Gauge Theory and Integrability

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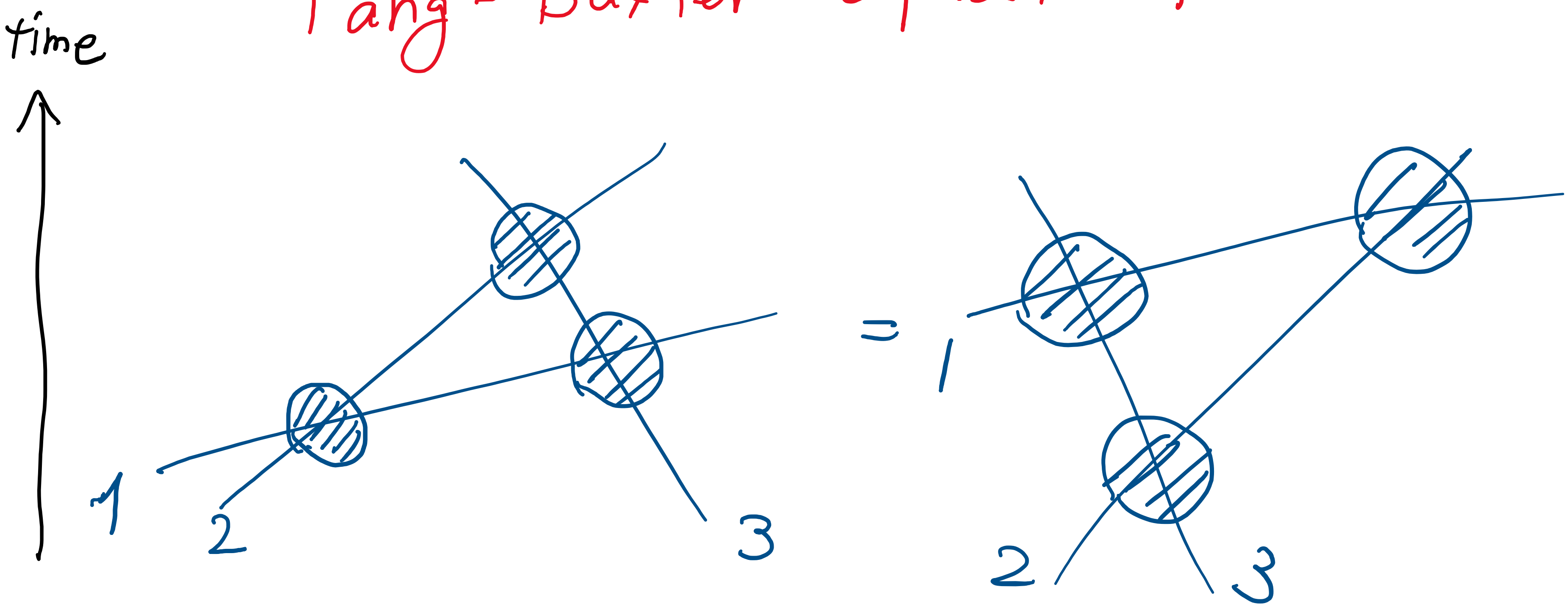
work w/ Kevin Costello
Edward Witten

part I, II, ...
1709, 1802

integrable models :

Yang - Baxter equation



$$R_{12}(z_1 - z_2)\, R_{13}(z_1 - z_3)\, R_{23}(z_2 - z_3)$$

$$= R_{23}(z_2 - z_3)\, R_{13}(z_1 - z_3)\, R_{12}(z_1 - z_2)$$

$$R_{12}(z_1-z_2)\,R_{13}(z_1-z_3)\,R_{23}(z_2-z_3)$$
$$= R_{23}(z_2-z_3)\,R_{13}(z_1-z_3)\,R_{12}(z_1-z_2)$$

in $\mathrm{End}(V_1 \otimes V_2 \otimes V_3)$

$R(\underbrace{z}_{\text{spectral parameter}})$: R-matrix $\qquad R_{12}(z) \in \mathrm{End}(V_1 \otimes V_2)$

highly overconstrained equations

Q: Can we explain

<u>why</u> integrable models exist?

[esp. spectral param]

- R-matrix
- sym $\quad Y_\hbar(\mathfrak{g}),\ U_q(\mathfrak{g}),\ U_{q,\tau}(\mathfrak{g})$
- classification

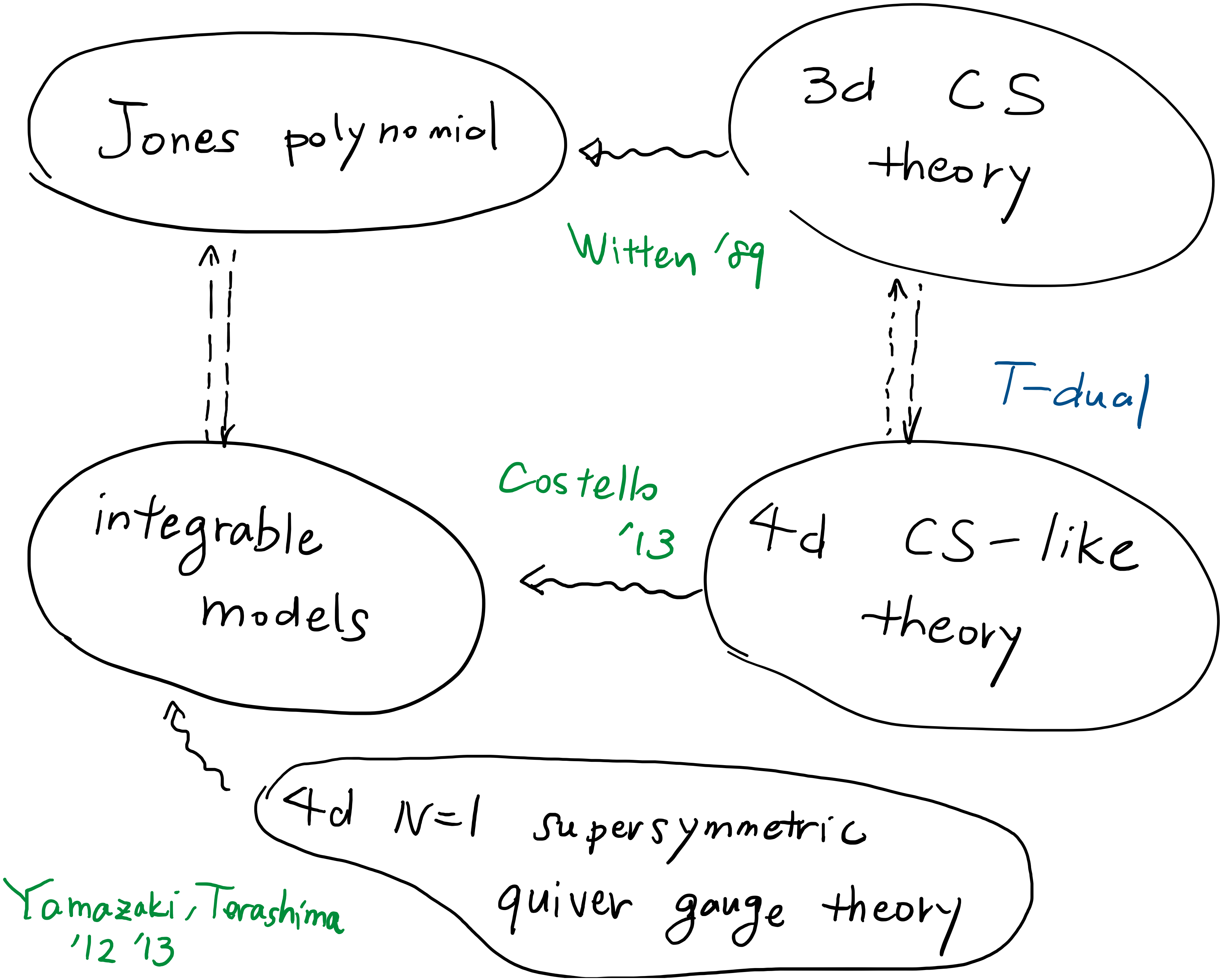
  ⋮

Witten's paper ('90)
"Gauge theories, vertex models, and quantum groups"

There are several obvious areas for further investigation. In terms of statistical mechanics, one compelling question is to understand the origin of the spectral parameter (and the elliptic modulus) in IRF and vertex models; this is essential for explaining the origin of integrability. Another question, which may or may not be related, is to understand the spin models formulated only rather recently [24] in which the spectral parameter is not an abelian variable (as in previous construc-

