Brane Tilings and Homological Mirror Symmetry

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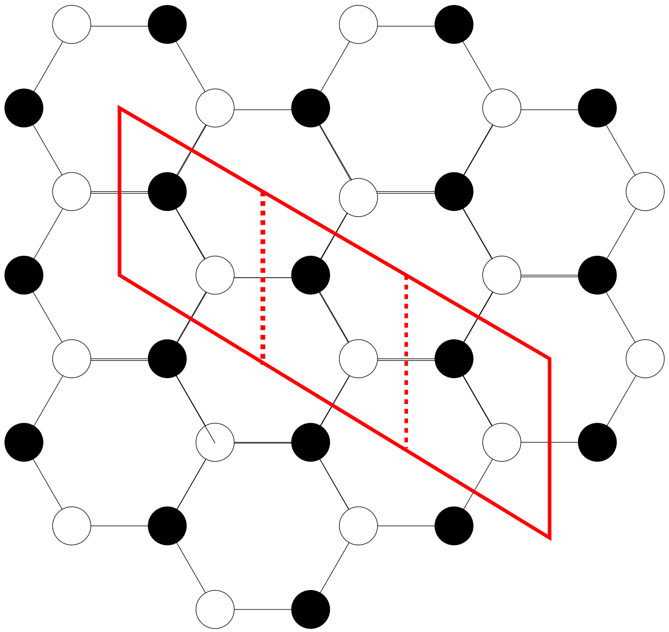
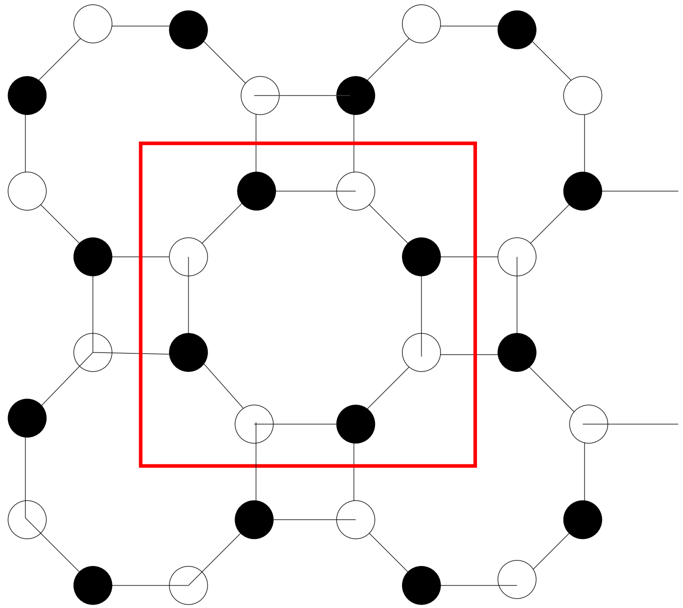
Based on math.AG/0605780, 0606548, 07?????
(In collaboration with Kazushi Ueda)

Introduction

- Recent development: "Brane Tiling" (introduced in 2005), a bipartite graph (dimer) on torus:



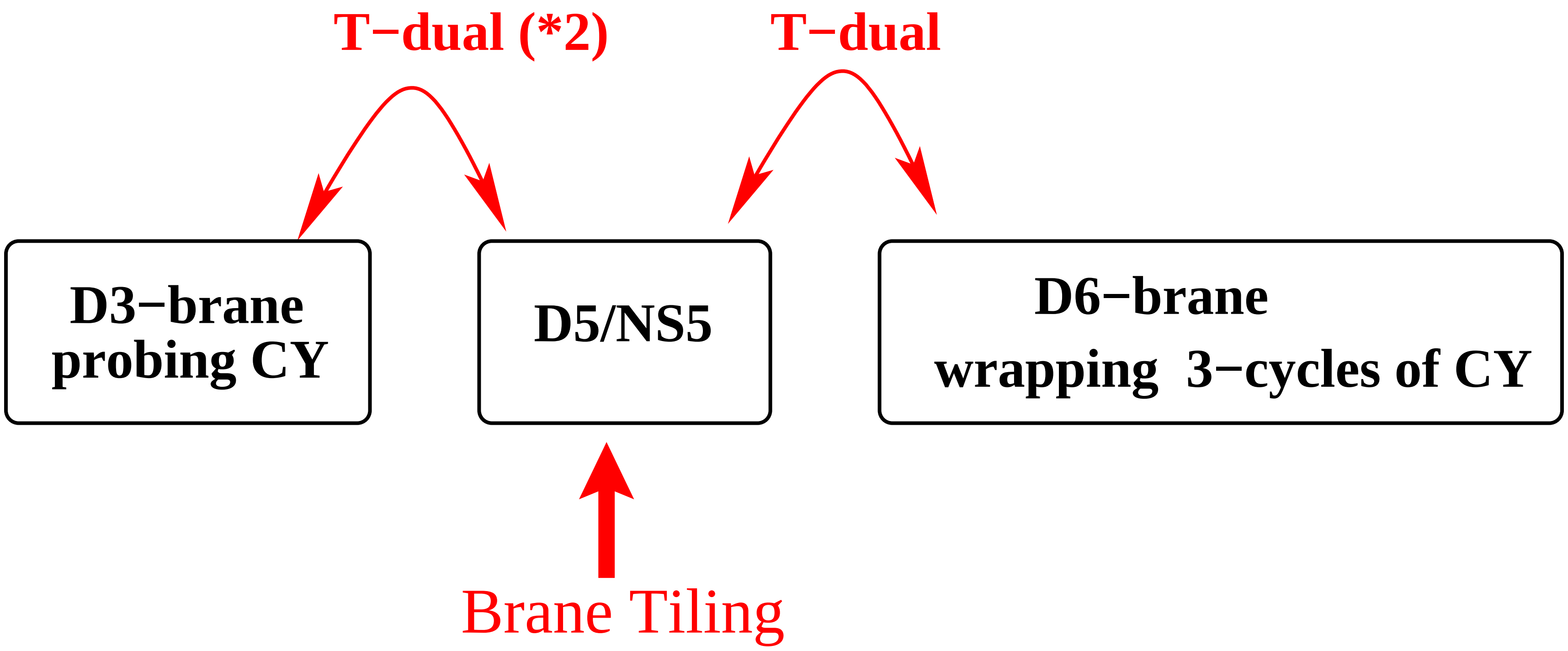


Brane Tiling

- As I will explain, *brane tilings* represent configuration of D5-branes and NS5-branes.
- On the world-volume of these D5-branes, we have $\mathcal{N} = 1$ superconformal *quiver gauge theory*, a version of gauge theory specified by an oriented graph (*quiver*).
- The brane tiling technique makes it possible to construct brane configurations for complicated quiver gauge theories.

