α,

and let Q be the corresponding Noether charge.

(Noethe charge corresponding to $\delta q = \frac{d}{d\alpha} g_\alpha(q) \Big|_{\alpha=0}$)

Show that

$$\mathrm{Tr}_{\mathcal{H}}\, e^{i\frac{\alpha}{\hbar}\hat{Q}}\, e^{-\frac{T}{\hbar}\hat{H}}$$

$$= \int_{q(\tau+T) = g_\alpha(q(\tau))} \mathcal{D}q\; e^{-\frac{1}{\hbar}\int_{S^1_T} d\tau\, L_E\left(q, \frac{dq}{d\tau}\right)}$$

$$=: Z_\alpha(S^1_T)$$

This is the symmetry-twisted partition function.

Examples

Compute $\mathrm{Tr}_{\mathcal{H}}\, e^{\frac{i\alpha}{\hbar}\hat{Q}}\, e^{-\frac{T}{\hbar}\hat{H}}$ in operator formalism

and $Z_\alpha(S^1_T)$ in path-integral,

and check that the results agree, in the following examples :

① 2d harmonic oscillator and rotational symmetry

Variables : $q = (q_1, q_2)$

Lagrangian : $L = \frac{m}{2}(\dot{q}_1^2 + \dot{q}_2^2) - \frac{m\omega^2}{2}(q_1^2 + q_2^2)$

Symmetry : $g_\alpha\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$ $\qquad \alpha \sim \alpha + 2\pi$

② Free particle in a circle (with a theta term)

Variable $q \sim q + 2\pi R$

Lagrangian : $L = \frac{m}{2}\dot{q}^2 \left(+ \theta \dot{q}\right)$

Symmetry : $g_\alpha(q) = q + \alpha$ $\qquad \alpha \sim \alpha + 2\pi R$

③ Free particle in a plane

Variables : $q = (q_1, q_2)$

Lagrangian : $L = \frac{m}{2}(\dot{q}_1^2 + \dot{q}_2^2)$

Symmetry : $g_\alpha \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$ $\qquad \alpha \sim \alpha + 2\pi$

※ In this example, the partition function was ill-defined.
(See example ① for lecture 1.)

However, the twisted partition function is well-defined as long as the twist is non-trivial, $\alpha \neq 0$.

This is an important use of twisted partition function.

A possibly convenient fact :

Generalized zeta function $\zeta(s,a) = \sum_{n=0}^{\infty} (n+a)^{-s}$

also is absolutely convergent for $\mathrm{Re}(s) > 1$ and has analytic continuation which is regular at $s=0$, with

$$\zeta(0,a) = \frac{1}{2} - a, \quad \zeta'(0,a) = \log\Gamma(a) - \frac{1}{2}\log 2\pi.$$

Here $\Gamma(a)$ is Gamma function.

(A convenient property : $\Gamma(a+1) = a\Gamma(a)$, $\Gamma(a)\Gamma(1-a) = \dfrac{\pi}{\sin\pi a}$.)

Using this

$$\prod_{n=1}^{\infty} \frac{X}{n+y} = \exp\left(\sum_{n=1}^{\infty} \left(\log X - \log(n+y)\right) \right)$$

$$= \exp\left(\log X \cdot \zeta(0,1) + \zeta'(0,1+y) \right)$$

$$= \exp\left(-\frac{1}{2}\log X + \Gamma(1+y) - \frac{1}{2}\log 2\pi \right)$$

$$= \frac{\Gamma(1+y)}{\sqrt{2\pi X}}.$$