Witten interview by Tohru Eguchi ('90)

うです(笑)。勇気づけられています。また，ここ2〜3年私がとりつかれている問題は可解格子模型におけるいわゆる spectral parameter の問題です。Chern-Simons 理論における Wilson line の期待値はいろいろな仕方で計算できますが，可解格子模型を用いる方法——Jones が初めて試みたわけですが——では spectral parameter を無限大にとばしてしまいます。しかし可解模型をより深く理解するには spectral parameter が本質的に重要です。したがって，もし我々が Chern-Simons 理論のアプローチを信ずるならば，spectral parameter を含むような一般化が必要です。また，次のような問題——解がないかも知れませ


Jones polynomial
3d CS theory
Witten '89
T-dual
integrable models
Costello '13
4d CS-like theory
4d N=1 supersymmetric quiver gauge theory
Yamazaki, Terashima '12 '13

the four-dimensional theory

[Costello '13]

4d theory [Costello]

$$\mathcal{L} = \frac{1}{2\pi k} \int_{\Sigma \times C} \omega \wedge \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right)$$

4d theory [Costello]

$$\mathcal{L} = \frac{1}{2\pi k} \int_{\Sigma \times C} \omega \wedge \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right)$$

* Defined on 4-mfd of the form

$$\underbrace{\Sigma}_{\substack{\text{topological} \\ x,\, y}} \times \underbrace{C}_{\substack{\text{holomorphic} \\ z,\, \bar{z}}}$$

4d theory [Costello]

$$\mathcal{L} = \frac{1}{2\pi K} \int_{\Sigma \times C} \overbrace{\omega}^{\text{1-form}} \wedge \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right)$$

* Defined on 4-mfd of the form

$$\underbrace{\Sigma}_{\substack{\text{topological} \\ x,\, y}} \times \underbrace{C}_{\substack{\text{holomorphic} \\ z,\, \bar{z}}}$$

* ω : hol. 1-form on C

[e.g. $\omega = dz$ for $C = \mathbb{C}$]

4d theory [Costello]

$$\mathcal{L} = \frac{1}{2\pi k} \int_{\Sigma \times C} \underbrace{\overbrace{\omega}^{1\text{-form}} \wedge \overbrace{\mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right)}^{\text{Chern-Simons 3-form}}}_{4\text{-form}}$$

* Defined on 4-mfd of the form

$$\underbrace{\Sigma}_{\substack{\text{topological} \\ x,\, y}} \times \underbrace{C}_{\substack{\text{holomorphic} \\ z,\, \bar{z}}}$$

* ω : hol. 1-form on C

e.g. $\omega = dz$ for $C = \mathbb{C}$

4d theory [Costello]

$$\mathcal{L} = \frac{1}{2\pi K} \int_{\Sigma \times C} \underbrace{\omega \wedge \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right)}_{\text{complex Lagrangian}}$$

* the action is complex

⇝ for full definition, need to specify integration contour Γ

$$Z = \int_{\Gamma} [\mathcal{D}A]\, e^{-L[A]}$$

Γ : middle-dim. int. cycle in the complexified gauge field config.

* here we do perturbation in $\hbar$ around isolated classical contribution

4d theory [Costello]

$$\mathcal{L} = \frac{1}{2\pi\hbar} \underbrace{\int_{\Sigma\times C}}_{\text{4d integral}} \omega \wedge \overbrace{\mathrm{Tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right)}^{[\text{mass}]^3}$$

* expansion parameter $\hbar$ is dimensionfull

$[\hbar] = [\text{mass}]^{-1}$

$\rightsquigarrow$ theory is IR free

non-renormalizable by power counting

Consider the simplest case $C = \mathbb{C}_z$, $\omega = dz$

$$\mathcal{L} = \frac{1}{2\pi\hbar}\int_{\Sigma\times\mathbb{C}} dz \wedge \mathrm{Tr}\left(A\wedge dA + \frac{2}{3}A^3\right)$$

locally always the case

* Since the integrand contains dz, only 3 components of A matter:

$$A = A_x\,dx + A_y\,dy + A_{\bar z}\,d\bar z + \cancel{A_z\,dz}$$

The gauge transformation as usual

$$A_i \to g^{-1}A_i g + g^{-1}\partial_i g, \quad g \in G$$

$$i = x, y, \bar z$$

(we disregard the A_z component)

Consider the simplest case $C = \mathbb{C}_z$, $\omega = dz$

$$\mathcal{L} = \frac{1}{2\pi\hbar}\int_{\Sigma\times\mathbb{C}} dz \wedge \mathrm{Tr}\left(A\wedge dA + \frac{2}{3}A^3\right)$$

* Indeed, by integrating by parts

$$\mathcal{L} = -\frac{1}{4\pi\hbar}\int_{\Sigma\times\mathbb{C}} \underset{\omega}{z}\; \overbrace{\mathrm{Tr}(F\wedge F)}^{\text{gauge-invariant}}$$

"holomorphic θ-angle"

Consider the simplest case $C = \mathbb{C}_z$, $\omega = dz$

$$\mathcal{L} = \frac{1}{2\pi\hbar}\int_{\Sigma\times\mathbb{C}} dz \wedge \mathrm{Tr}\left(A\wedge dA + \frac{2}{3}A^3\right)$$

