Some useful formulae

1 Exponential integral

We often encounter an integral of the form

$$E_1(z) \;:=\; \int_z^\infty \mathrm{e}^{-t}\,\frac{\mathrm{d}t}{t}, \tag{1.1}$$

especially for small positive real z. A useful formula for this is

$$E_1(z) \;=\; -\log(z) - \gamma + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n \cdot n!} z^n, \tag{1.2}$$

where γ is Euler's constant (A.13). The power series is absolutely convergent for any $z \in \mathbf{C}$ as seen by the ratio test. The formula can be proved as follows.

$$\begin{aligned}
E_1(z) &= E_1(1) + \int_z^1 \mathrm{e}^{-t}\,\frac{\mathrm{d}t}{t} \\
&\qquad \left[\text{ note } \frac{\mathrm{e}^{-t}}{t} = \frac{1}{t} + \frac{\mathrm{e}^{-t}-1}{t} \right] \\
&= E_1(1) - \log(z) + \int_z^1 \frac{\mathrm{e}^{-t}-1}{t}\mathrm{d}t \\
&= -\log(z) + \underbrace{E_1(1) + \int_0^1 \frac{\mathrm{e}^{-t}-1}{t}\mathrm{d}t}_{=:\,C} - \int_0^z \frac{\mathrm{e}^{-t}-1}{t}\mathrm{d}t.
\end{aligned} \tag{1.3}$$

The last integral gives the convergent series in (1.2),

$$-\int_0^z \frac{\mathrm{e}^{-t}-1}{t}\mathrm{d}t = -\int_0^z \sum_{n=1}^{\infty} \frac{1}{n!}\frac{(-t)^n}{t}\mathrm{d}t = -\sum_{n=1}^{\infty} \frac{1}{n!}\frac{(-z)^n}{n}. \tag{1.4}$$

The constant part is

$$\begin{aligned}
C &= \int_1^\infty \mathrm{e}^{-t}\,\frac{\mathrm{d}t}{t} + \int_0^1 (\,\mathrm{e}^{-t}-1)\frac{\mathrm{d}t}{t} \\
&= \int_1^\infty \mathrm{e}^{-t}\mathrm{d}\log(t) + \int_0^1 (\,\mathrm{e}^{-t}-1)\mathrm{d}\log(t) \\
&= -\int_1^\infty \mathrm{d}\,\mathrm{e}^{-t}\cdot\log(t) - \int_0^1 \mathrm{d}(\,\mathrm{e}^{-t}-1)\log(t) \\
&= \int_0^\infty \mathrm{d}t\,\mathrm{e}^{-t}\log(t).
\end{aligned} \tag{1.5}$$

In view of

$$\Gamma'(z) = \frac{\mathrm{d}}{\mathrm{d}z}\int_0^\infty \mathrm{e}^{-t}t^{z-1}\mathrm{d}t = \int_0^\infty \mathrm{e}^{-t}\log(t)t^{z-1}\mathrm{d}t, \tag{1.6}$$

this is nothing but $\Gamma'(1)$ which is found to be $-\gamma$ (see (A.12) and (A.14)).

2 Volume of the unit sphere

We give some expressions for the volume $Vol(S^{d-1})$ of the unit sphere S^{d-1} in $\mathbf{R}^d$. The Gaussian integral

$$I_d := \int_{\mathbf{R}^d} \frac{\mathrm{d}^d x}{(2\pi)^d}\, \mathrm{e}^{-\|x\|^2} = \left(\int_{\mathbf{R}} \frac{\mathrm{d}x}{2\pi}\, \mathrm{e}^{-x^2} \right)^d = \left(\frac{\sqrt{\pi}}{2\pi} \right)^d = \frac{1}{(4\pi)^{\frac{d}{2}}} \tag{2.1}$$

can be computed in polar coordinates as

$$\begin{aligned} I_d &= \frac{Vol(S^{d-1})}{(2\pi)^d} \int_0^\infty r^{d-1}\mathrm{d}r\, \mathrm{e}^{-r^2} \\ &\qquad \left[\text{substitute } r^2 \to t \right] \\ &= \frac{Vol(S^{d-1})}{2(2\pi)^d} \int_0^\infty t^{\frac{d}{2}-1}\mathrm{d}t\, \mathrm{e}^{-t} = \frac{Vol(S^{d-1})}{2(2\pi)^d} \Gamma\left(\tfrac{d}{2}\right). \end{aligned} \tag{2.2}$$

Comparing the two expressions, we find

$$\frac{Vol(S^{d-1})}{2(2\pi)^d} = \frac{1}{(4\pi)^{\frac{d}{2}}\, \Gamma\left(\frac{d}{2}\right)}. \tag{2.3}$$