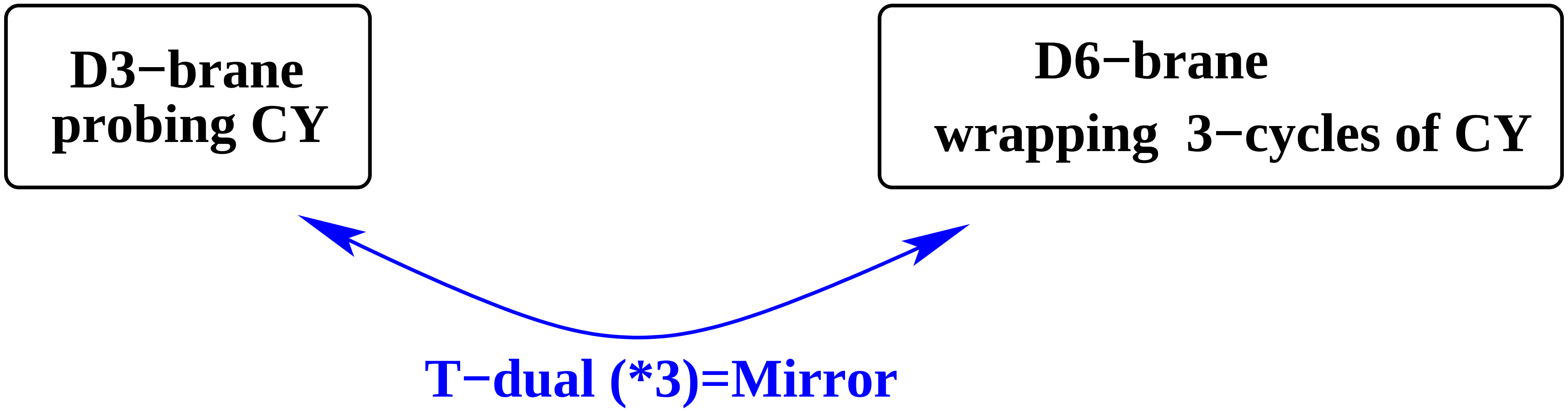
Various Realizations in String Theory

- Actually, quiver gauge theories have several realizations in string theory, which are all related by T-duality, and brane tiling is only one of them. (D3, D4/NS5, D5/NS5, D6...)



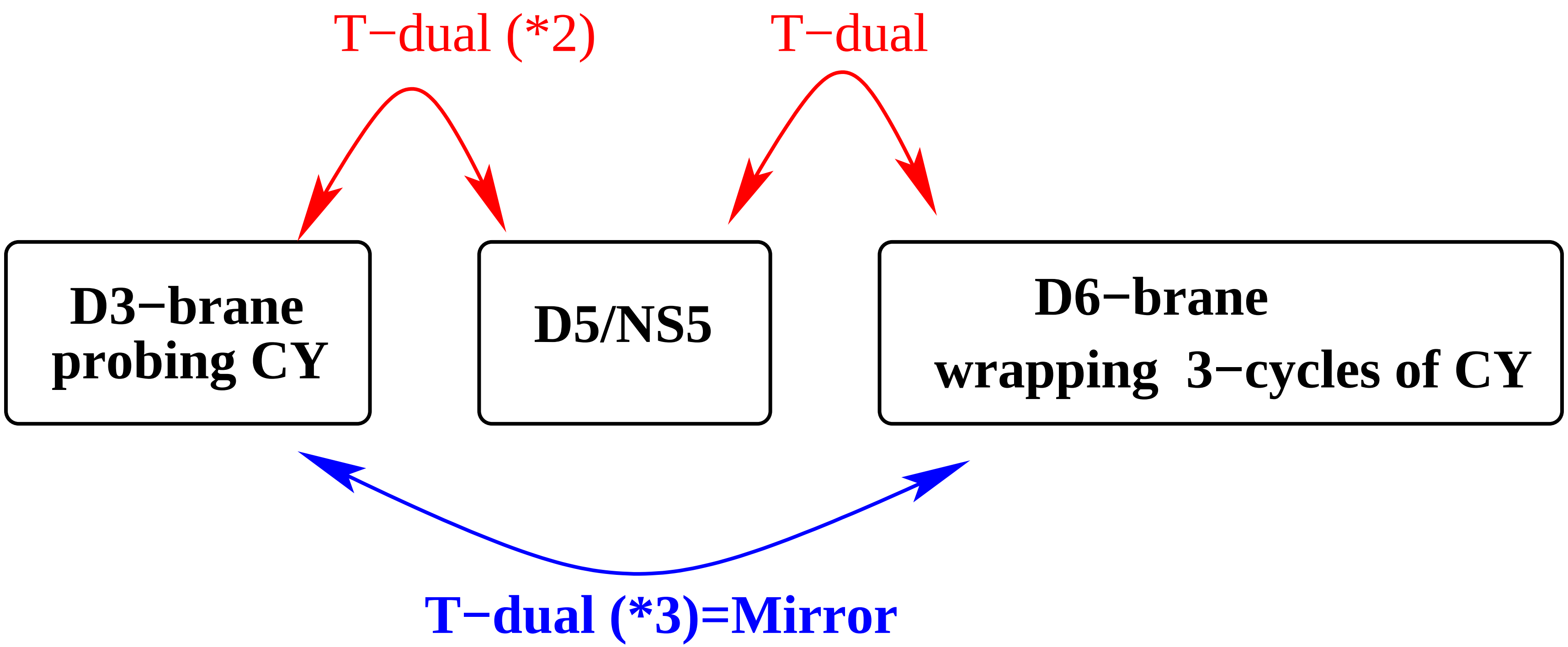
Various Realizations in String Theory

- Among these, D3-brane picture and D6-brane picture are related by T-duality 3 times, which is nothing but mirror transformation.



Various Realizations in String Theory

- This means D5/NS5-system (brane tiling) might be useful for investigating mirror symmetry.



- Our conclusion is that this is indeed the case!!

Today's Goal

Use Brane Tilings to Prove (Homological) Mirror Symmetry

We will show that this method gives

- Intuitive understanding from D-brane perspective
- Rigorous mathematical proofs
- Generalization to orbifold case (new result)

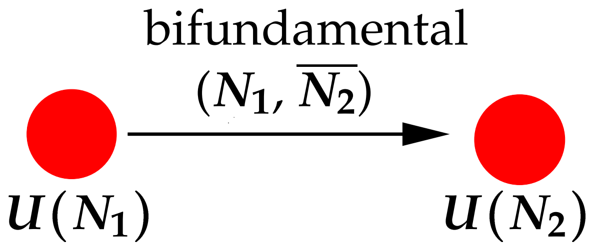
Plan of This Talk

1. Introduction
2. D-brane Realizations of Quiver Gauge Theories
   (a) D3-brane Probing CY
   (b) D5/NS5-picture (*Brane Tiling*)
   (c) D6-brane Wrapping 3-cycles of CY
3. Proof of Homological Mirror Symmetry
4. Summary and Outlook

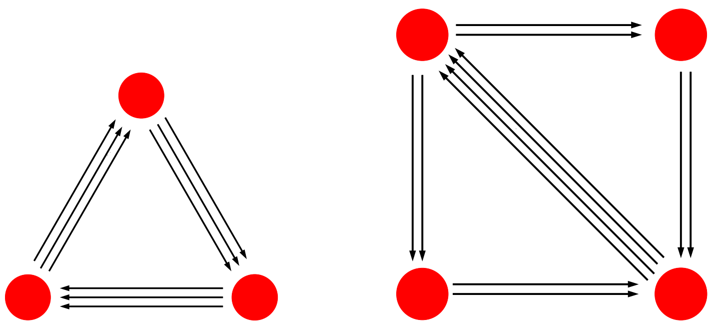
What is Quiver Gauge Theory?