* equation of motion

$F_{xy} = 0$: flat connection along Σ

$F_{x\bar{z}} = F_{y\bar{z}} = 0$: holomorphic along $\mathbb{C}$

$(\partial_{\bar{z}} A_x = \partial_{\bar{z}} A_y = 0$ if $A_{\bar{z}} = 0)$

We will consider pert. theory

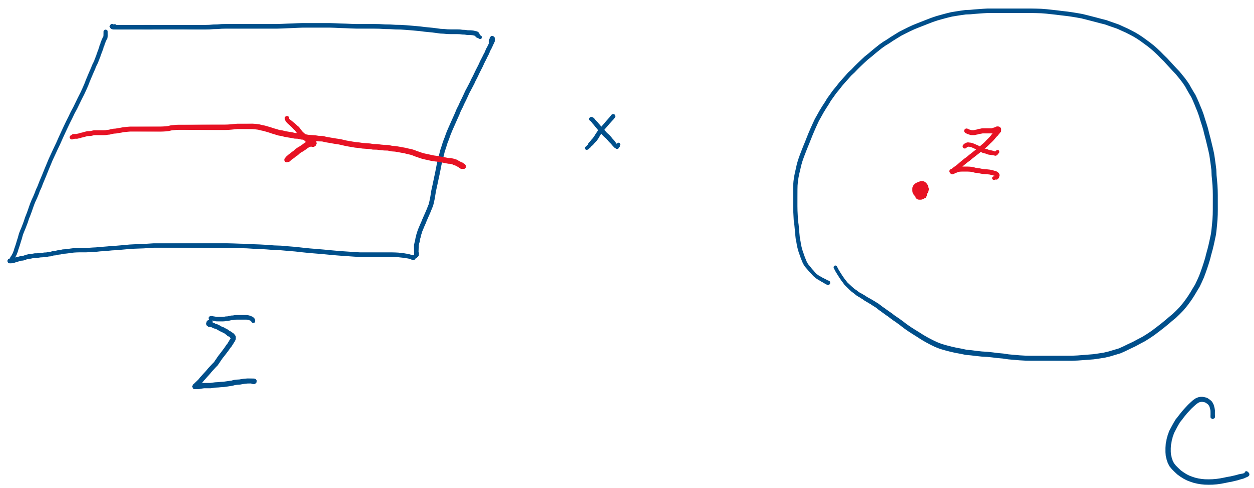
around the trivial solution $A = 0$

integrability

Wilson line

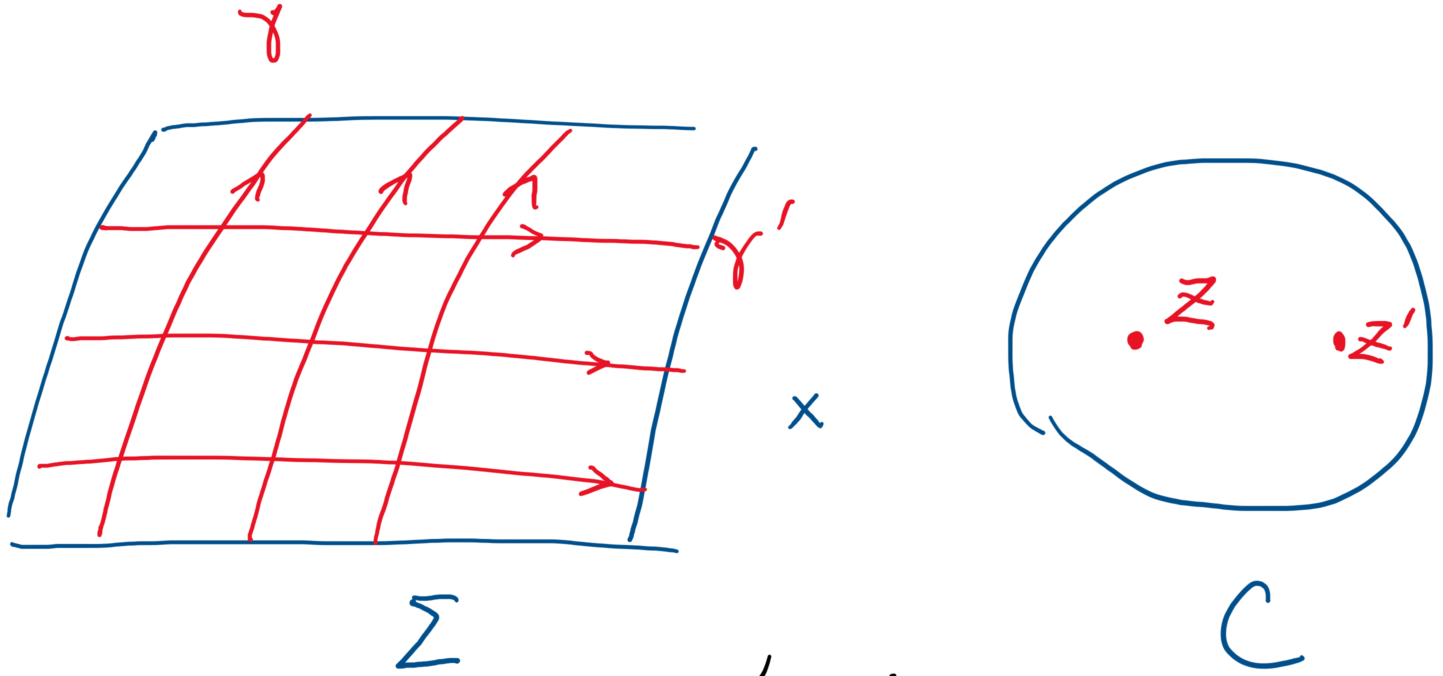
$$W_{\underset{\sim}{\gamma}} = \left\langle \mathrm{Tr}\, \exp \int_{\gamma} A \right\rangle$$

path along Σ, located at point z



generate the statistical lattice

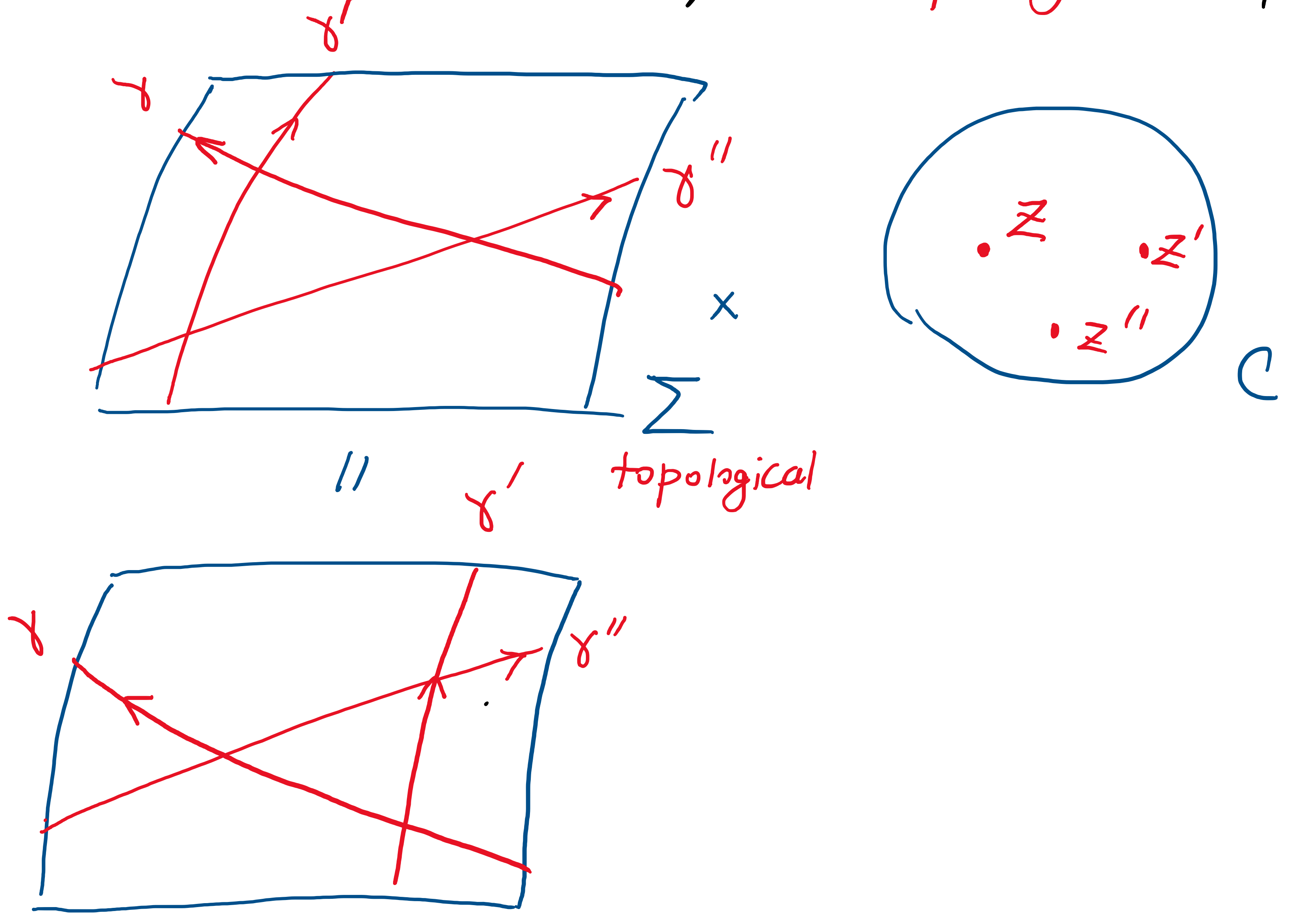
from Wilson lines



integrable lattice model

"extra dimension" for spectral parameter

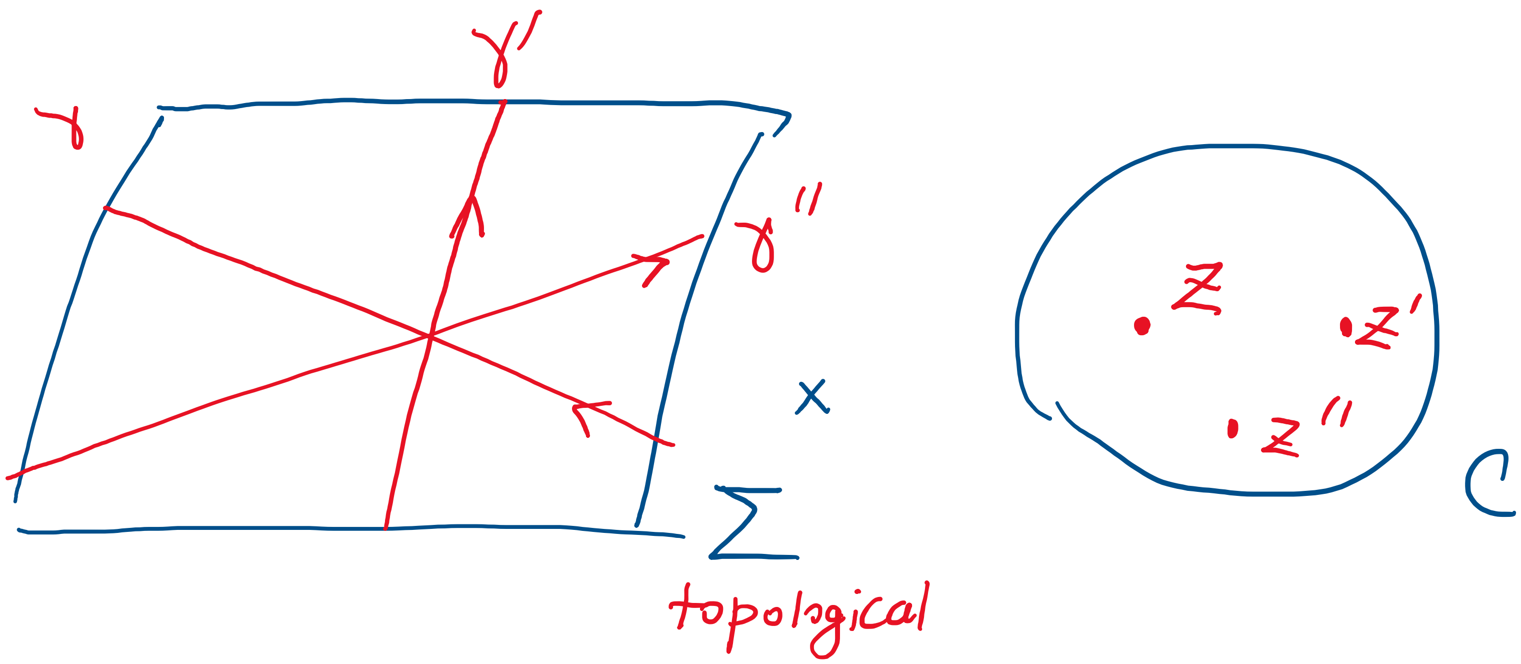
Yang-Baxter equation follows since the theory is topological along Σ



