Yang-Baxter Duality

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Baxter 2015 Jul/21/2015

based on my works since 2012

M.Y. + W. Yan 1504   M.Y 1307

D. Xie + M.Y. 1207   M.Y. 1203   Y. Terashima + M.Y. 1203

(+ pedagogical review & some new coming)

inspired by many works

Baxter ('86!)   Bazhanov + Baxter ('92)   Kashaev

Faddeev + Volkov ('93 '94)   Bobenko + Springborn ('02)

Bazhanov + Mangazeev + Sergeev ('07)   Bazhanov + Sergeev ('10 '11)

Spiridonov ('10)

Some related recent works

A. Kels 1504   J. Yagi 1504

this afternoon!

Barry McCoy

"No one can be said to understand a paper until they generalize it"

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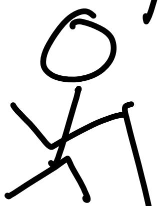

MY

"I wish to understand Yang-Baxter eq. from supersymmetric gauge theories and find new solutions to YBE!"

Summary (Gauge/YBE Correspondence)

YBE

find gauge theory interpretation

"categorification"

"gauge-theorification"

"Yang-Baxter Duality"

between two SUSY gauge theories

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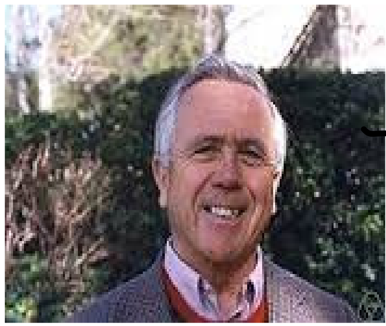

"I'm not a duality!"

Summary (Gauge/YBE Correspondence)

YBE

find gauge theory interpretation

"categorification"

"gauge-theorification"

Z_{M_1}

compute SUSY partition function on mfd M

"Yang-Baxter Duality"

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find gauge theory interpretation

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"Yang-Baxter Duality"