Quiver(箙): "portable case for holding arrows", an oriented graph

- vertex= gauge group
- oriented arrow= bifundamental



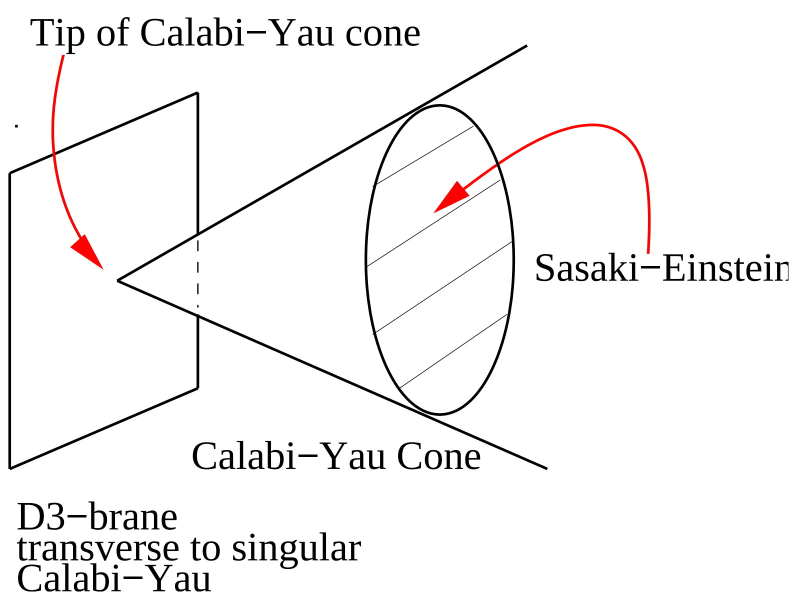
- each quiver specifies gauge theory



$\mathbb{C}^3/\mathbb{Z}_3$ $\qquad$ $\mathbb{F}_0 = \mathbb{P}_1 \times \mathbb{P}_1$

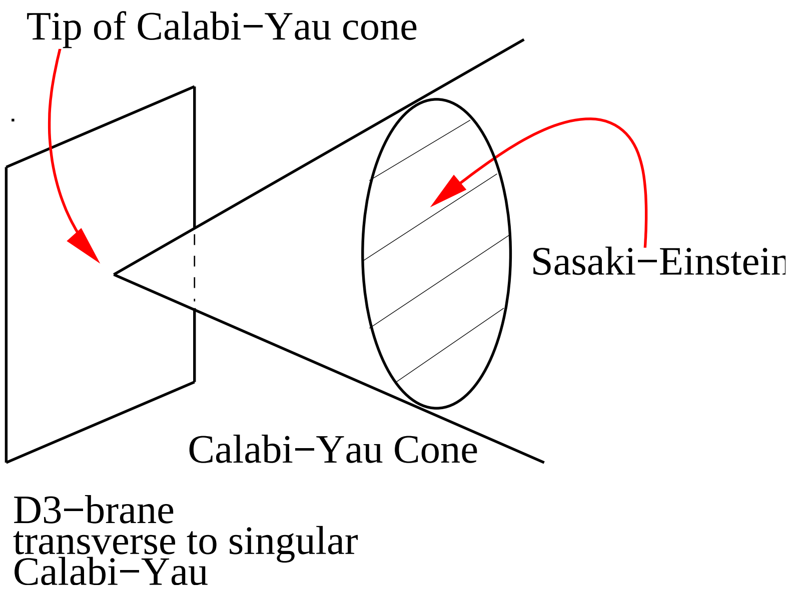
D3-brane Probing Singular Calabi-Yau

- Consider non-compact CY which has cone singularity.
- We further assume that CY is toric (specified by toric diagram).
- D3-brane is transverse to CY and placed at the apex of CY cone.



D3-brane Probing Singular Calabi-Yau

- Then it is long belived that we have 4d $\mathcal{N} = \mathbf{1}$ quiver gauge theory on D3-brane.
- Q: Which CY gives which quiver (and superpotential) ?
  A: Given by brane tiling



D5/NS5-System

- If we T-dualize along 2-cycle of CY, then we have configuration of D5-branes and NS5-branes:

	0	1	2	3	4	5	6	7	8	9
D5	○	○	○	○		○		○		
NS5	○	○	○	○	Σ (2-dim surface)					

  - D5-brane worldvolume: $\mathbf{R}^4 \times T^2$
  - NS5-brane worldvolume: $\mathbf{R}^4 \times \Sigma$

- Real shape of branes: difficult to determine in general (we need to solve EOM), but can be analyzed when $g_s \to 0$ and $g_s \to \infty$.

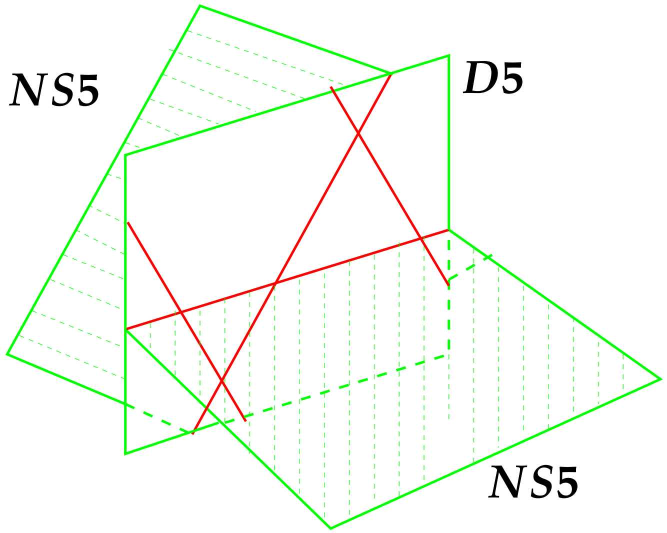
Strong Coupling

- Consider the strong coupling limit $g_s \to \infty$. Then

$$T_{D5} \gg T_{NS5}$$

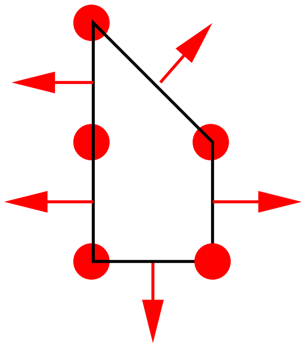
  Then D5-branes become flat and NS5-branes are orthogonal to D5.

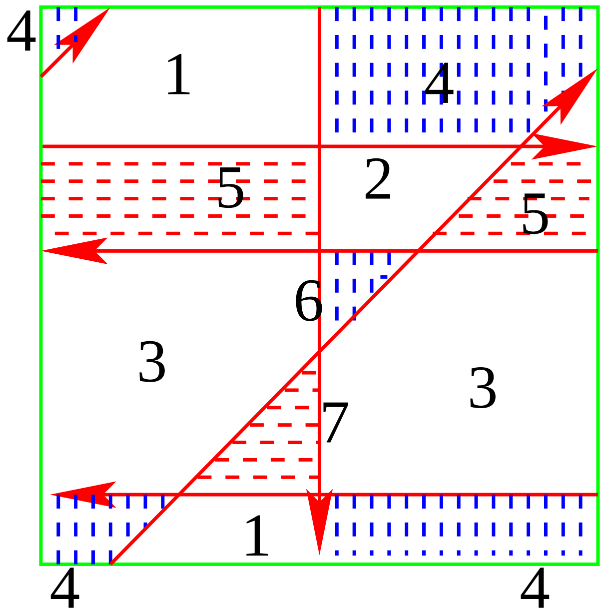
- Stack of N D5-branes are divided by NS5-branes into several regions, thus we have multiple gauge groups.