no singularity when 3 line cross,

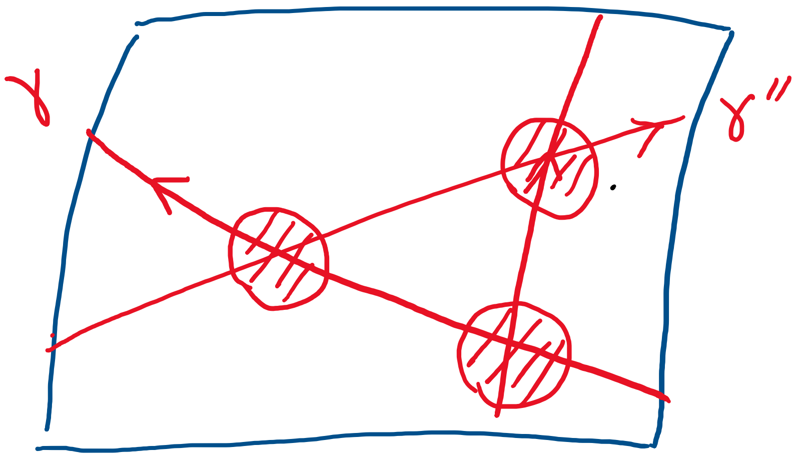
since the Wilson lines are separate

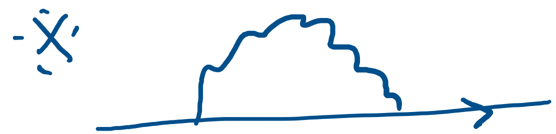
along C and never touch



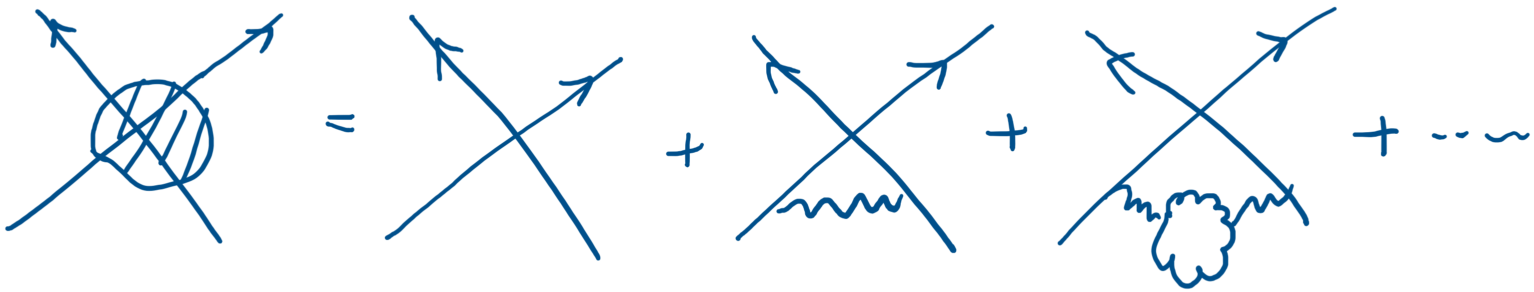
moreover, since the theory is IR free and topological along Σ,

we have factorized contribution from each crossing



( $= 0$ for straight line)

R-matrix :



perturbative

classification

$$\mathcal{L} = \frac{1}{2\pi\hbar} \int_{\Sigma \times C} \omega \wedge \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A^3\right)$$

ω and $\hbar$ appear in combination

$\hbar \to \infty$ around zero of ω : $\omega \to 0$

$\rightsquigarrow$

let's therefore impose ω has no zero,

for perturbation in $\hbar$

C in general has boundary points

$$C = \underbrace{\bar{C}}_{\text{compactification}} \setminus \{\text{points}\}$$

ω : hol. 1-form on C, no zero

$\downarrow$ Riemann-Roch theorem

3 possibilities

$\omega = dz \qquad C = \mathbb{C} \qquad$ rational

$\omega = \dfrac{dz}{z} \qquad C = \mathbb{C}^\times \qquad$ trigonometric

$\omega = dz \qquad C = E \qquad$ elliptic

matches with classification of Belavin & Drinfeld

identify w/ $\hbar$ in $\mathcal{L}$

$$\underbrace{R_{\hbar}(z)}_{\text{quasi-classical R-matrix}} = Id + \hbar \underbrace{r(z)}_{\text{classical R-matrix}} + \hbar^2 r'(z) + \cdots$$

today: $C = \mathbb{C}$ (rational)

$C = \mathbb{C}^\times, E$ cases similar,

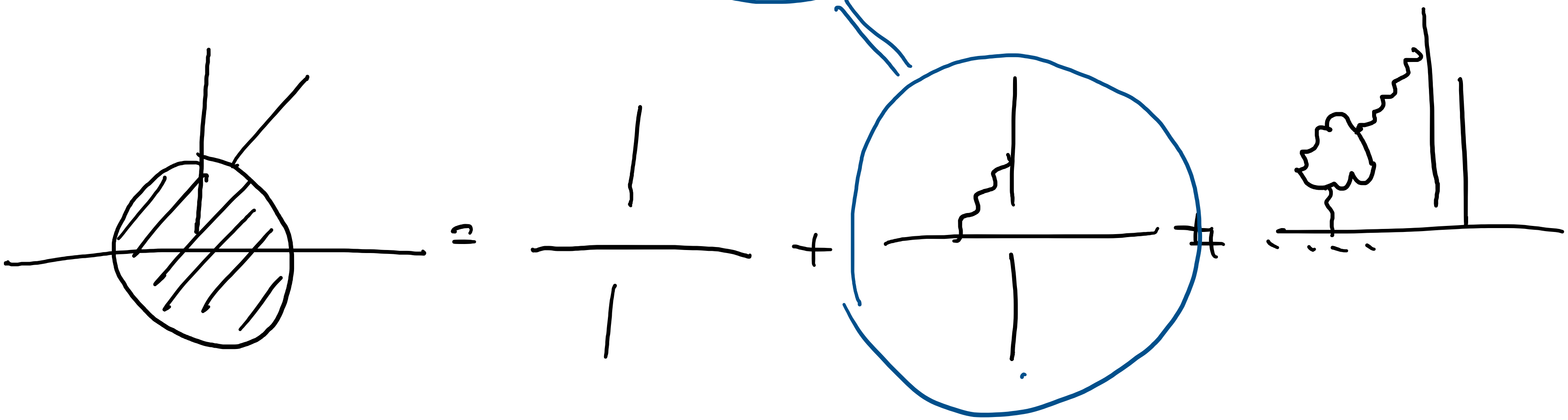
but with interesting twists

R - matrix

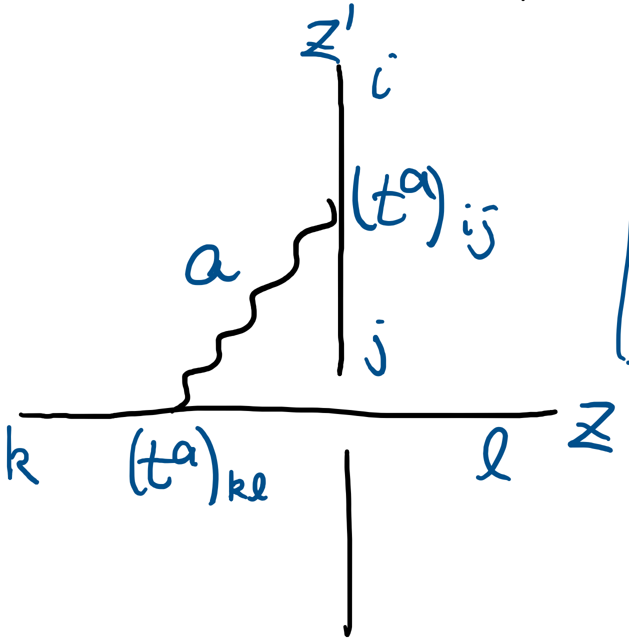
perturbative Feynman diagram computation