between two SUSY gauge theories

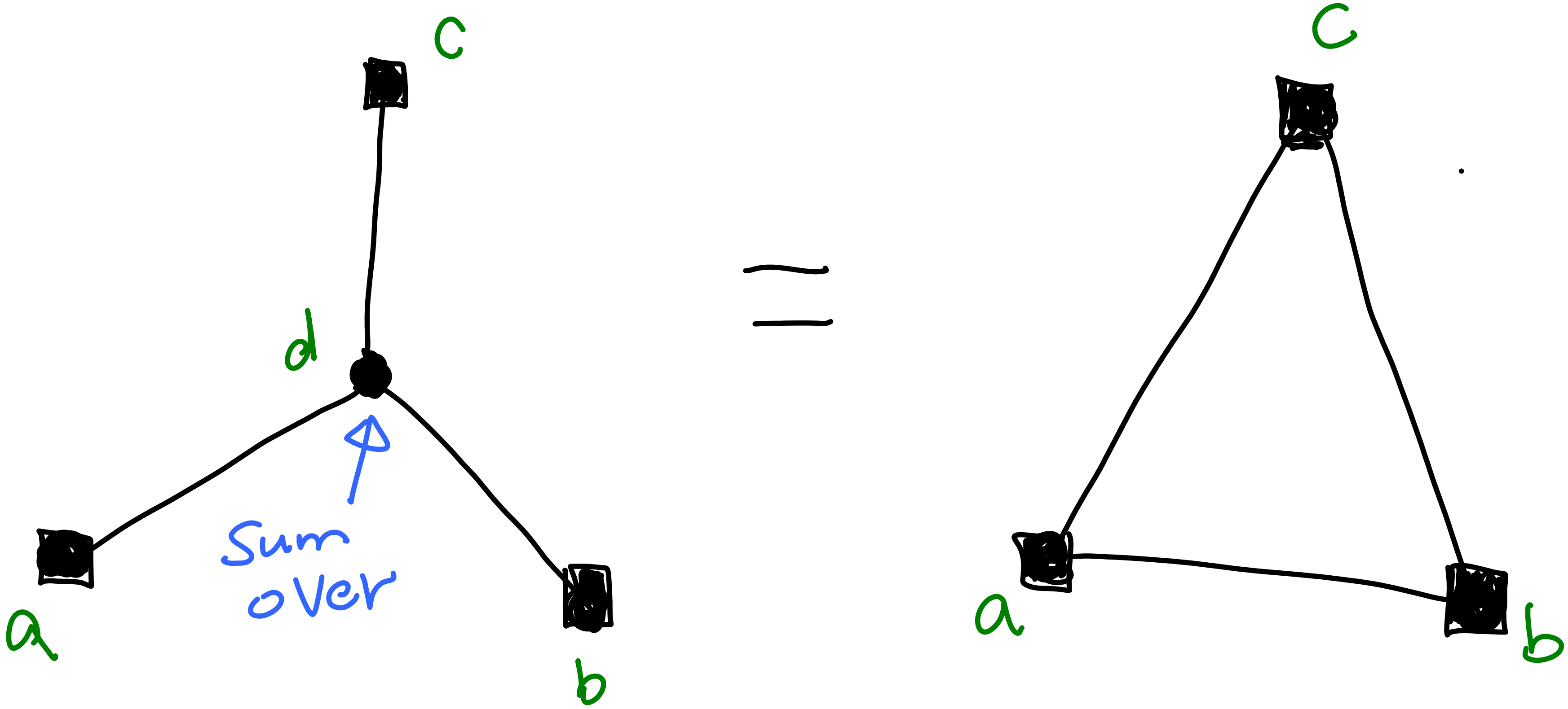
Z_{M_2}

YBE_2

Z_{M_3}

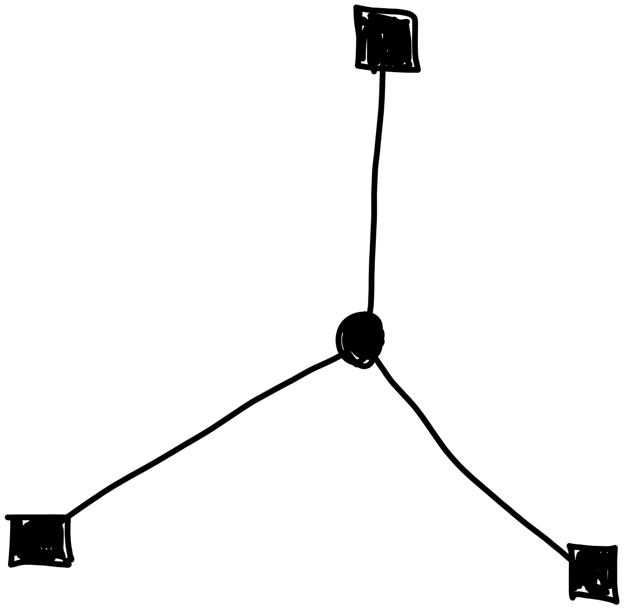
YBE_3

Star-Triangle Relation $\Rightarrow$ YBE

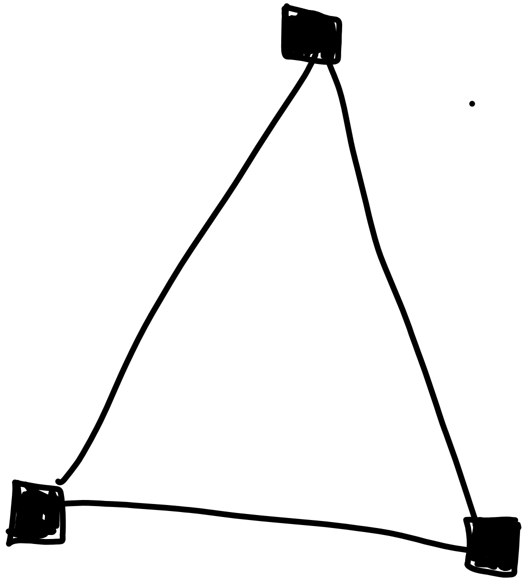


$$\sum_d S^d W_{ad} W_{bd} W_{cd} = R\, W_{ab} W_{bc} W_{ca}$$

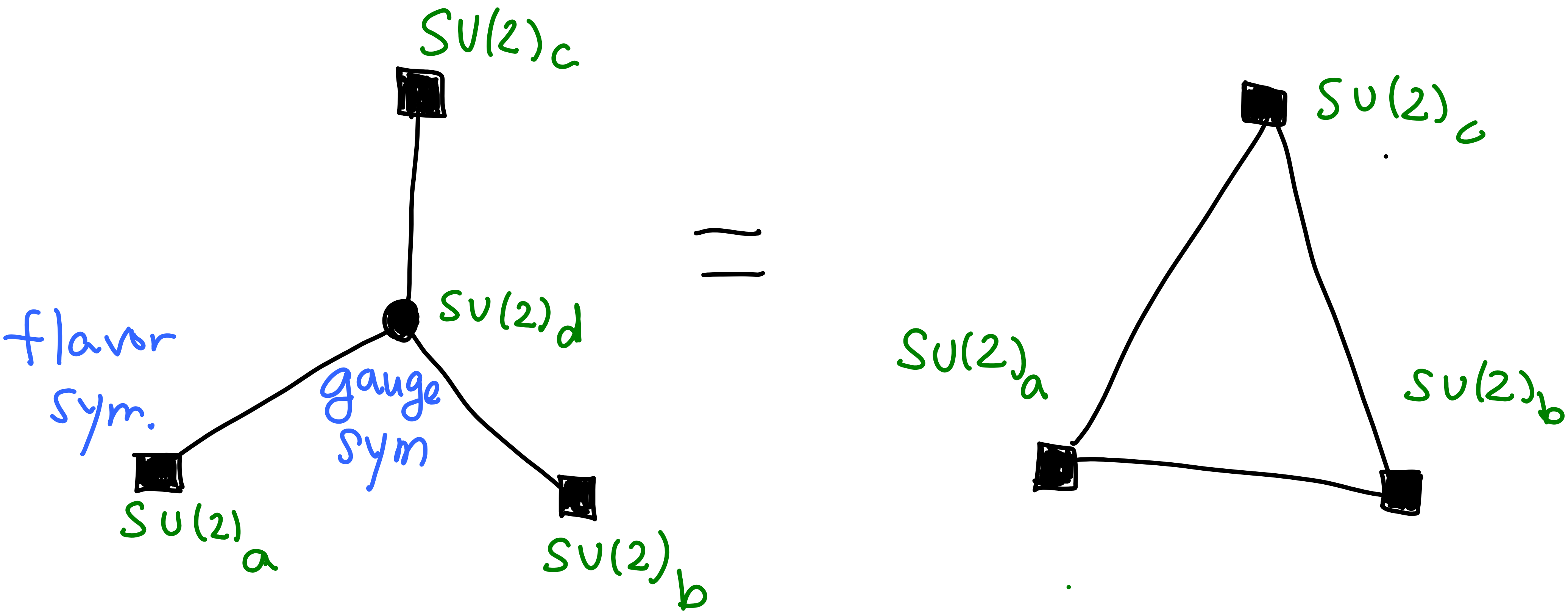
Star-Triangle Duality



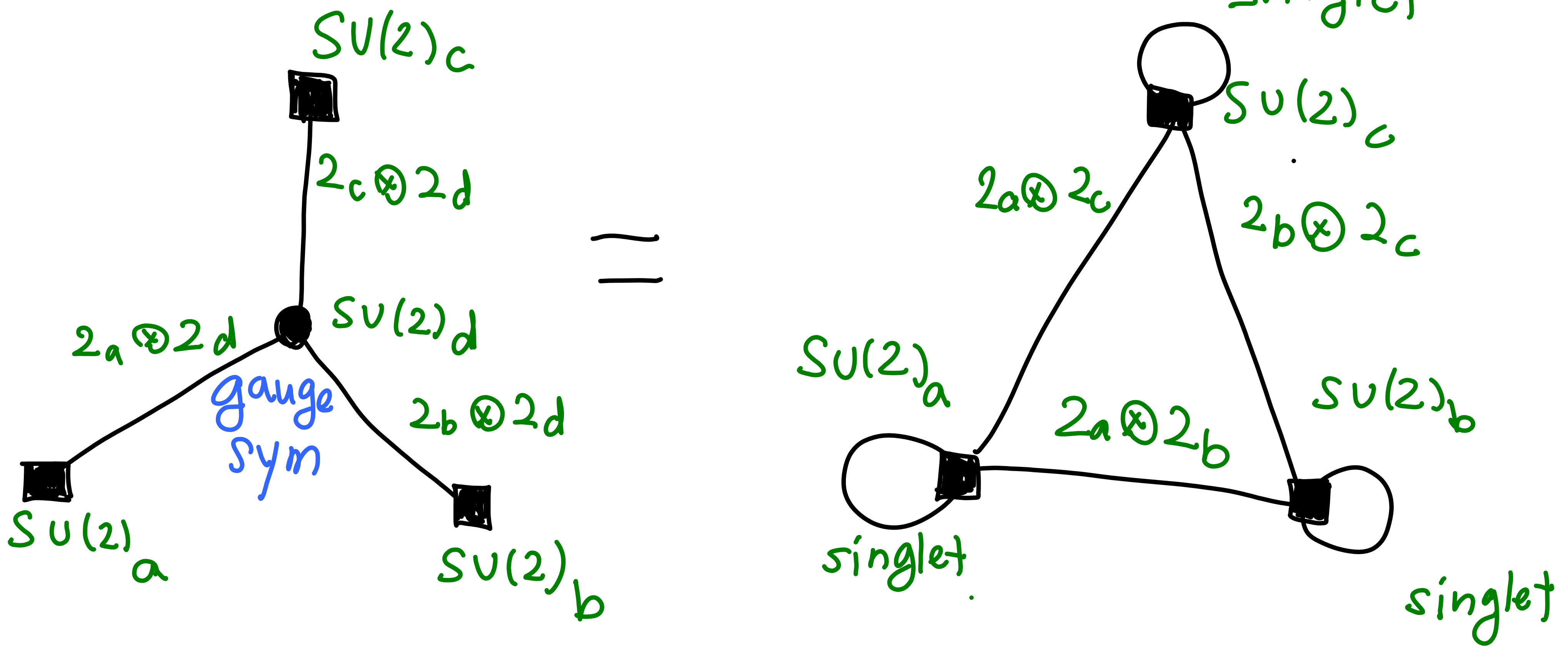
$=$