The combination on the left hand side appears quite often in the loop integrals, and this formula is useful. Using duplication formula (A.5) with $z = \frac{d-1}{2}$,

$$\Gamma(d-1)\, Vol(S^{d-1}) = \frac{2^{d-1}}{\sqrt{4\pi}} \Gamma\left(\tfrac{d-1}{2}\right) \Gamma\left(\tfrac{d}{2}\right) \cdot \frac{2(2\pi)^d}{(4\pi)^{\frac{d}{2}}\, \Gamma\left(\frac{d}{2}\right)} = (4\pi)^{\frac{d-1}{2}} \Gamma\left(\tfrac{d-1}{2}\right),$$

we obtain an alternative expression

$$Vol(S^{d-1}) = (4\pi)^{\frac{d-1}{2}} \frac{\Gamma(\frac{d-1}{2})}{\Gamma(d-1)}. \tag{2.4}$$

For integer values of the dimension,

$$Vol(S^d) = (4\pi)^{\frac{d}{2}} \frac{\Gamma(\frac{d}{2})}{\Gamma(d)} = \begin{cases} \dfrac{2(2\pi)^m}{(2m-1)!!} & \text{for } d = 2m, \\[2ex] \dfrac{(2m-1)!!(2\pi)^{m+1}}{(2m)!} & \text{for } d = 2m+1, \end{cases} \tag{2.5}$$

where we used the values (A.7) and (A.10) of Gamma function.

3 Feynman's parametric formula

$$\frac{1}{AB} = \int_0^1 \frac{\mathrm{d}x}{(xA + (1-x)B)^2}. \tag{3.1}$$

Proof:

$$\begin{aligned}
\frac{1}{AB} &= \int_0^\infty \mathrm{d}\alpha\ \mathrm{e}^{-\alpha A} \int_0^\infty \mathrm{d}\beta\ \mathrm{e}^{-\beta B} \\
&\qquad \left[\ \text{insert } \textstyle\int_0^\infty \mathrm{d}\lambda\, \delta(\lambda - \alpha - \beta) = 1\ \right] \\
&= \int_0^\infty \int_0^\infty \int_0^\infty \mathrm{d}\alpha\, \mathrm{d}\beta\, \mathrm{d}\lambda\, \delta(\lambda - \alpha - \beta)\, \mathrm{e}^{-\alpha A - \beta B} \\
&\qquad \left[\ \text{substitute } \alpha \to \lambda x,\ \beta \to \lambda y\ \right] \\
&= \int_0^\infty \int_0^\infty \int_0^\infty \mathrm{d}x\, \mathrm{d}y\, \lambda^2 \mathrm{d}\lambda\, \delta(\lambda(1 - x - y))\ \mathrm{e}^{-\lambda(xA + yB)} \\
&\qquad \left[\ \text{use } \delta(\lambda(1 - \cdots)) = \lambda^{-1}\delta(1 - \cdots) \text{ and substitute } \lambda(xA + yB) \to \lambda'\ \right] \\
&= \int_0^\infty \int_0^\infty \frac{\mathrm{d}x\, \mathrm{d}y}{(xA + yB)^2} \delta(1 - x - y) \int_0^\infty \lambda' \mathrm{d}\lambda'\ \mathrm{e}^{-\lambda'} \\
&= \int_0^1 \frac{\mathrm{d}x}{(xA + (1-x)B)^2}.
\end{aligned}$$

Similarly,

$$\frac{1}{A_1 \cdots A_n} = \int_0^\infty \!\!\cdots\! \int_0^\infty \frac{\mathrm{d}x_1 \cdots \mathrm{d}x_n\, \delta(1 - x_1 - \cdots - x_n)}{(x_1 A_1 + \cdots + x_n A_n)^n} (n-1)!, \tag{3.2}$$

$$\frac{1}{A_1^{p_1} \cdots A_n^{p_n}} = \int_0^\infty \!\!\cdots\! \int_0^\infty \frac{x_1^{p_1 - 1} \mathrm{d}x_1 \cdots x_n^{p_n - 1} \mathrm{d}x_n\, \delta(1 - x_1 - \cdots - x_n)}{(x_1 A_1 + \cdots + x_n A_n)^{p_1 + \cdots + p_n}} \frac{\Gamma(p_1 + \cdots + p_n)}{\Gamma(p_1) \cdots \Gamma(p_n)}. \tag{3.3}$$

Appendix

A Gamma function

The gamma function is defined by

$$\Gamma(z) \ := \ \int_0^\infty \mathrm{e}^{-t}\, t^{z-1} \mathrm{d}t. \tag{A.1}$$