reproduce perturbative R- matrix

$$R_{\hbar}(z) = \mathrm{id} + \hbar\, r(z) + \cdots$$



lowest order computation gives classical R-matrix $r(z)$



$$\boxed{r(z) = \frac{(t^a)_{ij}(t^a)_{kl}}{z - z'}}$$

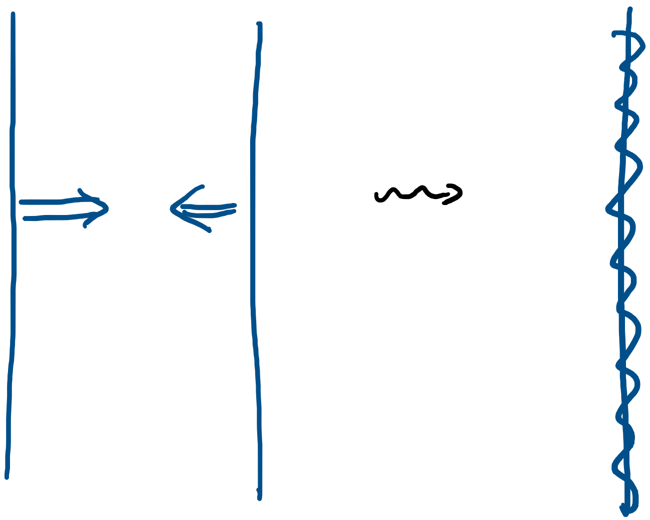
reproduces the known answer

this together w/ YBE is enough for reproducing the full R-matrix, all order in $\hbar$

[Drinfeld, also CWY II]

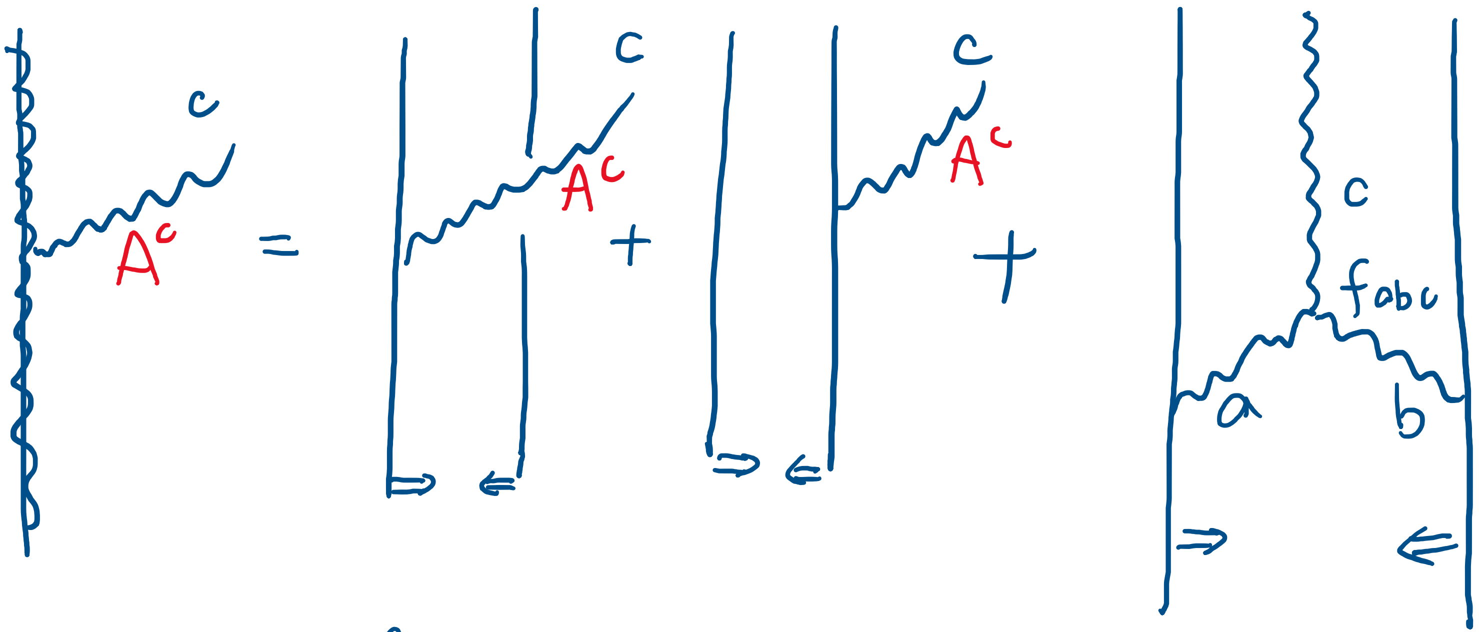
Yangian

Yangian arises from OPE of Wilson lines



W_1 $\quad$ W_2 $\quad$ $W_1 \circ W_2$: product Wilson line

associative



$\dfrac{\partial A^c}{\partial z}$ ~ derivative coupling

$$A^c : \quad \underbrace{t^c \otimes 1 + 1 \otimes t^c}_{\text{level 0 generater}}$$

$$\frac{\partial A^c}{\partial z} : \quad \underbrace{-\frac{\hbar}{2} f^c{}_{ab}\, t^a \otimes t^b}_{\text{level 1 generator}}$$

matches with coproduct of Yangian (up to $O(\hbar^2)$)

the symmetry algebra here is

$$\text{Yangian} \quad Y_\hbar(\mathfrak{g}) \underset{\hbar \to 0}{\rightsquigarrow} \mathcal{U}(\mathfrak{g}[[z]])$$

$$\{ t^a z^{n \geq 0} \}$$

$\hbar$ deformation

the Wilson line allows for derivative couplings

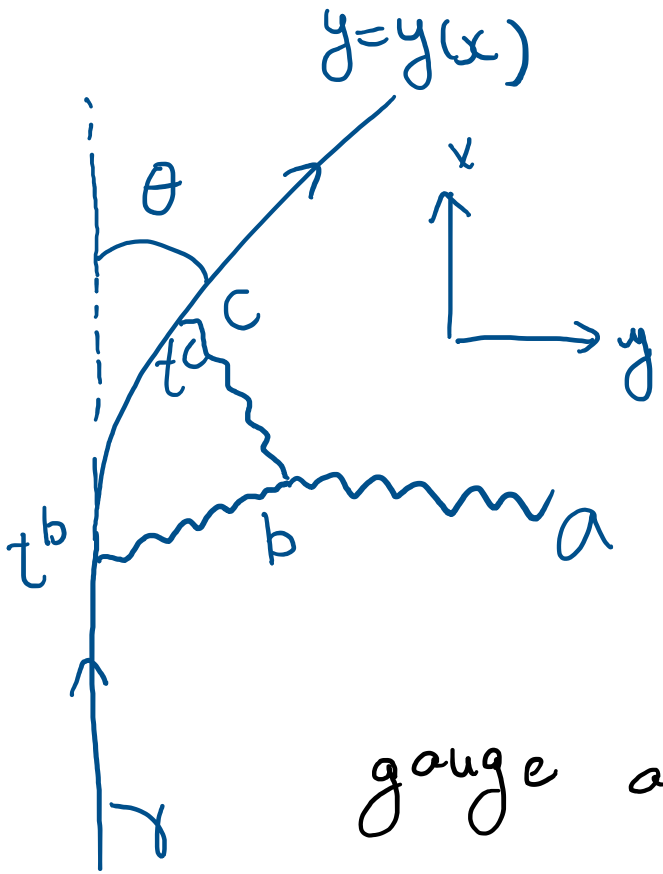
$$\langle \underbrace{W_\gamma}_{\substack{\text{along } \Sigma \\ \text{not along } C}} \rangle = \left\langle \mathrm{Tr}_{\hat\rho} \exp \int_{\gamma \times \{z_0\}} \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{n!} (\partial^n A)(z_0) \right\rangle$$