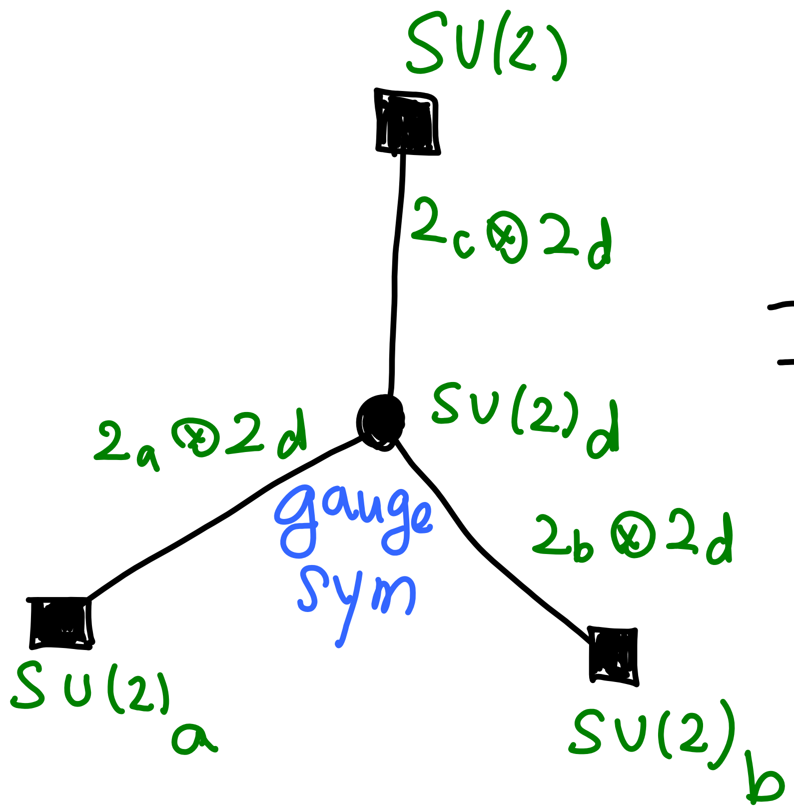
Star-Triangle Duality



Star-Triangle Duality



Star-Triangle Duality [Seiberg '94]



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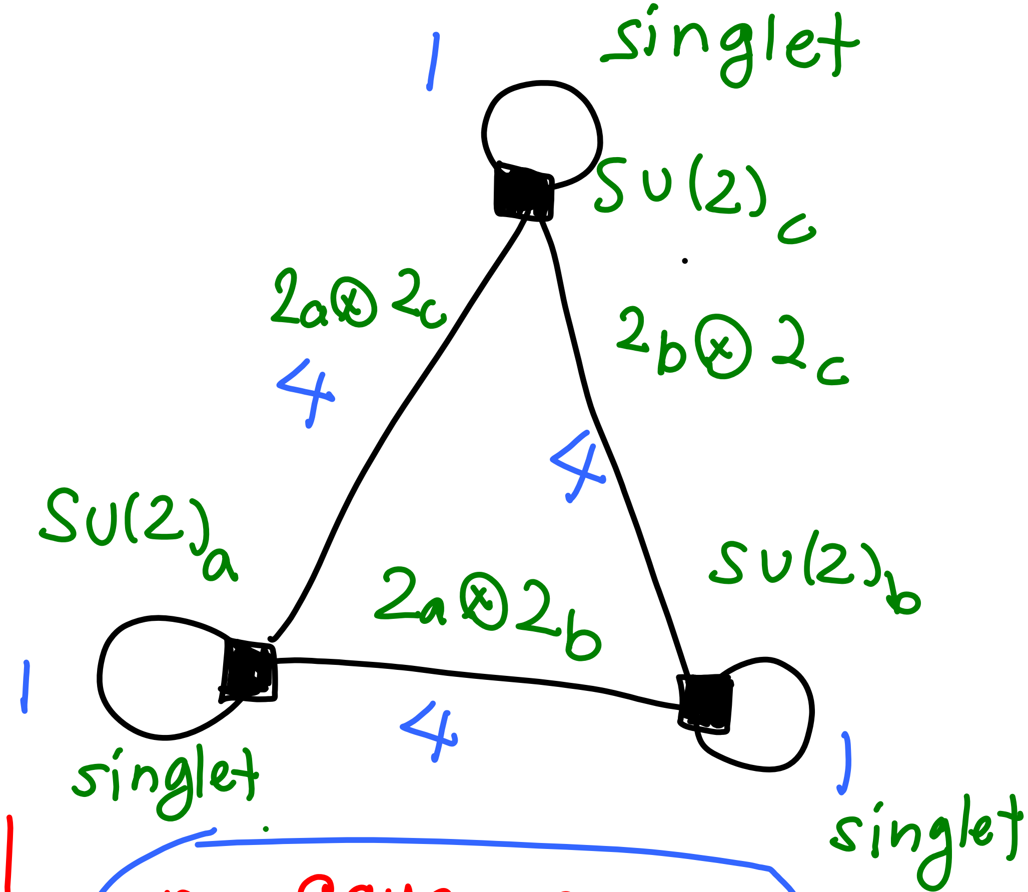
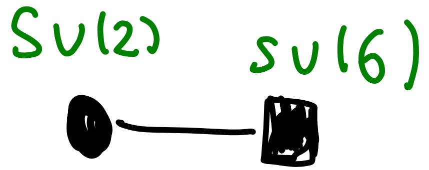


$G = SU(2) = Usp(2)$
w/ $2N_f = 6$ flavors Q_i

dual =

no gauge group
${}_6C_2 = 15$ mesons
$M_{ij} = \epsilon_{ij} Q^i Q^j$



4D $\mathcal{N}=1$

"s-confining"