To be precise, the integral on the right hand side is convergent for $\mathrm{Re}(z) > 0$ and $\Gamma(z)$ is defined as its analytic continuation. It is a nowehere vanishing meromorphic function on the whole complex plane with simple poles at non-positive integers. There is an alternative definition as an infinite product

$$\Gamma(z) = \lim_{n\to\infty} \frac{n^z (n-1)!}{z(z+1)\cdots(z+n-1)}. \tag{A.2}$$

Functional equations:

$$\Gamma(z+1) = z\Gamma(z). \tag{A.3}$$

This follows from partial integration in (A.1) for $\mathrm{Re}(z) > 0$ and is obvious in (A.2) for all z. We also have Euler's reflection formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}, \tag{A.4}$$

and Legendre's duplication formula

$$\Gamma(2z) = \frac{2^{2z}}{\sqrt{4\pi}} \Gamma(z)\Gamma(z+\tfrac{1}{2}). \tag{A.5}$$

Special values:
By (A.1),

$$\Gamma(1) \ = \ 1. \tag{A.6}$$

Combining this with (A.3), we find for $n = 0, 1, 2, \ldots$

$$\Gamma(n+1) \ = \ n!, \tag{A.7}$$

$$\Gamma(-n+z) \ = \ \frac{1}{z(z-1)\cdots(z-n)} \Gamma(z+1) \sim \frac{(-1)^n}{n!z} \qquad z \sim 0. \tag{A.8}$$

By (A.1) and Gaussian integral,

$$\Gamma\left(\frac{1}{2}\right) \ = \ \sqrt{\pi}, \tag{A.9}$$

and for $n = 0, 1, 2, \ldots$

$$\begin{aligned}
\Gamma\left(n+\tfrac{1}{2}\right) &= \left(n-\tfrac{1}{2}\right)\left(n-\tfrac{3}{2}\right)\cdots\tfrac{3}{2}\cdot\tfrac{1}{2}\Gamma(\tfrac{1}{2}) = \tfrac{(2n-1)!!}{2^n}\sqrt{\pi} && \text{(A.10)}\\
\Gamma\left(-n+\tfrac{1}{2}\right) &= \frac{1}{\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\cdots\left(\frac{1}{2}-n\right)}\Gamma(\tfrac{1}{2}) = \tfrac{(-2)^n}{(2n-1)!!}\sqrt{\pi}. && \text{(A.11)}
\end{aligned}$$

Also,

$$\Gamma'(1) = -\gamma, \tag{A.12}$$

where γ is *Euler's constant*

$$\gamma := \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log(n)\right) = 0.57721\ldots. \tag{A.13}$$

This can be seen from the product formula (A.2),

$$\begin{aligned}
\Gamma'(1) &= \lim_{n\to\infty}\frac{n^z(n-1)!}{\prod_{k=0}^{n-1}(z+k)}\left(\log(n) - \sum_{k=0}^{n-1}\frac{1}{z+k}\right)\Bigg|_{z=1}\\
&= \lim_{n\to\infty}\left(\log(n) - \sum_{k=0}^{n-1}\frac{1}{1+k}\right) = -\gamma. && \text{(A.14)}
\end{aligned}$$

The special values $\Gamma(1) = 1$ and $\Gamma'(1) = -\gamma$ means

$$\Gamma(1+z) = 1 - \gamma z + O(z^2). \tag{A.15}$$

Combined with $\Gamma(z+1) = z\Gamma(z)$ or (A.8), we obtain

$$\begin{aligned}
\Gamma(z) &= \frac{1}{z} - \gamma + O(z), && \text{(A.16)}\\
\Gamma(-1+z) &= -\frac{1}{z} + \gamma - 1 + O(z), && \text{(A.17)}\\
\ldots\\
\Gamma(-n+z) &= \frac{(-1)^n}{n!}\left(\frac{1}{z} - \gamma + 1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) + O(z), && \text{(A.18)}\\
\ldots
\end{aligned}$$

Stirling's formula

Asymptotic expansion as $|z| \to \infty$

$$\Gamma(z) \sim \sqrt{2\pi}\,\mathrm{e}^{(z-\frac{1}{2})\log(z)-z}\left(1 + \frac{1}{12z} + \frac{1}{288z^2} + \cdots\right) \tag{A.19}$$

for a fixed $\mathrm{Arg}(z) = \mathrm{Im}\log(z)$ in $(-\pi, \pi)$, Related functions

Beta function

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\mathrm{d}x = \int_0^\infty \frac{y^{p-1}\mathrm{d}y}{(1+y)^{p+q}} \tag{A.20}$$

can be expressed by Gamma function as

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}. \tag{A.21}$$