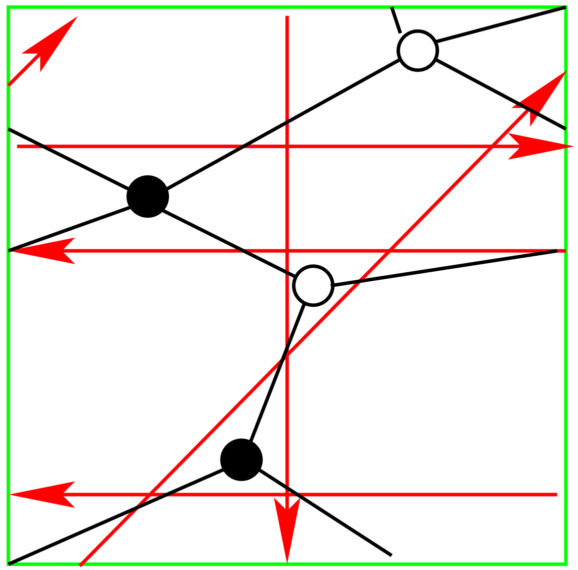
Strong Coupling

- Due to conservation of NS-charge, D5-brane actually becomes (N, k)-branes. ($k = 1, 0, -1$ in this talk)





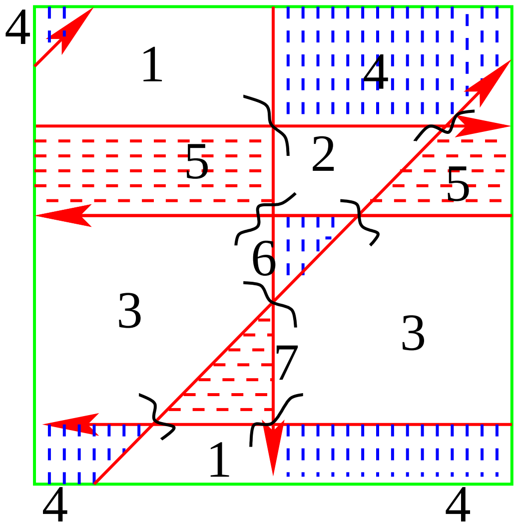
Blue Region: (N,1)−brane
Red Region: (N,−1)−brane
White Region: (N,0)−brane



Dimer Model

Strong Coupling

- We have a bifundamental for each intersection pt of $(N, 0)$-branes.
- From this we can read off quiver!



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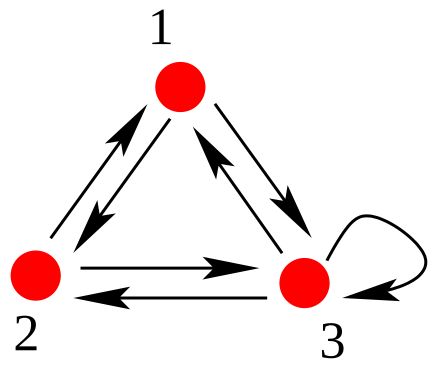


Quiver

Weak Coupling

Consider the weak coupling limit $g_s \to 0$. Then

$$T_{NS5} \gg T_{D5}$$

Then NS5-brane worldvolume Σ is a holomolophic curve $W(x,y) = 0$ in $\left(\mathbb{C}^\times\right)^2$, where

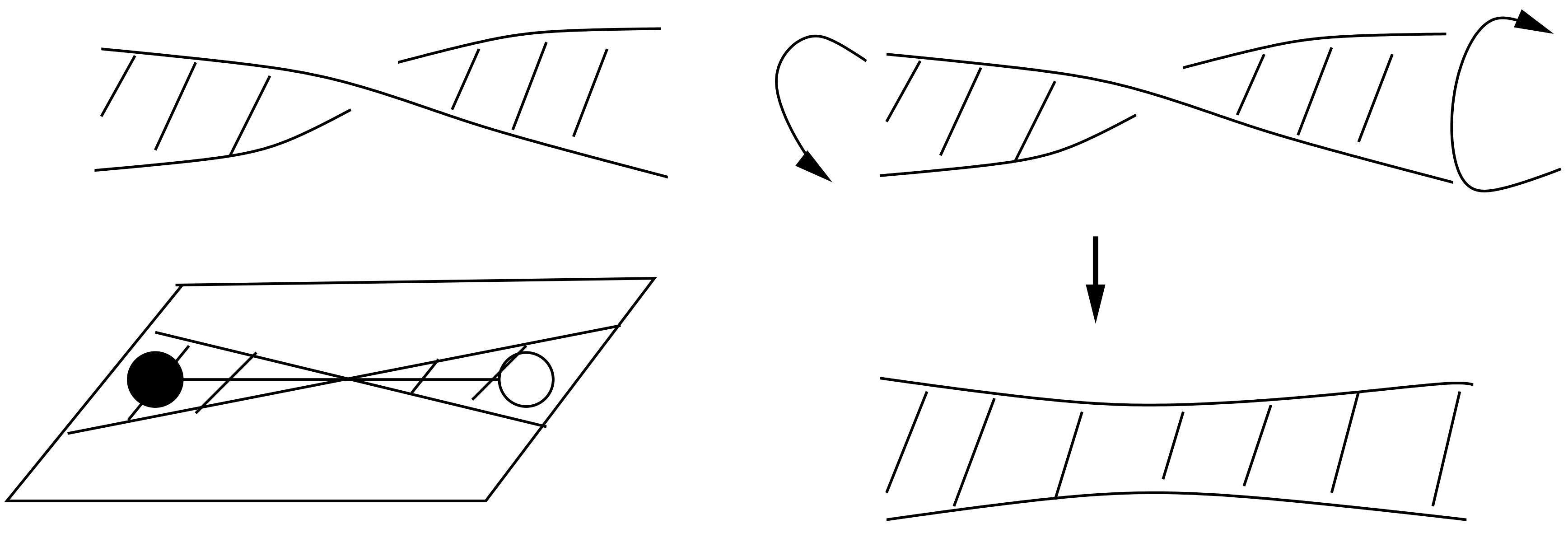
- $x = \exp(x_4 + ix_5), y = \exp(x_6 + ix_7)$
- $W(x,y)$ is a Newton Polynomial of the toric diagram