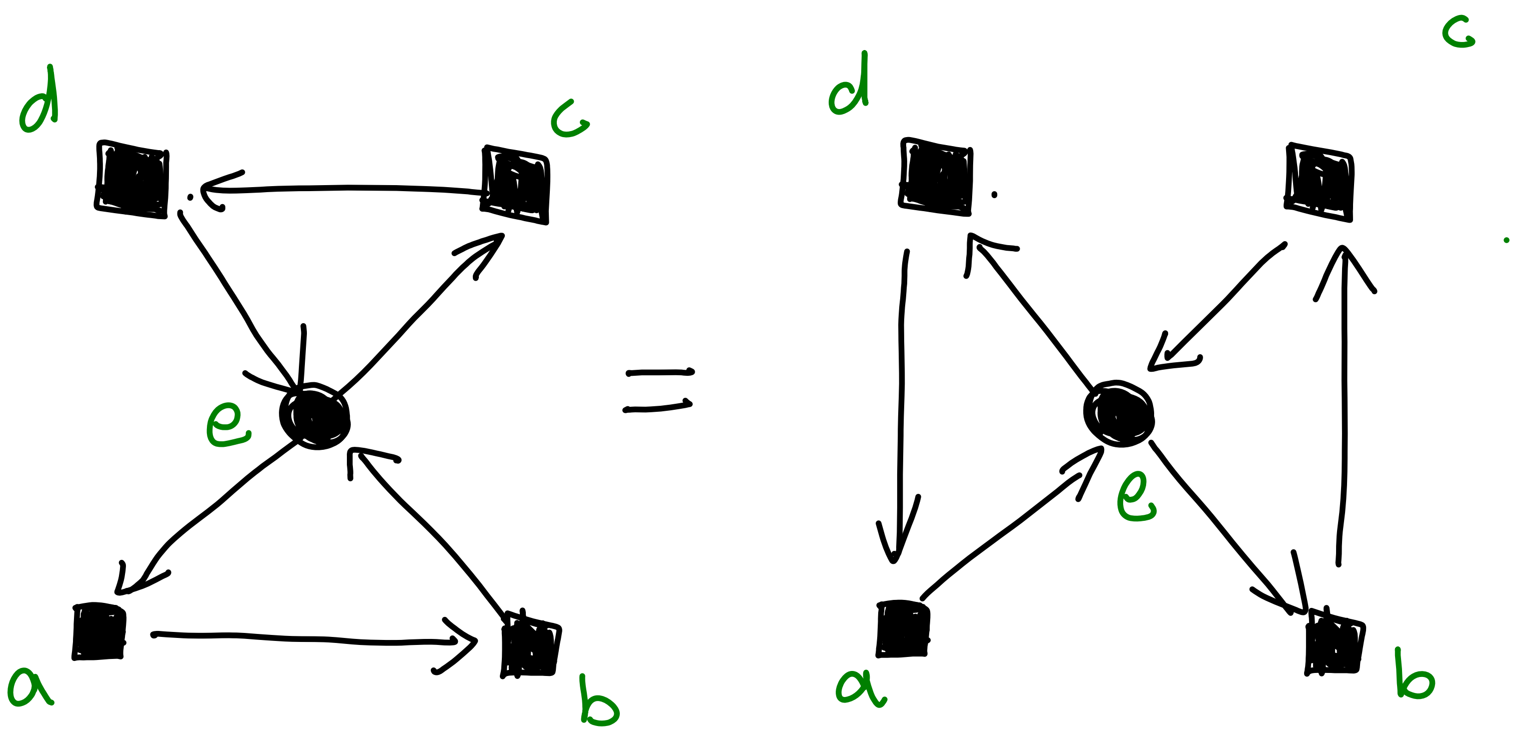
Star-Star Relation

[Baxter; Bazhanov-Baxter] $\Longrightarrow$ YBE

chiral model

$a \longrightarrow b \qquad a \longleftarrow b$

$W_{ab} \neq W_{ba}$



$$W_{ab} W_{cd} \sum_{e} S^{e} W_{ea} W_{be} W_{ec} W_{de}$$

$$= W_{da} W_{bc} \sum_{e} S_{e} W_{ae} W_{eb} W_{ce} W_{ed}$$

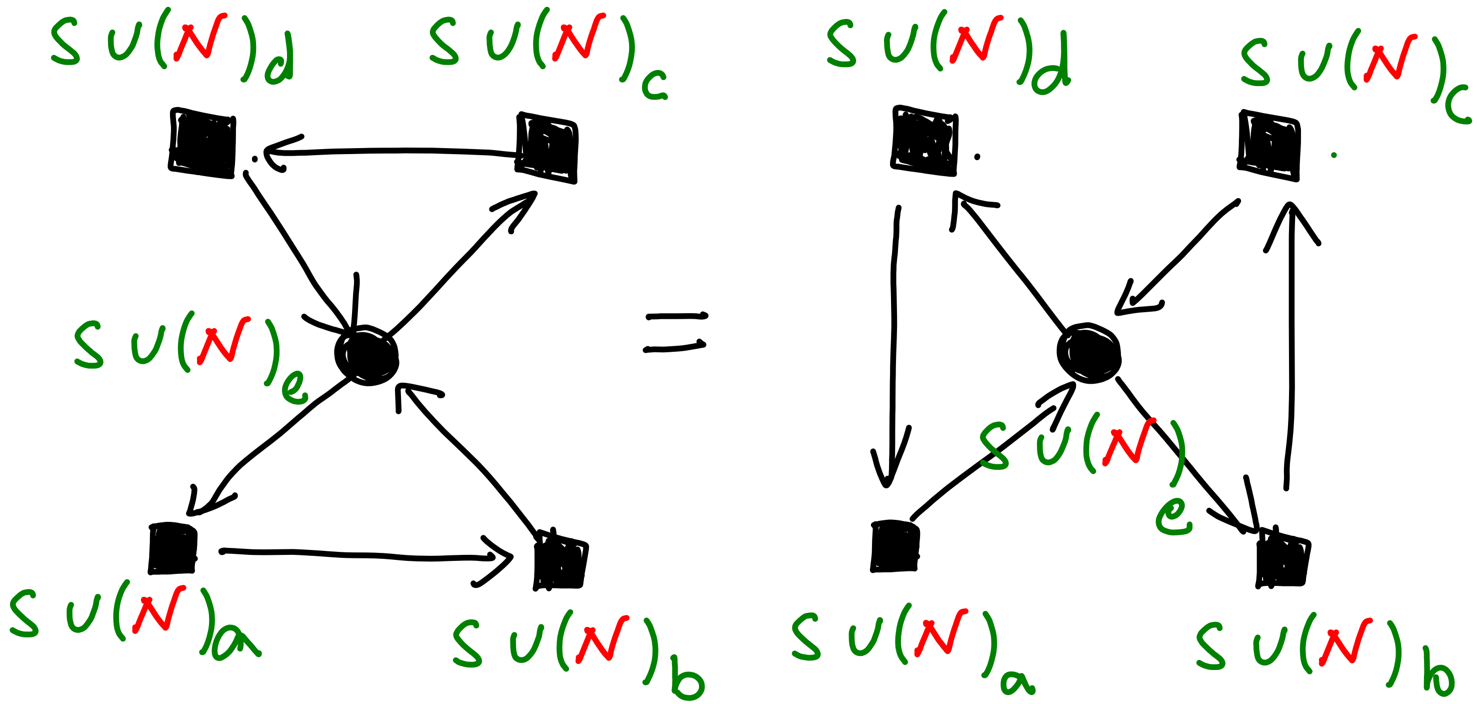
Star-Star Duality

[Seiberg ('94); Quiver mutation Fomin Zelevinski ('02)]

chiral theory

$a \rightarrow b$ $\quad a \leftarrow b$

$(N_a, \overline{N_b}) \neq (\overline{N_a}, N_b)$

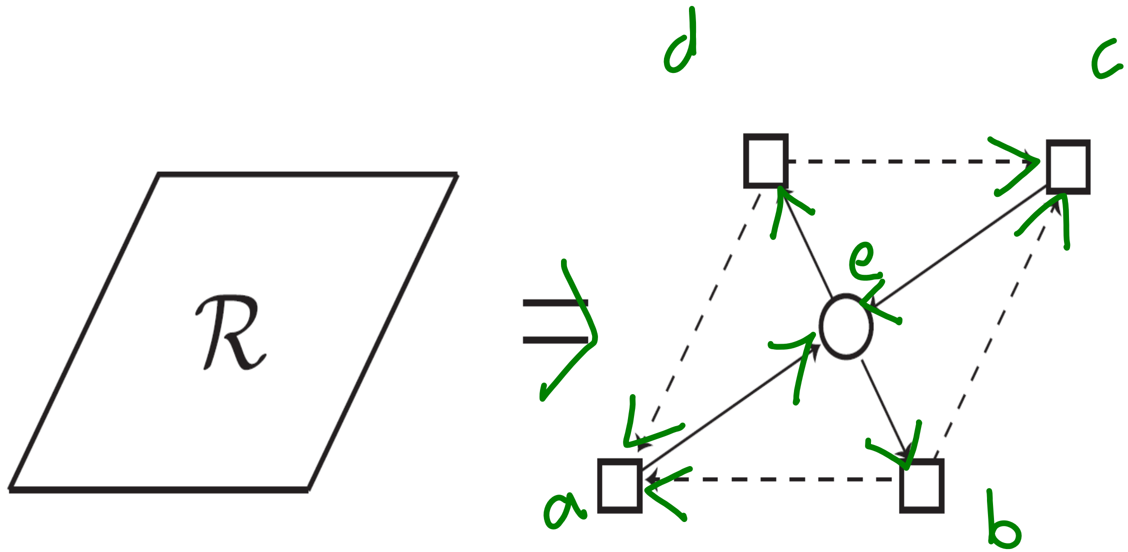


$G = SU(N)$, $N_f = 2N$ + superpotential $\underset{\text{dual}}{=}$ $G = SU(N)$, $N_f = 2N$ + superpotential