$$= \left\langle \mathrm{Tr}_{\underbrace{\hat\rho}_{\text{repr. of } \mathcal{U}(\mathfrak{g}[[z]])}} \exp \int_{\gamma \times \{z_0\}} A(z) \right\rangle$$

and hence we have $\mathcal{U}(\mathfrak{g}[[z]])$ at $\hbar = 0$

Framing Anomaly

consider curved Wilson line



Color factor

$$f^{abc}\, t_b t_c = \frac{1}{2} f^{abc} [t_b, t_c]$$

$$= \frac{1}{2} f^{abc} f_{bcd}\, t^d = \underbrace{h^\vee}_{\text{dual Coxeter number}} t^a$$

gauge anomaly

$$\delta I = -\frac{\hbar h^\vee}{\pi} \int_\gamma dx \, \frac{d^2y}{dx^2} \, \partial_z \varepsilon(x, 0)$$

for

$$A \to A + d\varepsilon$$

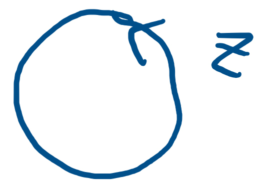
Fortunately, gauge anomaly

$$\delta I = -\frac{\hbar h^\vee}{\pi}\int_\gamma dx \, \frac{d^2y}{dx^2}\, \partial_z \varepsilon(x,0)$$

can be canceled by shifting z along Wilson line:

$$I = \int_\gamma dx \, A_x\left(x,\ z - \frac{\hbar h^\vee}{\pi}\frac{dy}{dx}\right)$$

i.e. $z - \frac{\hbar h^\vee}{\pi}\theta$ is constant along WL

[※ in particular no closed loop 

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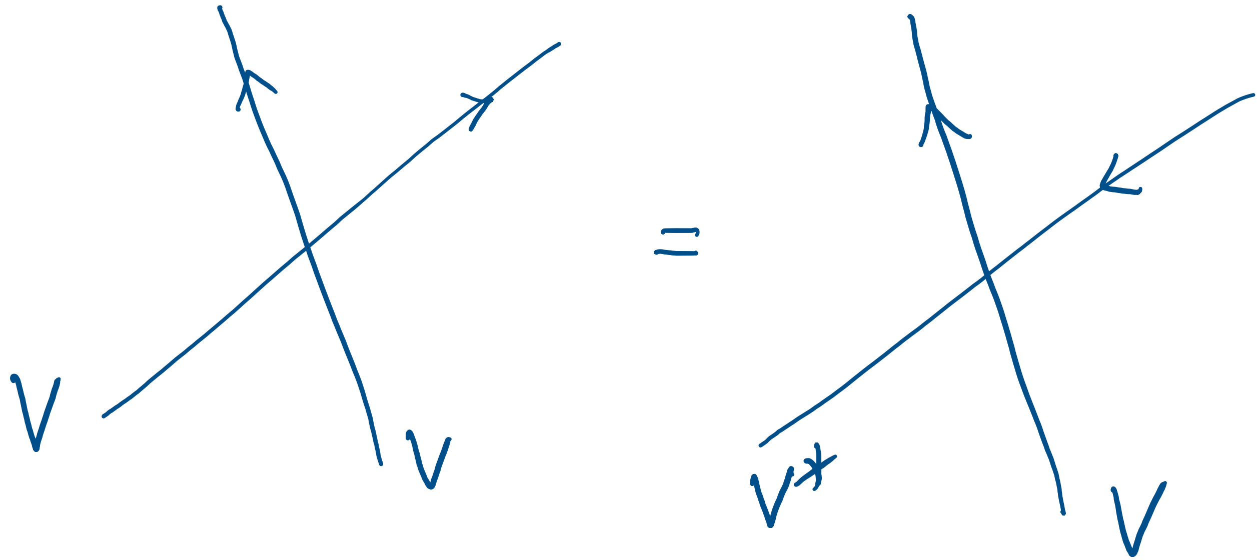
For $\theta = \pi$ we need to shift z by $\hbar h^\vee$

this is indeed the case:



but

$$R_{VV}(z) \neq R_{V^*V}(-z)$$

For $\theta = \pi$ we need to shift z by $\hbar h^\vee$

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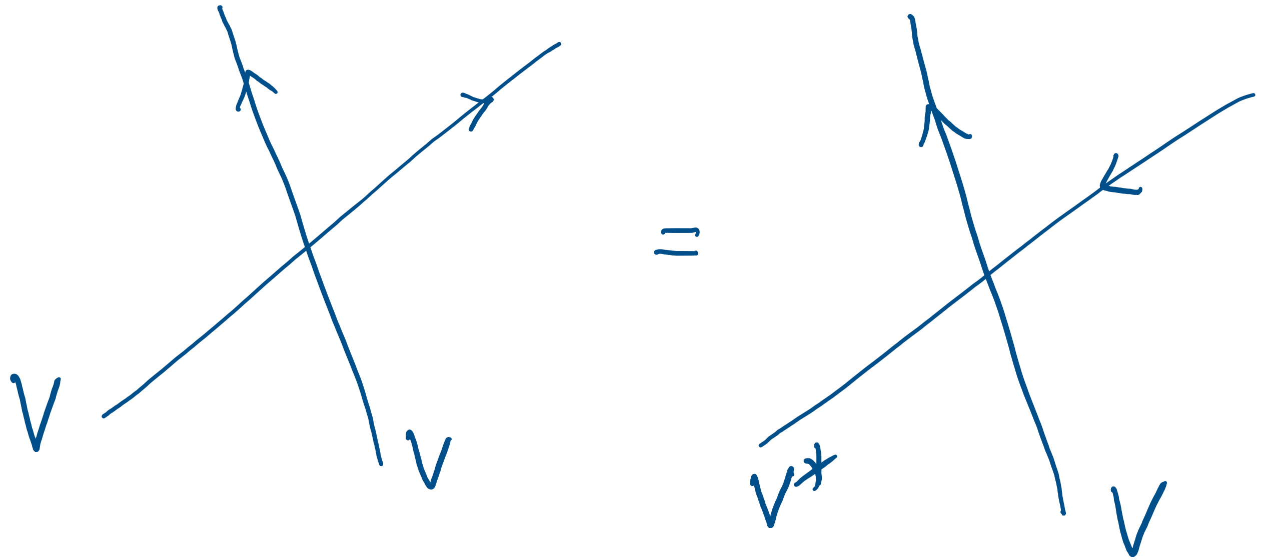

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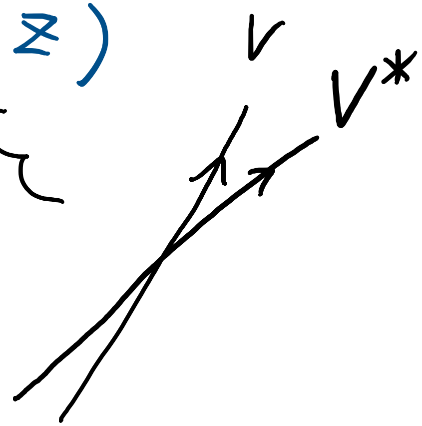
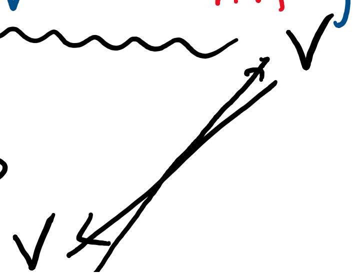


$$R_{VV}(z-\hbar h^\vee) = R_{V^*V}(-z)$$



2-loop anomaly

$$\mathcal{U}(\mathfrak{g}[[z]]) \underset{\hbar=0}{\rightsquigarrow} Y_\hbar(\mathfrak{g})$$

not all representation of $\mathcal{U}(\mathfrak{g}[[z]])$

lifts to representation of $Y_\hbar(\mathfrak{g})$:

anomaly for quantizing Wilson lines

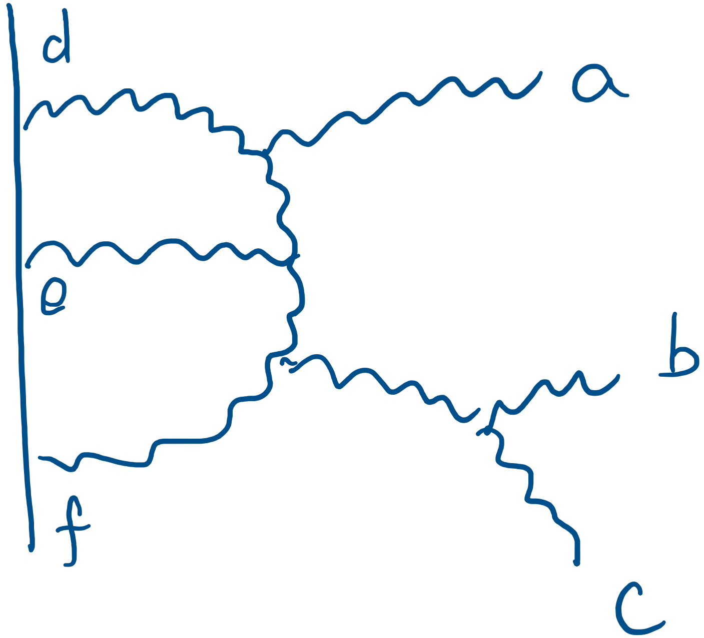
Yangian relation:

(J: $t_a = t_a^0 \to t_a^1$
level 0 → level 1)

$$\underbrace{\left[J(t_a), J([t_b, t_c])\right] + (\text{cyclic})}_{\text{Jacobi identity}}$$

$$= \frac{\hbar^2}{24} \underbrace{\left([t_a, t_d], [[t_b, t_e], [t_c, t_f]]\right) \sum \left[t_d t_e t_f + (\text{cyclic})\right]}_{\text{non-zero e.g. for adj. repr. of } so_N}$$

we find that the 2-loop diagram



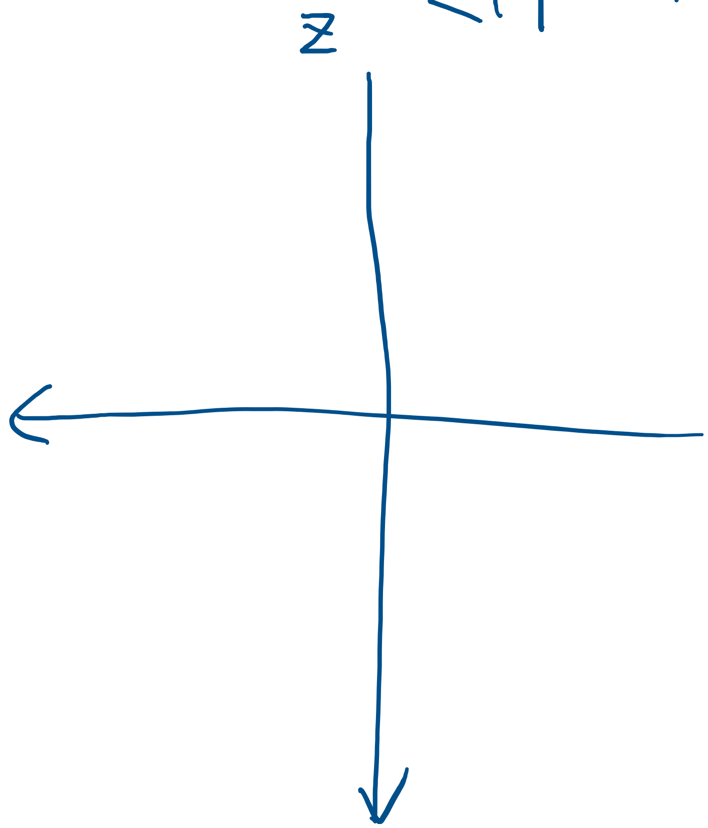
indeed generates the RHS of the Yangian relation

and all other 2-loop diagram canceled by counterterms

RTT presentation

(all-order $Y_\hbar(\mathfrak{g})$)

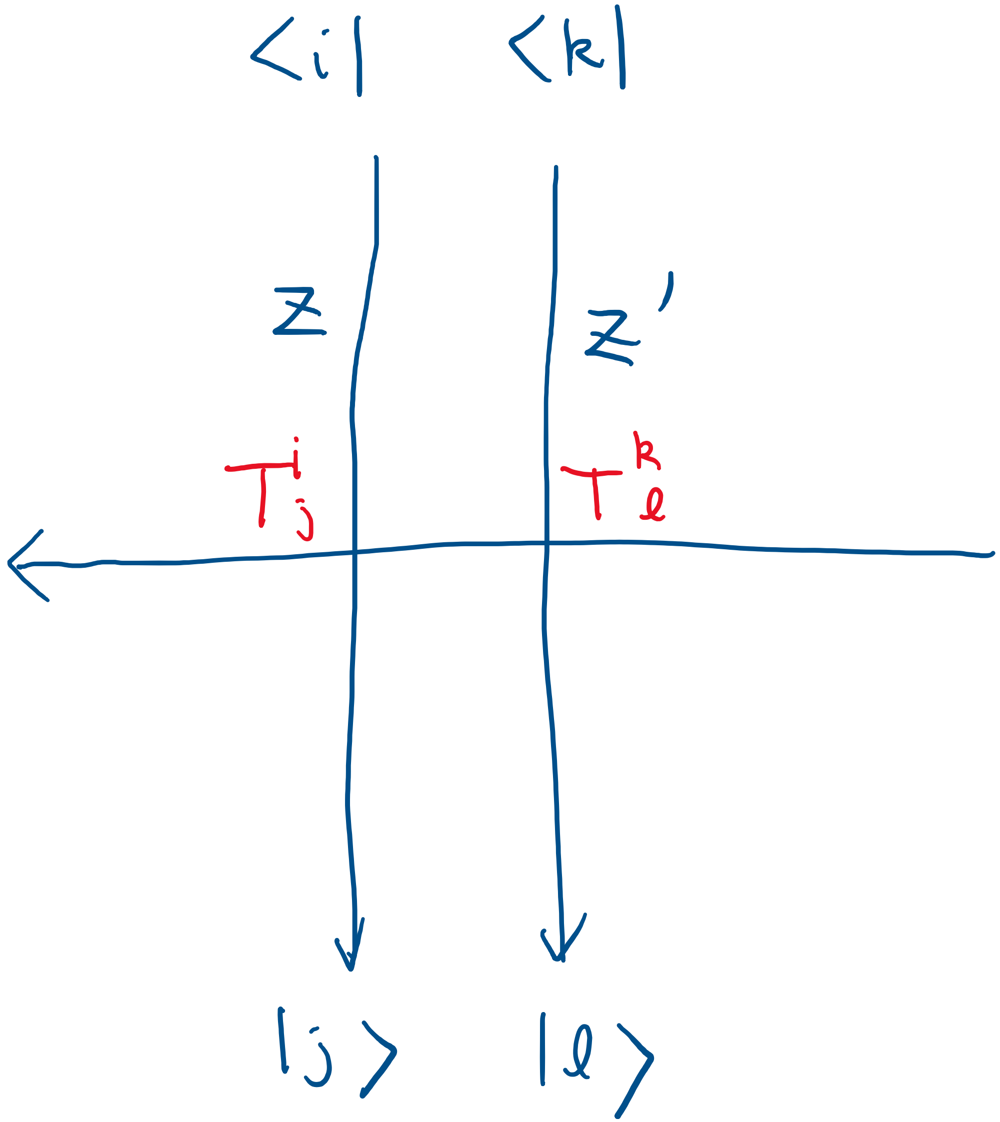
we can regard one Wilson line as an operator acting on another WL:



$\langle i|$ incoming state of vertical WL

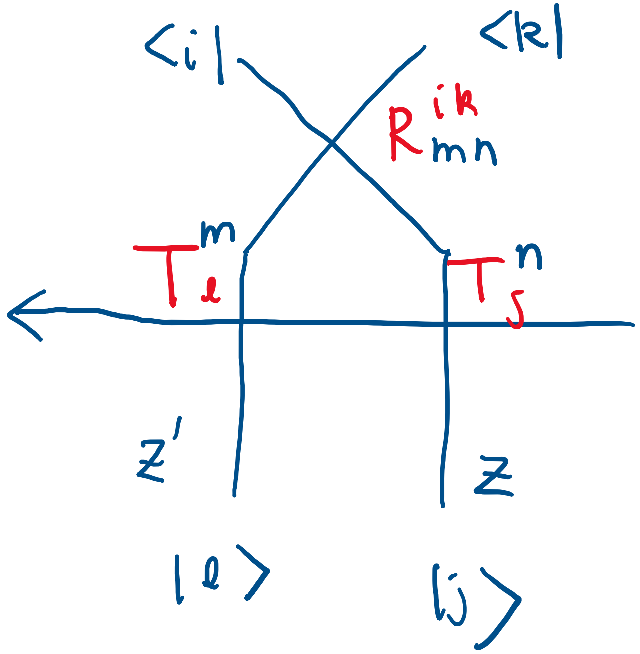
horizontal WL $\{T^i_j(z)\}$

$|j\rangle$ outgoing state of outgoing WL



$$T^i_j(z)\cdot T^k_\ell(z')$$

We can derive



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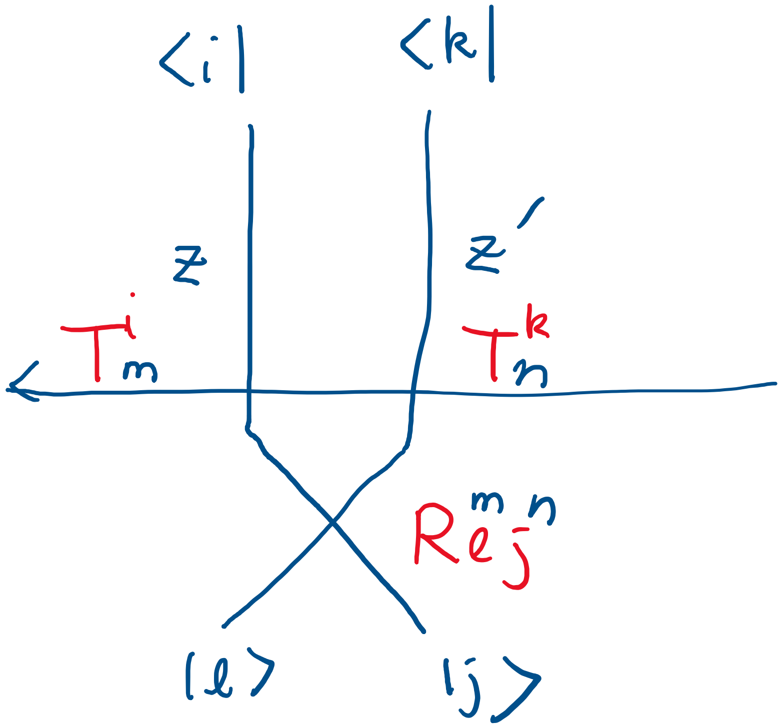


namely, we have

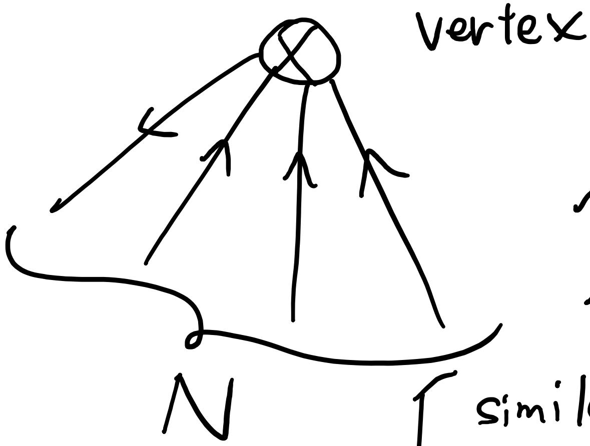
$$\sum_{mn} R^{ik}_{mn}(z-z')\, T^{m}_{l}(z')\, T^{n}_{j}(z) = \sum_{mn} R^{mn}_{lj}(z-z')\, T^{i}_{m}(z)\, T^{k}_{n}(z')$$

when we expand $T^i_j(z)$ in powers of z^{-1}

this gives defining rel. of Yangian for $\mathfrak{gl}_N$ [Drinfeld]

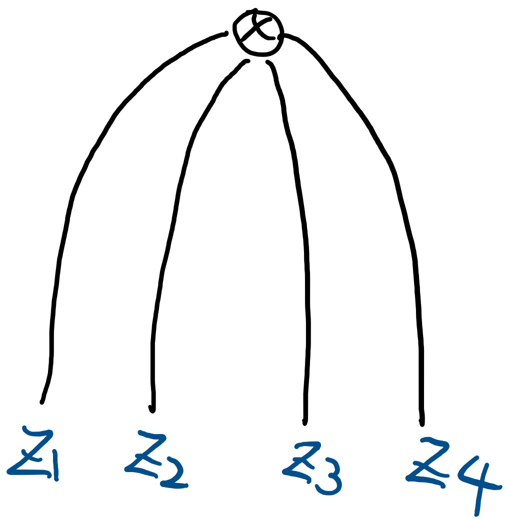
We can generalize the discussion to other $\mathfrak{g} \neq e_8$

e.g. $\mathfrak{g} = \mathfrak{sl}_N$



$\rightsquigarrow$ $\hbar \neq 0$

[similar to framing anomaly]

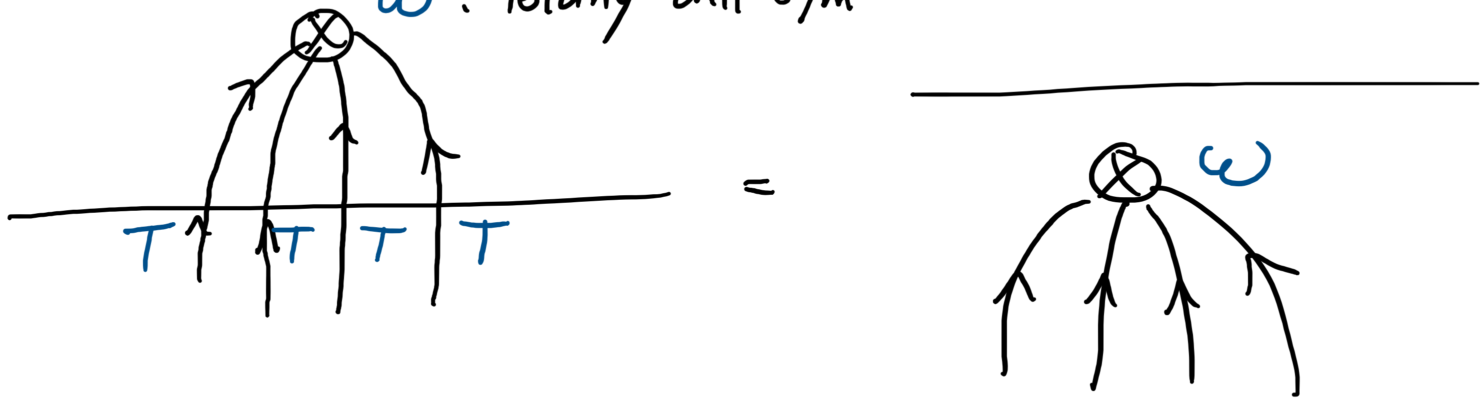


"Network of Wilson lines"

$$z_{i+1} - z_i = \frac{2}{N} \hbar h^\vee$$

an extra relation for $\mathfrak{sl}_N$:

ω : totally anti-sym



$$\boxed{\sum_{k_1 \ldots k_N} \omega_{k_1, \ldots, k_N} T_{i_1}^{k_1}(z) T_{i_2}^{k_2}\left(z + \frac{2}{N}\hbar h^\vee\right) \cdots T_{i_N}^{k_N}\left(z + \frac{2(N-1)}{N}\hbar h^\vee\right) = \omega_{i_1 \ldots i_N}}$$

i.e. $\quad q\,\mathrm{Det}\, T(z) = 1$

we obtained similar relations for all $\mathfrak{g} \neq \mathfrak{e}_8$

Summary

4d Chern-Simons-like theory

perturbative quantization

integrable models

- conceptual explanation of integrability
- known & new results
  * R-matrix
  * Yangian relation (also RTT)
  * perturbative classification
  * vertex for WL ⟵ fusion of R

  ⋮

framing anomaly

many questions [more to come !]

- 2d σ- model

- AdS/CFT integrability [string]

- non-pert. completion [on D4-NS5]

  e.g. root of unity degeneration

- relation to other works

  e.g. Gauge / Bethe [Nekrasov, Shatashvili '09-, ..., Aganagic, Okounkov '17-]

  Gauge / YBE [Yamazaki, Terashima etc '12-]