$$W(x,y) = \sum_{(i,j)\in\Delta} c_{(i,j)} x^i y^j$$

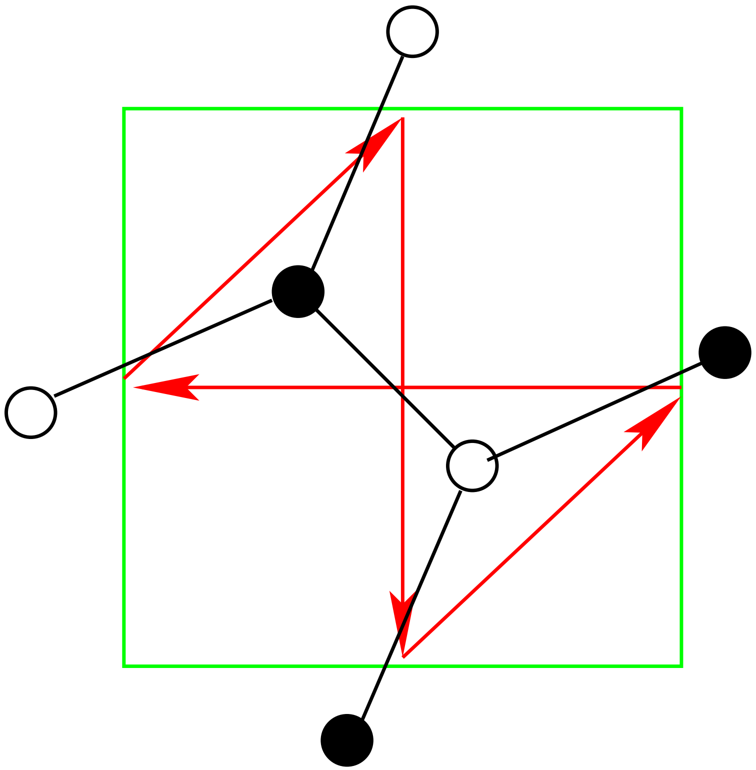
where $\Delta \in \mathbb{Z}^2$ is the toric diagram.

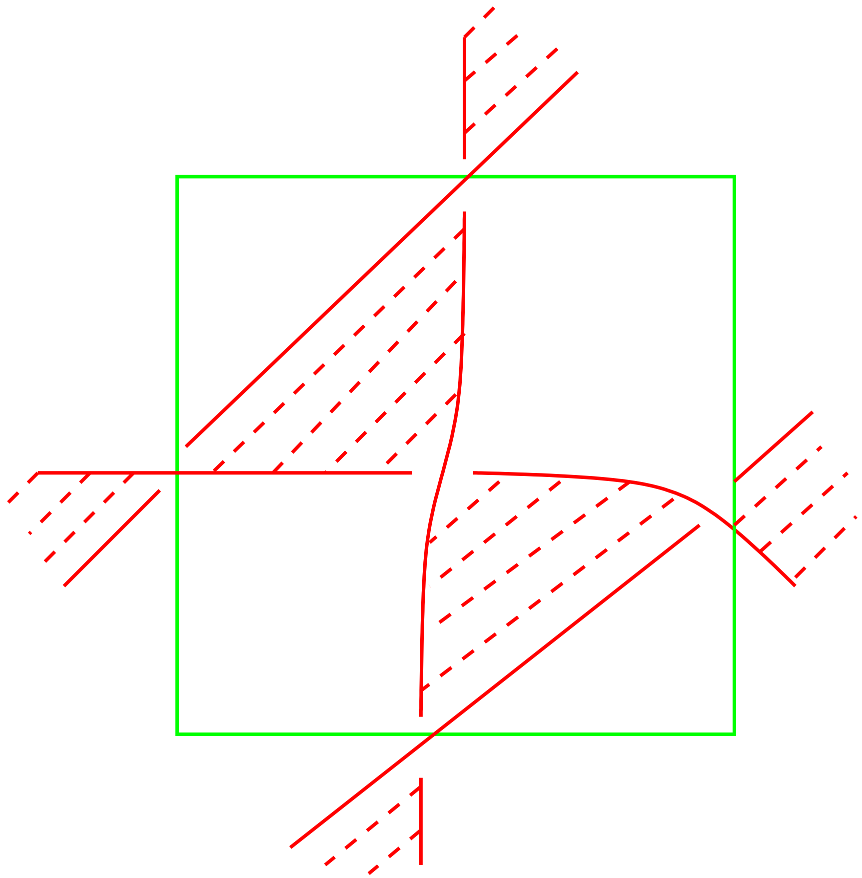
Untwisting

Actually, there exists a method to relate weak coupling to strong coupling, which is known as *untwisting* [Feng-He-Kennaway-Vafa].