Yang-Baxter Duality

R-matrix

quiver theory $T[R]$

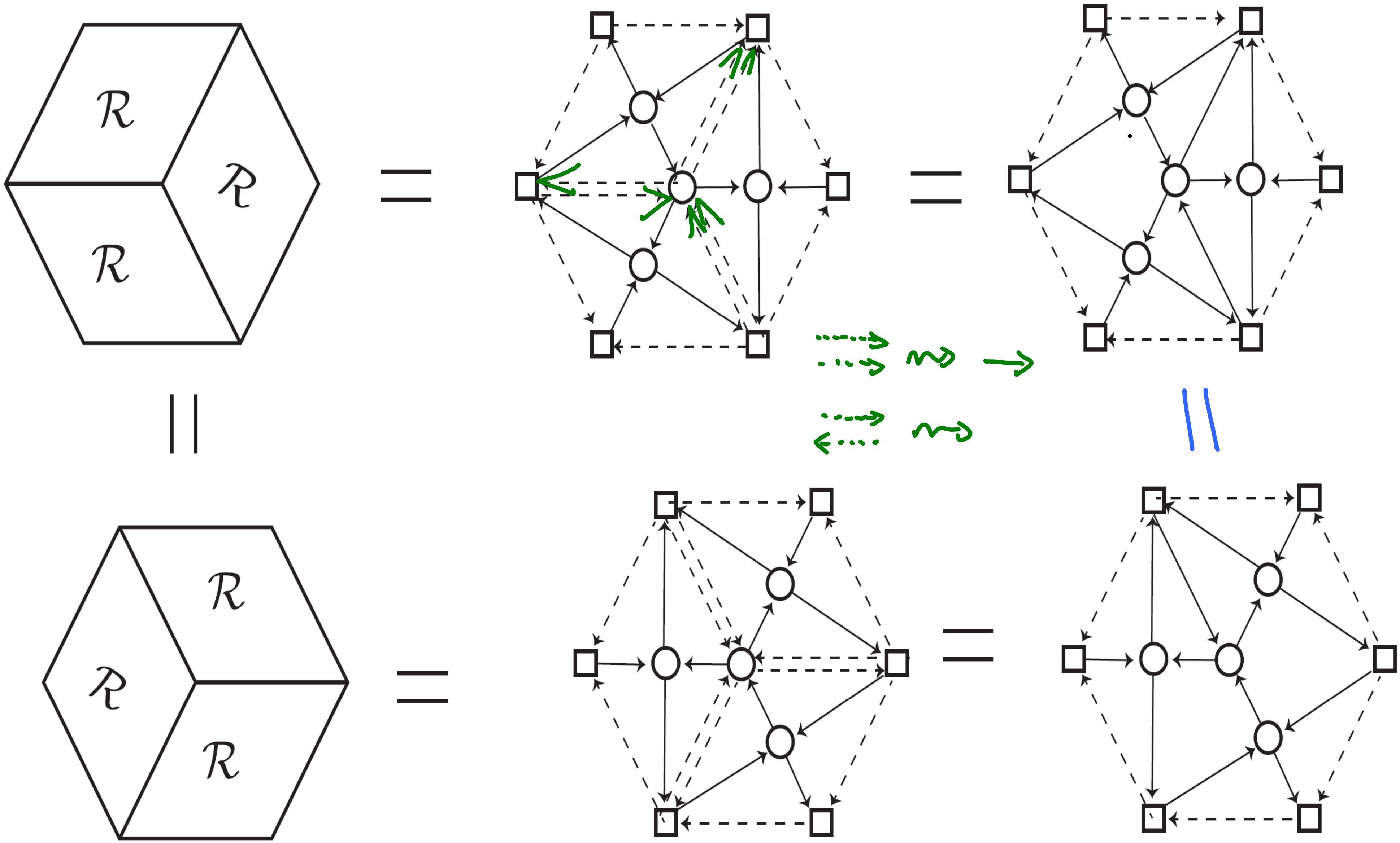


R-charge /spectral param

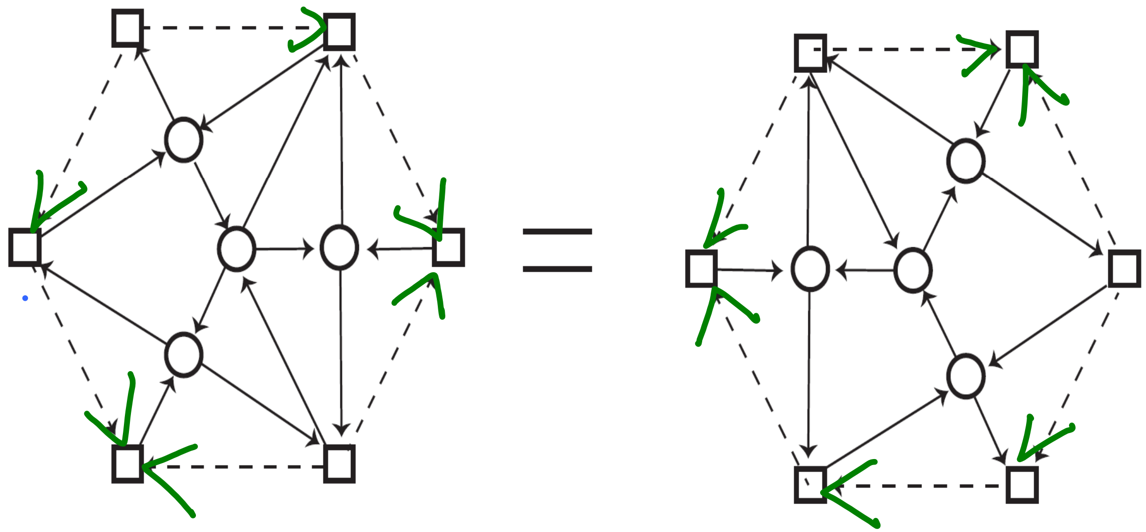
α

$$W^{\alpha}_{ab}\, W^{2-\alpha}_{ba} = 1$$

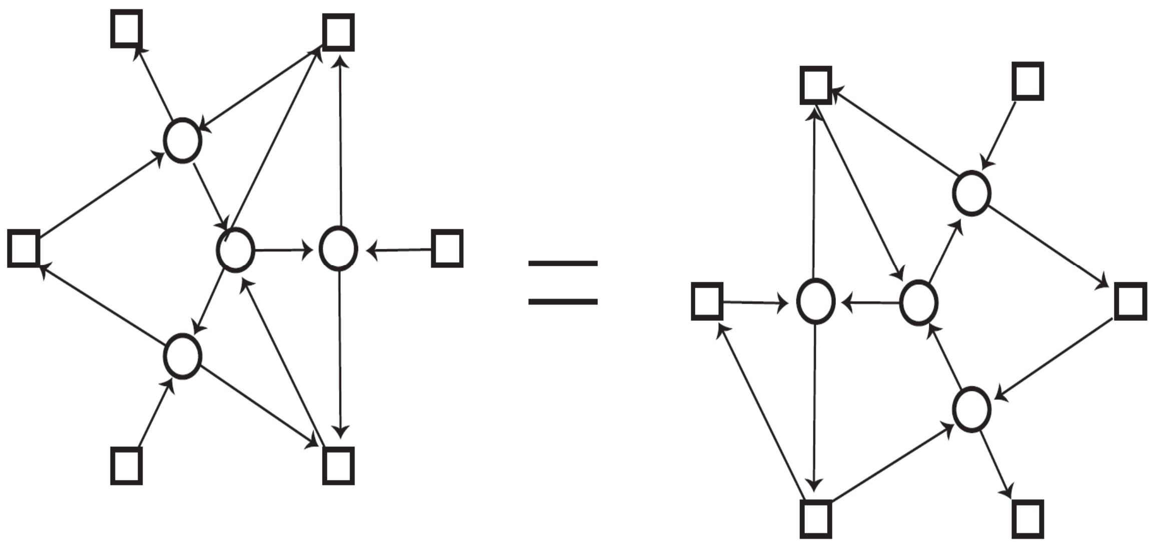
$$Z[T(R)] = \sqrt{W_{ba}W_{bc}W_{dc}W_{da}}\,\sum_e S^e\, W_{ae}W_{eb}W_{ce}W_{ed}$$


R
R
R
R
R
R

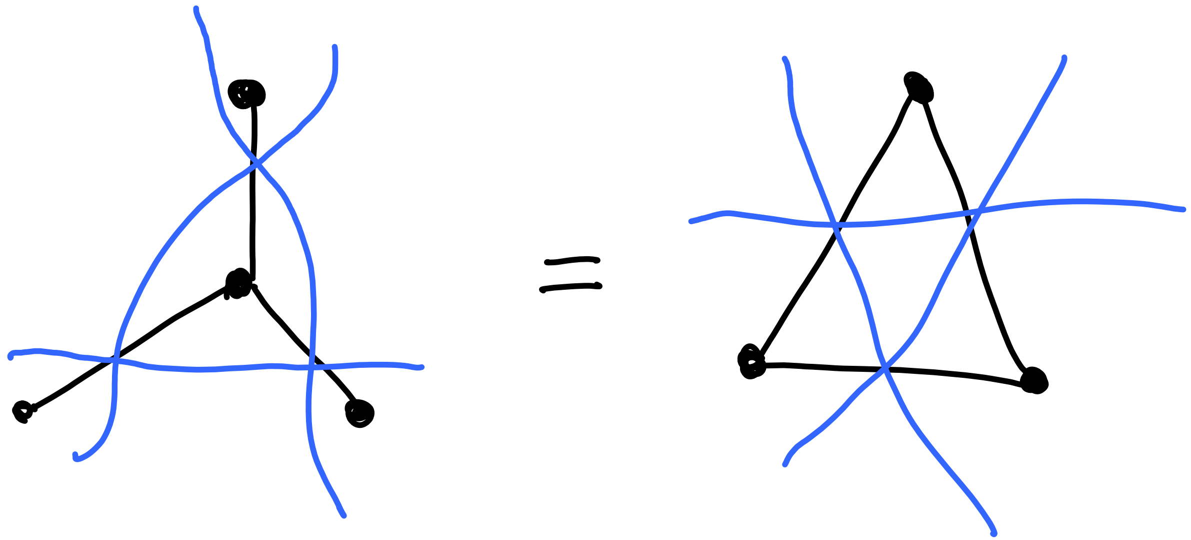
Therefore



When half-chirals are combined into chirals,



Baxter's Z-invariant lattice



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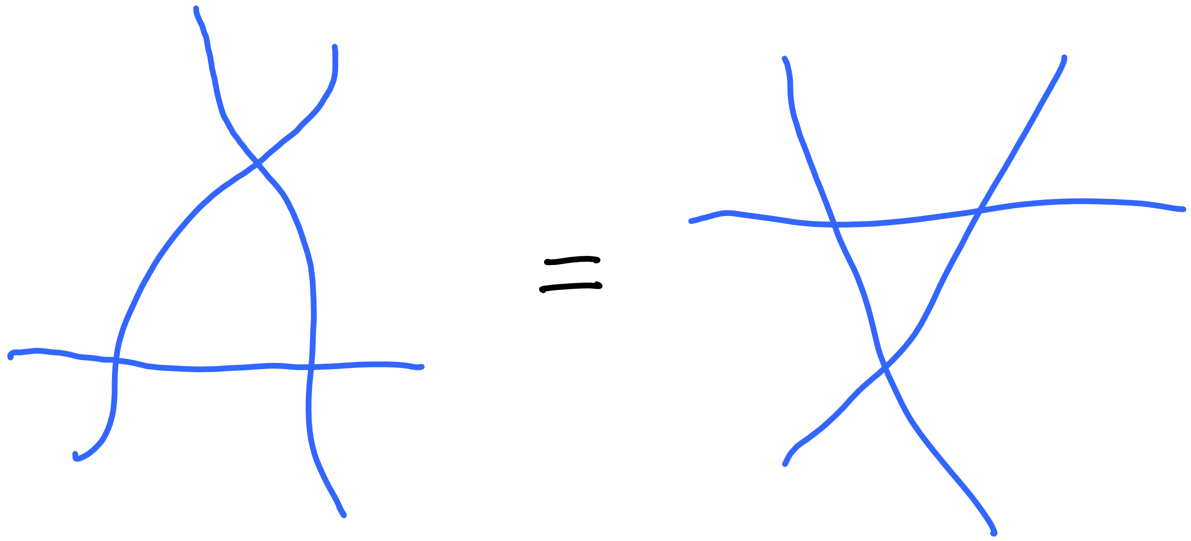

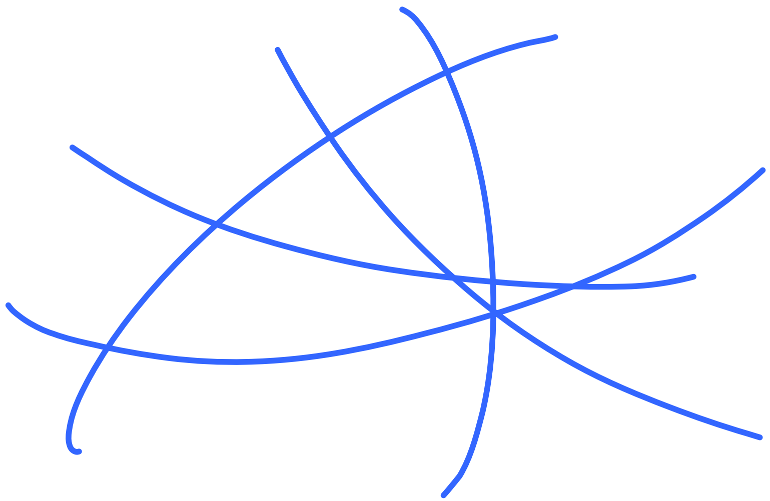

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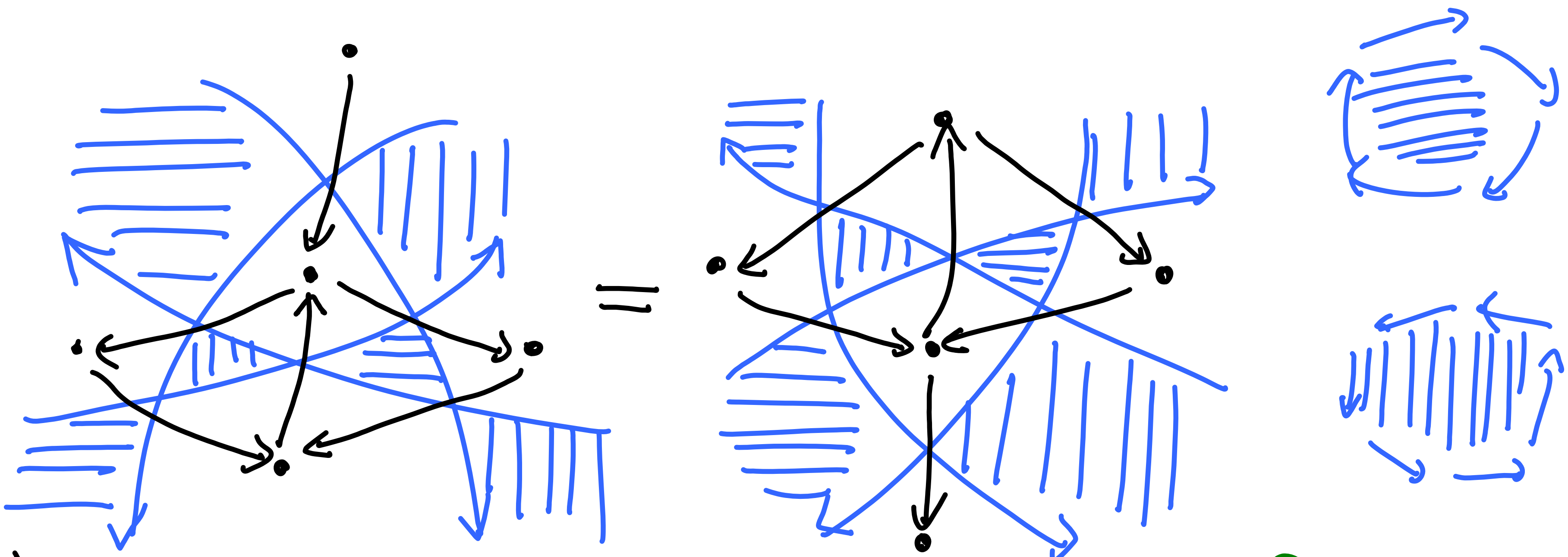


rule Hanany-Vegh ('05) Thurston ('04) Postnikov ('06)

brane Feng-He-Kennaway-Vafa ('05) MY ('08)