Example of untwisting: $\mathbb{C}^3$





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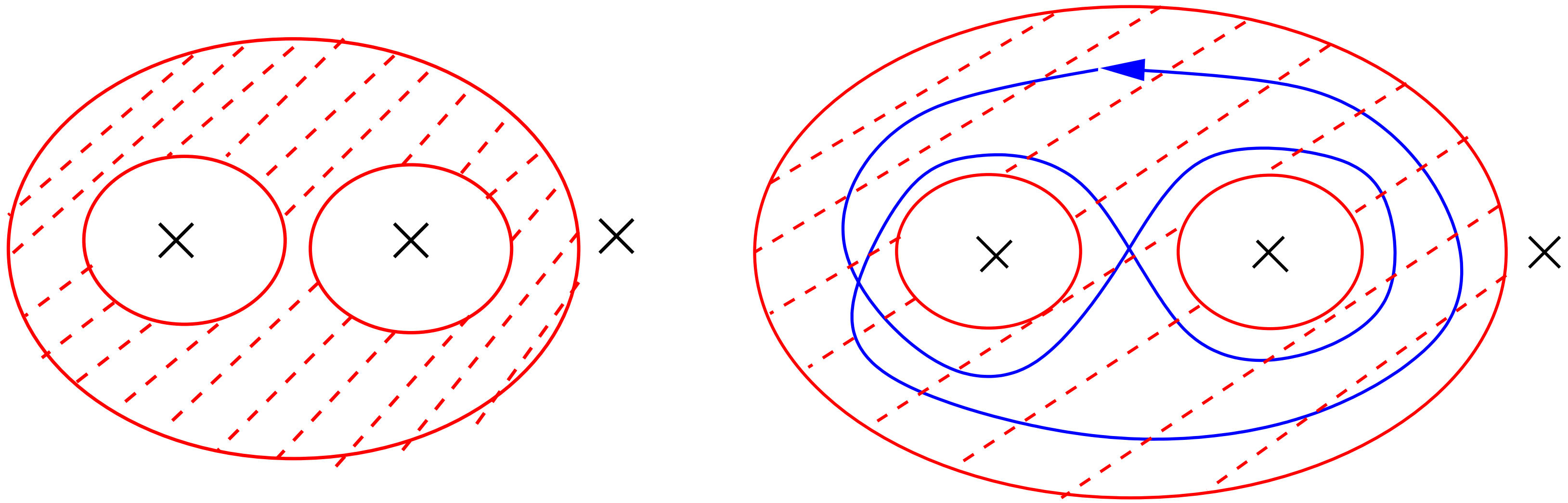
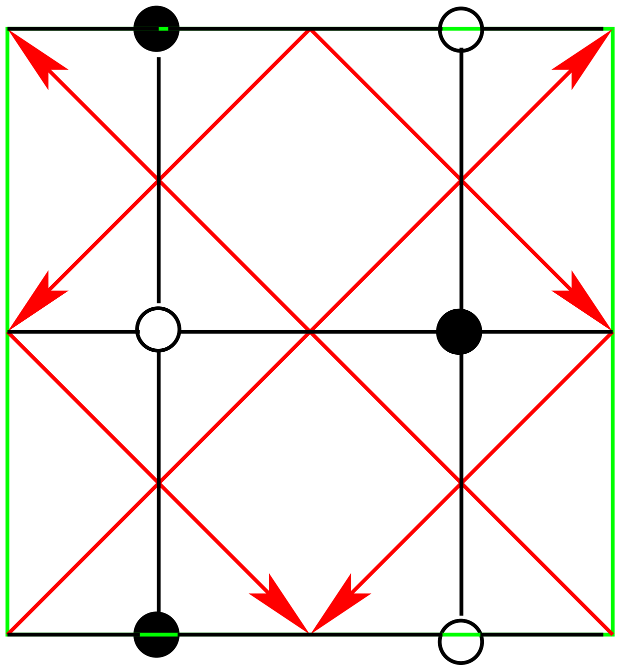
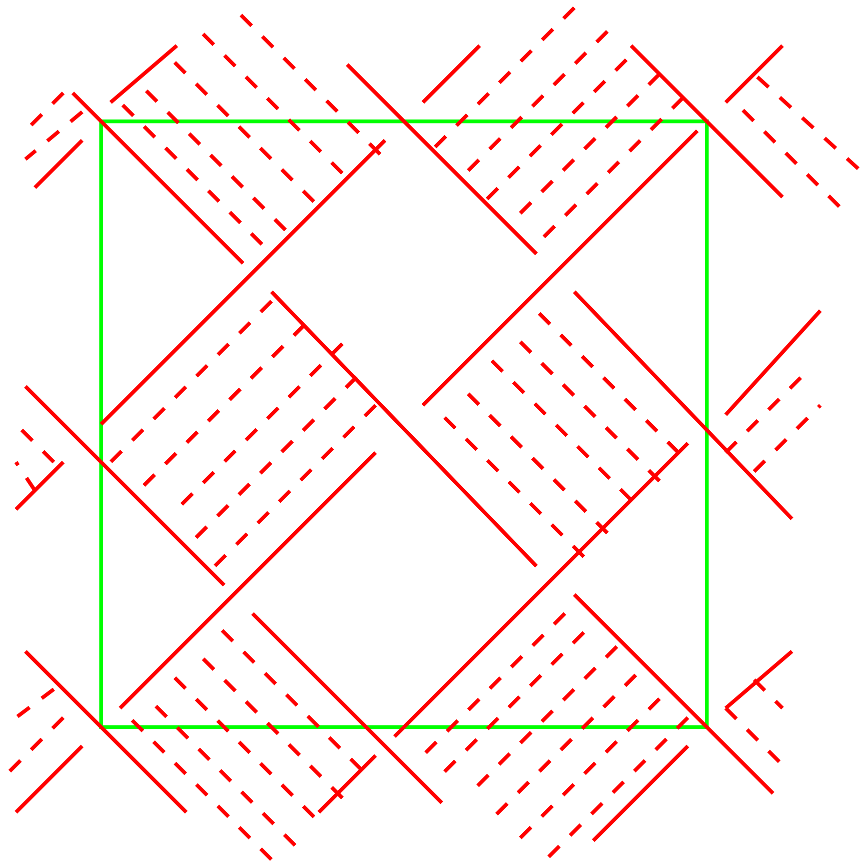


Example of untwisting: $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$





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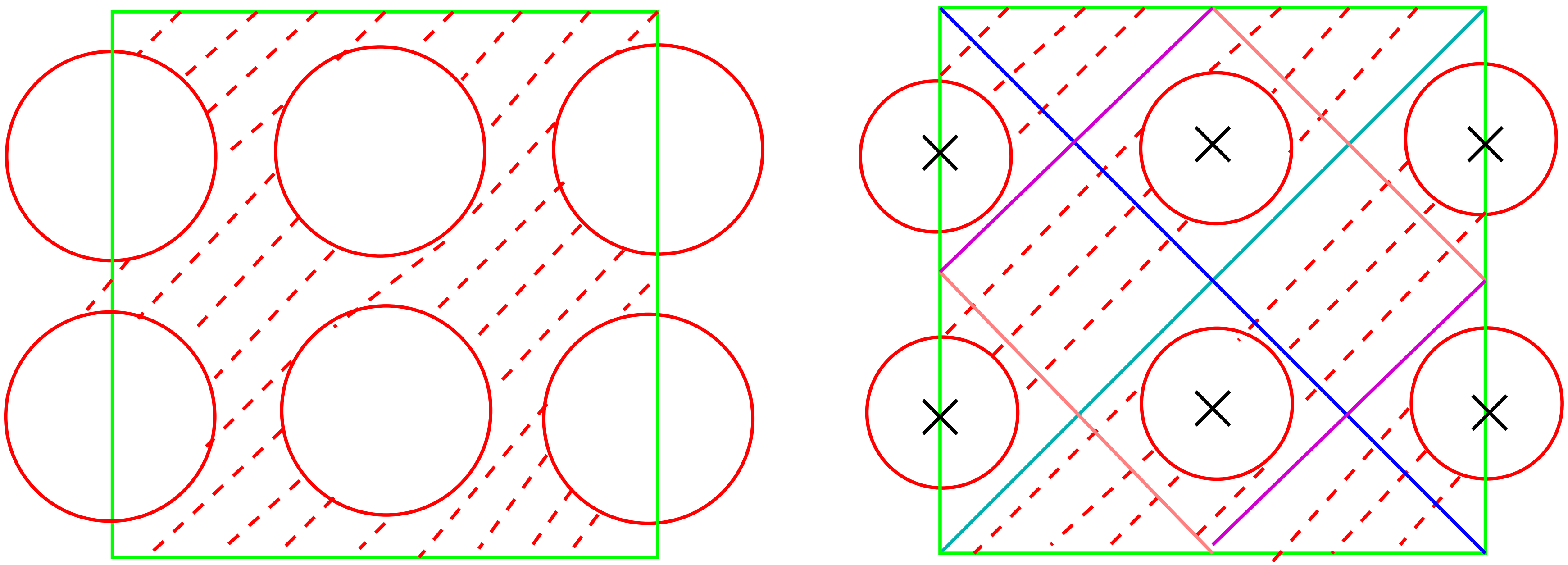

Result of Untwisting

- The surface Σ we obtain after twisting is the curve of NS5-brane (i.e. $W(x, y) = 0$). Therefore, by untwisting, we can go to weak coupling.
- Winding cycles of $\mathbb{T}^2$ are mapped to boundaries of Σ
- D5-branes (regions of $\mathbb{T}^2$) are mapped to winding cycles of Σ.

D6-brane Picture

- Suppose we have $g_s \to 0$ (weak coupling). Then, as we have seen, NS5-brane is Riemann surface $\Sigma : W(x, y) = 0$.
- Take T-duality, then NS5-brane turns into CY, which is often written as double fibration over W-plane.

$$W = W(x, y), \quad W = uv$$

where $w, z \in \mathbb{C}^{\times}$ and $u, v \in \mathbb{C}$.