Heckman-Vafa-Xie-MY ('12)

$(Gr_{k,n})_{\geq 0}$

also applications to mirror symmetry Ueda-MY ('06-'07)

$D^b \mathrm{Fuk}\, X$

4d $\mathcal{N}=1$ Seiberg dual

$S^1 \times S^3$, $S^1 \times S^3/\mathbb{Z}_r$, $S^2 \times T^2$, ...

$\Gamma(x;p,q)$ $\Gamma_r(x;p,q)$ $\theta(x;q)$

[elliptic]

3d $\mathcal{N}=2$ Aharony dual $S^1 \times S^2 \to (x;q)_\infty$

2d $\mathcal{N}=(2,2)$ Seiberg-like dual $T^2 \to \theta(x;q)$

[trigonometric]

1d $\mathcal{N}=4$ Seiberg-like dual $S^1 \to \sin(x)$

[rational]

Works in progress: 5d $\mathcal{N}=1$

2d $\mathcal{N}=(0,2)$

$4d\ \mathcal{N}=1$, $(S^1\times S^3/\mathbb{Z}_r)_{p,q}$

parameters $N, r; \underbrace{p, q}_{\text{elliptic}}$

We can consider specializations

$S^1\times S^3/\mathbb{Z}_r \xrightarrow{r=1} S^1\times S^3 \xrightarrow{p,q\to 1\ +\text{Higgsing}} S^3$ Faddeev-Volkov $\to$ $R^2\times S^1$ Bobenko Springborn

trigonometric

Benini Nishioka

$S^1\times S^2$ $\to$ S^2

$S^3/\mathbb{Z}_r$ $\to$ S^2

Bazhanov Sergeev model $\to$ Faddeev-Volkov

Bazhanov Sergeev model $\to$ Kashiwara Miwa

Bazhanov Sergeev model $\to$ chiral Potts

root of unity

Kashiwara Miwa $\to$ Fateev-Zamolodchikov

chiral Potts $\to$ Fateev-Zamolodchikov

Fateev-Zamolodchikov $\to$ Ising

elliptic

$U_{p,q}(sl_N)$

sklyanin

cherednik

$U_{p,q;r}(sl_N)$?

Feigin, Odesskii??

2D (2,2) answer from T^2:

Probably new

[W. Yan + MY '15]

spin variables

$$\mathcal{R}\begin{pmatrix} \delta & \gamma \\ \alpha & \beta \end{pmatrix}\begin{bmatrix} d & c \\ a & b \end{bmatrix}(q,y)$$

deformation parameters

spectral param

eta/theta function

$$= \sqrt{\frac{\prod_{i,j=1}^{N} \overline{\Delta}_{2-\delta-\alpha}(d_i a_j^{-1})\overline{\Delta}_{2-\beta-\gamma}(b_i c_j^{-1})}{\prod_{i,j=1}^{N} \overline{\Delta}_{\alpha+\beta}(a_i b_j^{-1})\overline{\Delta}_{\gamma+\delta}(c_i d_j^{-1})}} \left[\frac{1}{N!}\left(\frac{\eta(q)^3}{i\theta_1(t^{-1};q)}\right)^N\right]^2 \tag{3.12}$$

$$\times \sqrt{\prod_{i\neq j}\frac{1}{\Delta(a_i a_j^{-1};q,y)}\prod_{i\neq j}\frac{1}{\Delta(c_i c_j^{-1};q,y)}} \oint \prod_i \frac{dz_i}{2\pi i z_i}\prod_{i\neq j}\frac{1}{\Delta(z_i z_j^{-1};q,y)}$$

$$\times \prod_{i,j=1}^{N} \overline{\Delta}_{\alpha}(a_j z_i^{-1})\overline{\Delta}_{\beta}(z_i b_j^{-1})\overline{\Delta}_{\gamma}(c_j z_i^{-1})\overline{\Delta}_{\delta}(z_i d_j^{-1}) .$$

N-comp. spin for G = U(N)

continuous spin (sum → integral)

$$\Delta(a;q,y) := \frac{\theta_1(y^{-1}a;q)}{\theta_1(a;q)}$$

theta function

$$\overline{\Delta}_r(x;q,y) := \Delta\left(y^{\frac{r}{2}}x;q,y\right)$$

(YBE can be proven directly)

For $G = U(1)$, and after dim. red to

1D QM, simplifies to [W. Yan + MY '15]

$$\mathcal{R}^{U(1)}\begin{pmatrix}\delta & \gamma\\ \alpha & \beta\end{pmatrix}\begin{bmatrix}\boldsymbol{d} & \boldsymbol{c}\\ \boldsymbol{a} & \boldsymbol{b}\end{bmatrix}(z)$$

$$=\sqrt{\frac{\overline{\mathbf{\Delta}}_{\alpha+\delta}(\boldsymbol{a}-\boldsymbol{d},z)\overline{\mathbf{\Delta}}_{\alpha+\beta}(\boldsymbol{a}-\boldsymbol{b},z)}{\overline{\mathbf{\Delta}}_{\gamma+\beta}(\boldsymbol{c}-\boldsymbol{b},z)\overline{\mathbf{\Delta}}_{\gamma+\delta}(\boldsymbol{c}-\boldsymbol{d},z)}}\overline{\mathbf{\Delta}}_{\gamma-\alpha}(\boldsymbol{c}-\boldsymbol{a},z) \tag{3.31}$$

$$+\sqrt{\frac{\overline{\mathbf{\Delta}}_{\gamma+\beta}(\boldsymbol{c}-\boldsymbol{b},z)\overline{\mathbf{\Delta}}_{\gamma+\delta}(\boldsymbol{c}-\boldsymbol{d},z)}{\overline{\mathbf{\Delta}}_{\alpha+\beta}(\boldsymbol{a}-\boldsymbol{b},z)\overline{\mathbf{\Delta}}_{\alpha+\delta}(\boldsymbol{a}-\boldsymbol{d},z)}}\overline{\mathbf{\Delta}}_{\alpha-\gamma}(\boldsymbol{a}-\boldsymbol{c},z)\,.$$

$$\mathbf{\Delta}(\boldsymbol{x},z) := \frac{\sinh(\boldsymbol{x}-z)}{\sinh(\boldsymbol{x})}\,,$$

$$\overline{\mathbf{\Delta}}_r(\boldsymbol{x},z) := \frac{\sinh\left(\boldsymbol{x}+(\frac{r}{2}-1)z\right)}{\sinh\left(\boldsymbol{x}+\frac{r}{2}z\right)}\,.$$

Q: Is this new?

simply sine functions!

Summary (Gauge/YBE Correspondence)

YBE $\rightsquigarrow$ "Yang-Baxter Duality"

Bazhanov-Sergeev $\xleftarrow{S^1 \times S^3}$ 4d $\mathcal{N}=1$

new solution $\xleftarrow{S^1 \times S^3/\mathbb{Z}_r}$

[cf. talk by Kels]

new? $\xleftarrow{S^2 \times T^2}$

tip of iceberg?

new? $\xleftarrow{T^2}$ 2d $\mathcal{N}=(2,2)$

new? $\xleftarrow{S^1}$ 1d $\mathcal{N}=4$

⋮

Outlook

* mathematical proof ← [Kel's talk]
* $U_{p,q;r}(sl_N)$ underlying $(S^1 \times S^3/\mathbb{Z}_r)_{p,q}$ index ?
* cluster-algebra enriched YBE
* BAE, fusion, boundary YBE, ...
* tetrahedron equation? (in progress)
* 2d CFT in the continuum limit? ← [Faddeev - Volkov]
* root-of-unity degeneration [generalization of chiral Potts?]

Yang-Baxter duality:

"integrability in theory space"

intimately tied with fundamental principles of our Nature

locality / unitarity / Lorentz sym.

From YB-duality we can go back to YB-equation by computing its SUSY partition function

path integral replaced by finite-dim integral over saddle pt

- Lagrangian

$$\mathcal{L} = \sum_{v \in V} \mathcal{L}_v(A_\mu^v) + \sum_{e \in E} \mathcal{L}(\Phi_e, A_\mu^v, A_\mu^{v'})$$

- partition function

$$Z = \int \prod_{v \in V} \mathcal{D}A_\mu^v \prod_{e \in E} \mathcal{D}\Phi_e \, e^{i\mathcal{L}}$$

$\Longrightarrow$

- energy

$$\mathcal{E} = \sum_{v \in V} \mathcal{E}_v(\vec{S}_v) + \sum_{e \in E} \mathcal{E}_e(\vec{S}_v, \vec{S}_{v'})$$

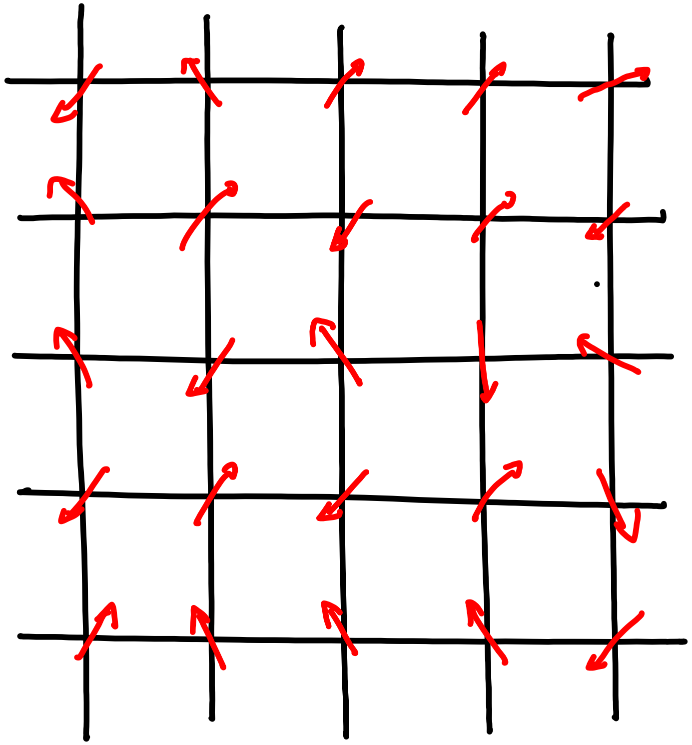
edge e between v & v'

- partition function

$$Z = \sum_{\{\vec{S}_v\}} e^{-\beta \mathcal{E}(\{S_v\})}$$

e.g. Cartan of SU(N)

stat-mech



- spin $\vec{S}_v$ at vertex $v \in V$
- energy

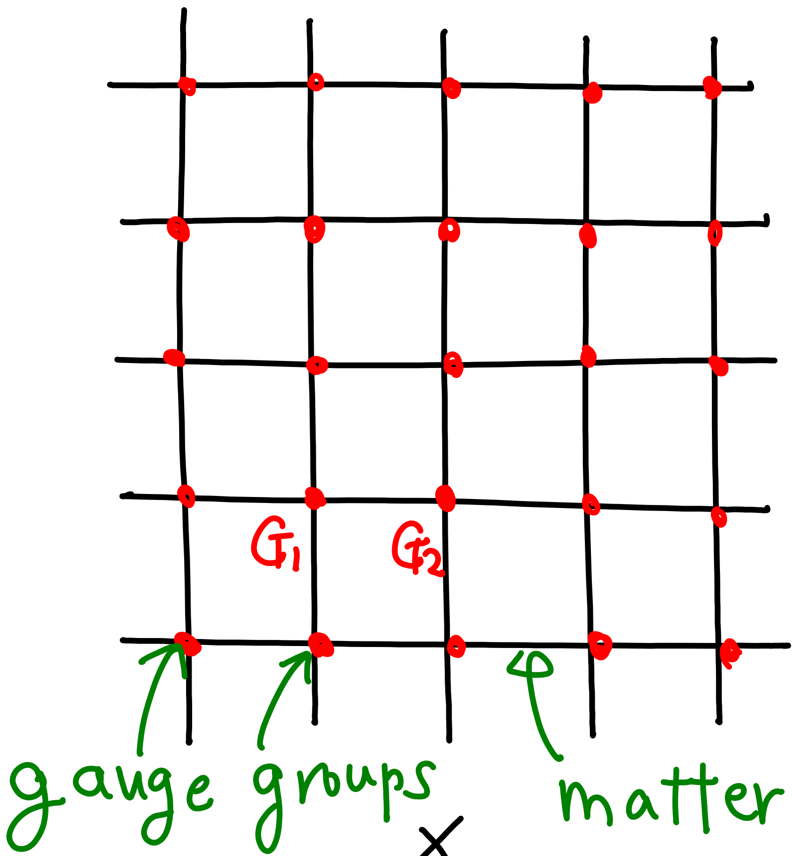
$$\mathcal{E} = \sum_{v \in V} \mathcal{E}_v(\vec{S}_v) + \sum_{e \in E} \mathcal{E}_e(\vec{s}_v, \vec{s}_{v'})$$

edge e between v & v'

- partition function

$$Z = \sum_{\{\vec{S}_v\}} e^{-\beta \mathcal{E}(\{s_v\})}$$

(quiver) gauge theory



$\times\ \mathbb{R}^{d,1}$

- gauge group G_v at vertex $v \in V$

  $A_\mu^v(x)$

- matter charged under $G_v \times G_{v'}$ for an edge e between v, v'

  $\Phi_e(x)$

- Lagrangian

$$\mathcal{L} = \sum_{v \in V} \mathcal{L}_v(A_\mu^v) + \sum_{e \in E} \mathcal{L}(\Phi_e, A_\mu^v, A_\mu^{v'})$$

- partition function

$$Z = \int \prod_{v \in V} \mathcal{D}A_\mu^v \prod_{e \in E} \mathcal{D}\Phi_e \; e^{i\mathcal{L}}$$

Generalization of YBE

Q: any interest?

2D $\mathcal{N}=(2,2)$ for S^2

$$\rightsquigarrow Z^{\text{electric}}_{S^2}(t_e) = Z^{\text{magnetic}}_{T^2}(t_m = -t_e)$$

complexified FI

cluster. y-variable [Benini Park Zhao]

$$R_{12}(u, x_{12})\, R_{13}(u+v, x_{13})\, R_{23}(v, x_{23})$$
$$= R_{23}(v, x_{23}')\, R_{13}(u+v, x_{13}')\, R_{12}(u, x_{12}')$$

x_{ij}' : rational function of x_{ij}

cluster y-variables