- D5-brane is mapped to D6-branes wrapping 3-cycles of CY.
- Now we have D6-branes wrapping 3-cycles of CY!!

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Mirror Symmetry

Mirror Symmetry:

$$\text{B-model on CY} = \text{A-model on another CY}$$

How to give a mathematical formulation of mirror symmetry?

Homological Mirror Symmetry

Partial Answer: Homological Mirror Symmetry [Kontsevich]:

$$\underbrace{D^b(\mathbf{coh}\,Y)}_{(B-brane)} \cong \underbrace{D^b\,\mathfrak{Fuk}^{\to}(W)}_{(A-brane)},$$

$D^b(\mathbf{coh}\,Y)$: derived category of coherent sheaves
$\mathfrak{Fuk}^{\to}(W)$: (directed) Fukaya category

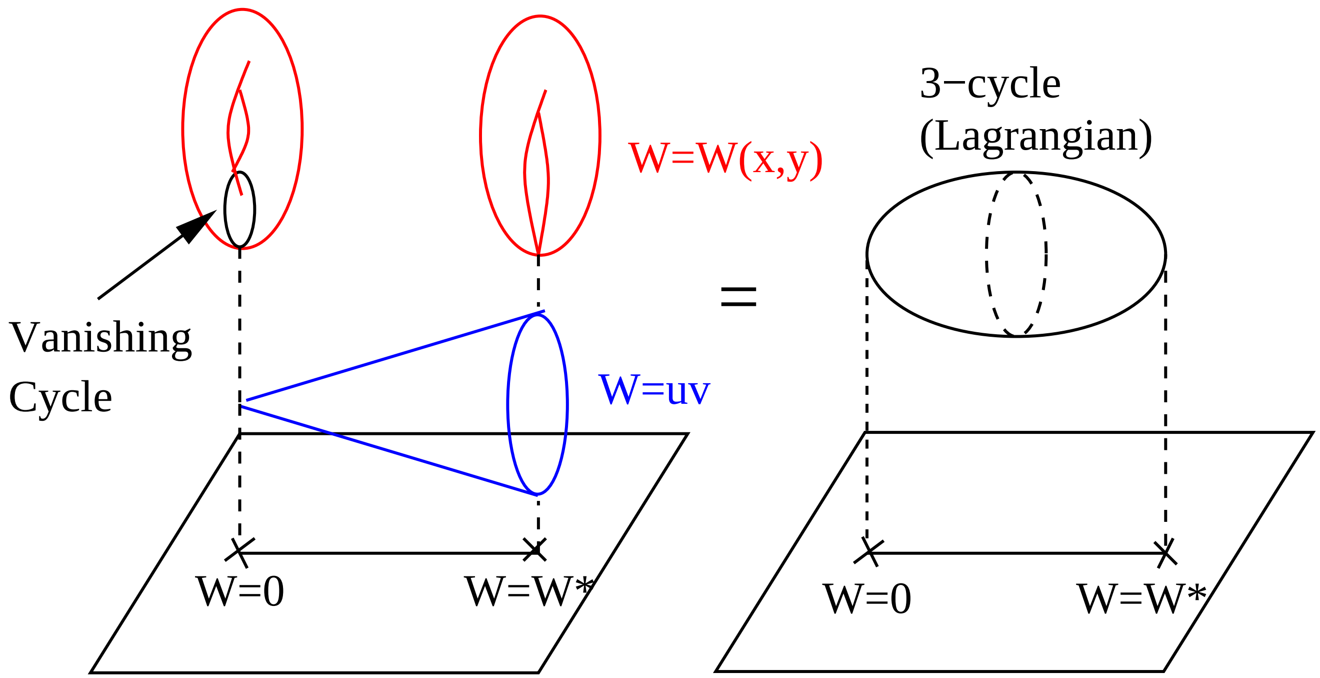
- It is known that B-model side can be computed from exceptional collections (Bondal's theorem)
- A-model side (Fukaya category) is much more difficult to compute

What is Category?

- Fukaya category is a so-called A^∞-category
- (Roughly speaking) an A^∞-category consists of
  1. Objects: $\mathcal{O}_i$
  2. Morphism: $Mor(\mathcal{O}_1, \mathcal{O}_2)$ for 2 objects $\mathcal{O}_1$ and $\mathcal{O}_2$
  3. Composition of Morphism:
     $m_k : Mor(\mathcal{O}_0, \mathcal{O}_1) \times Mor(\mathcal{O}_1, \mathcal{O}_2) \times \ldots \times Mor(\mathcal{O}_{k-1}, \mathcal{O}_k) \to Mor(\mathcal{O}_0, \mathcal{O}_k)$

What is Fukaya Category?

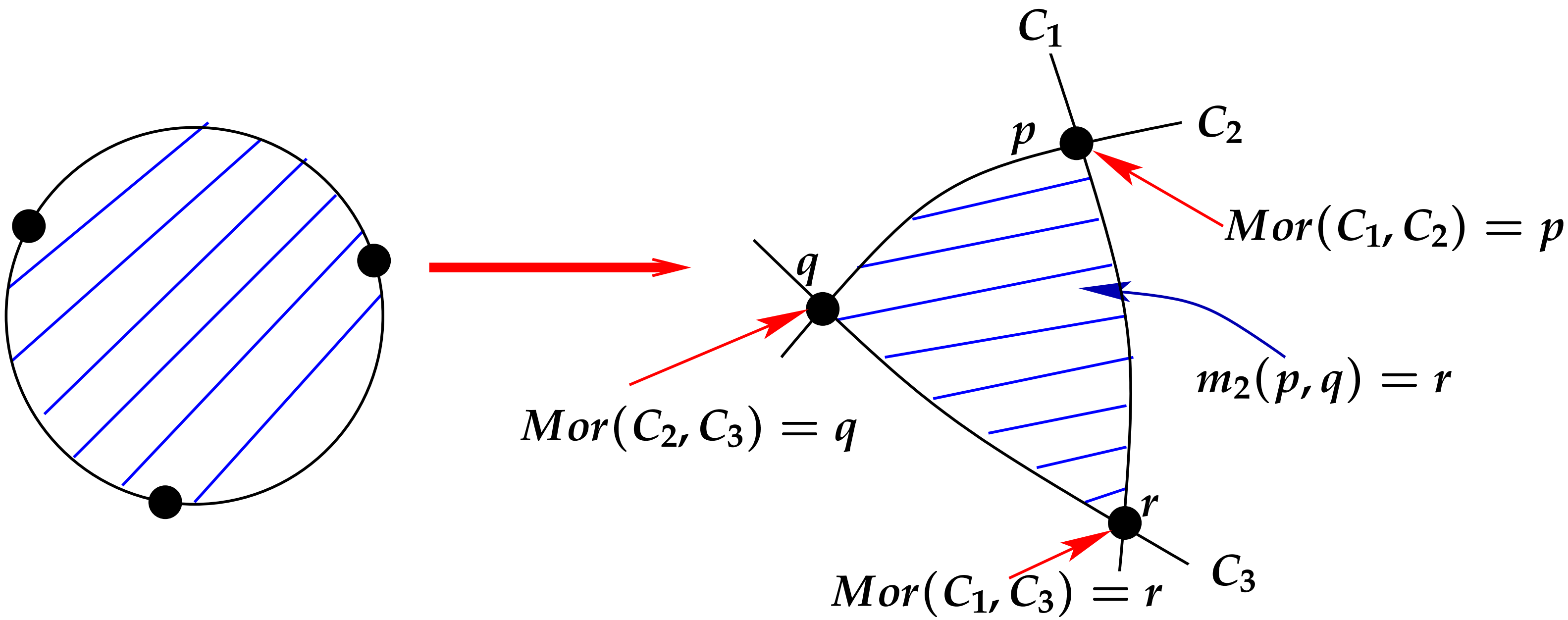
1. Objects: vanishing cycles $\{C_i\}_{i=1}^{N}$
(Mirror fiber degenerates at critical points of W and at $W = 0$)



D6-branes wrap these 3-cycles: Thus Objects=D-branes

What is Fukaya Category?

2. Morphism: $Mor(C_i, C_j)$ is the intersection points of C_i and C_j. (open strings)
3. Composition: non-zero when we can span a (pseudo-homorphic) disk. (disk amplitude)



What is Fukaya Category?

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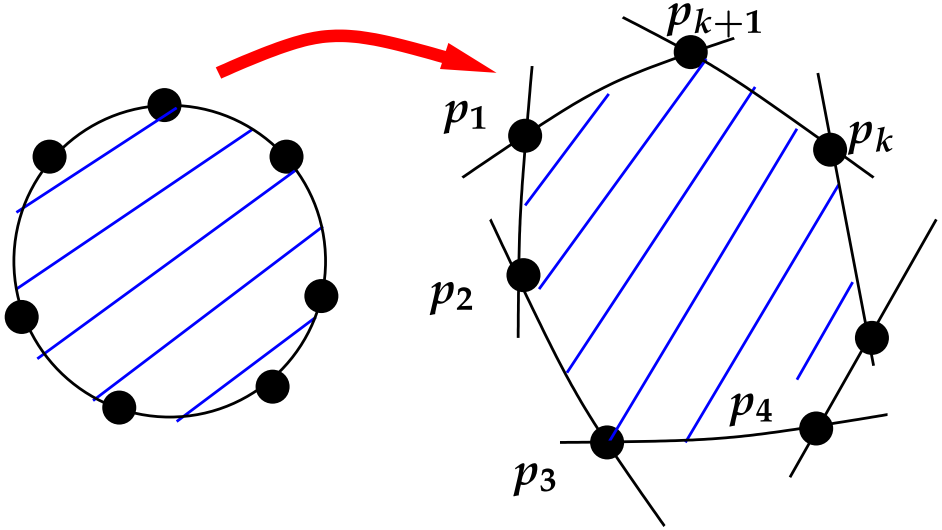
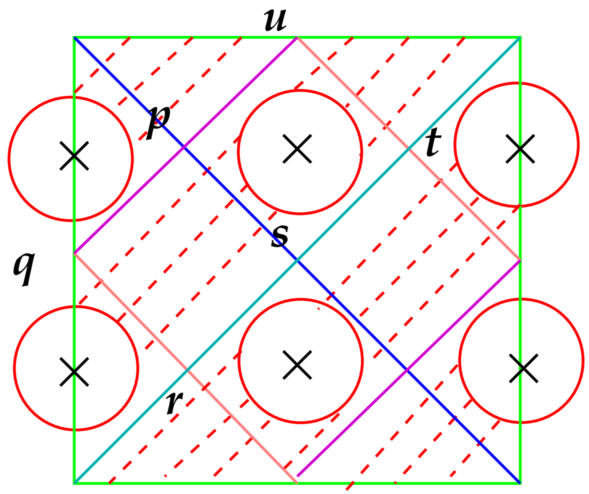




$m_k(p_1, p_2, \ldots p_k) = p_{k+1}$

$m_3(p, q, r) = s$

but $m_3(u, p, s) = 0$

Summary of Strategy

1. From toric diagram, draw D5/NS5 configuration in strong coupling (graph on $\mathbb{T}^2$).
2. Untwist to go to weak coupling. (graph on Σ)
3. Read off Fukaya Category.

Mathematical Formulation

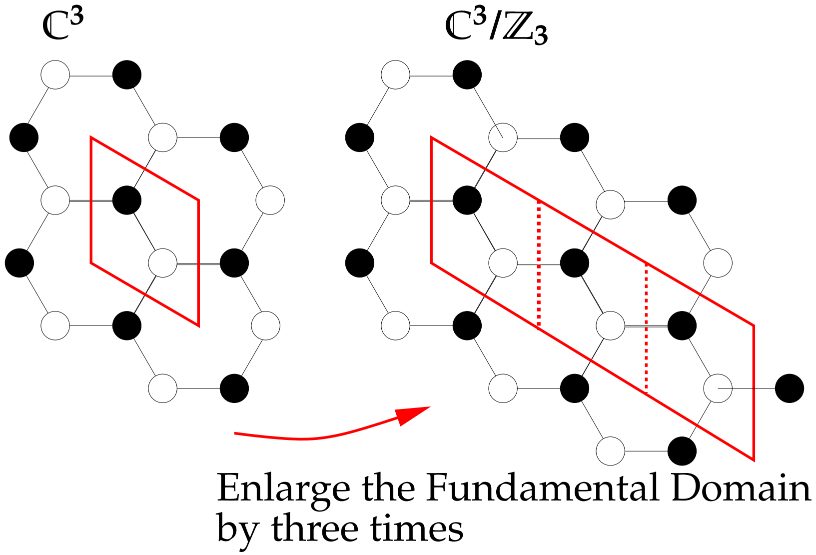
- This intuition can be given directly translated into mathematical formalism and now we have rigorous mathematical theorem!!

Main Theorem [Ueda-M.Y, math.AG/0606548]

Homological Mirror Symmetry is correct for $\mathbb{P}^1 \times \mathbb{P}^1$ and their $\mathbb{Z}_n$ orbifold.

Bonus: Generalization to $\mathbb{Z}_n$ Orbifold

Orbifolding by $\mathbb{Z}_n$ simply corresponds to enlarging the fundemental domain by n.



This means we can almost trivially extend our argument to $\mathbb{Z}_n$ orbifold case!

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Summary

- Brane tilings represent D5/NS5-brane system, and related by T-duality to Calabi-Yau geometry.
- Brane tiling technology is useful to prove homological mirror symmetry
- This method gives
  - Intuitive understanding from D-brane perspective
  - Rigorous mathematical proofs
  - Generalization to orbifold case (new result)

Outlook

- More general toric CY (e.g. toric del Pezzo), (work in progess with Kazushi Ueda)
- Moduli spaces of brane tilings? "Phases" of $\mathcal{N} = \mathbf{1}$ quiver gauge theories? (work in progress with Y.Imamura, H.Isono, and K.Kimura)
- Other dimensions? (cf. 3d brane tiling [Lee])
- Dynamical SUSY breaking, metastable vacuum, gaugino condensation from brane tiling and D-branes?
- More direct relation with BPS state counting as in topological vertex, instanton couting in SYM (cf. amoeba, tropical geometry?)
- Relation with tachyon condensation?