τ-function for Lens-elliptic discrete Painlevé equation

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Refs

"lens" case (r>1)

Andrew P. Kels + MY

1810.12103 [nlin.SI]

(1709.09993 [math-ph]

1704.03159 [math-ph])

Noumi 1604.06869 [math.QA]

"non-lens" case (r=1)

"Gauge/YBE" ('12–)

review: 1808.04374 [hep-th]

[also many thanks to Y. Yamada]

※ Today's main content:

100% mathematics (w/ proofs)

(combinatoric, algebraic)

but motivated by physics of

4d $\mathcal{N}=1$ supersymmetric gauge theory

(geometry ?)

Introduction

discrete Painleve eqn [Sakai '01]

$$\begin{tikzcd}
\text{elliptic} & E_8 \arrow[ddd] & & & & & & & & \\
\text{trigonometric} & E_8 \arrow[r] \arrow[ddd] & E_7 \arrow[r] & E_6 \arrow[r] & D_5 \arrow[r] & A_4 \arrow[r] & A_2+A_1 \arrow[r] & A_1+A_1 \arrow[r] \arrow[dr] & A_1' \arrow[r] & A_0 \\
 & & & & & & & & A_1 & \\
 & & & & P_{VI} & P_{V} & P_{III} & & & \\
\text{rational} & E_8 \arrow[r] & E_7 \arrow[r] & E_6 \arrow[r] & D_4 \arrow[r] & A_3 \arrow[r] \arrow[dr] & 2A_1 \arrow[r] \arrow[dr] & A_1' \arrow[r] & A_0 & \\
 & & & & & & A_2 \arrow[r] & A_1 \arrow[r] & A_0 & \\
 & & & & & & P_{IV} & P_{II} & P_{I} &
\end{tikzcd}$$

discrete Painleve eqn [Sakai '01]

$$\begin{tikzcd}
\text{elliptic} & E_8 \arrow[dd] \\
\theta(x,\tau) & \\
\text{trigonometric} & E_8 \arrow[r] \arrow[ddd] & E_7 \arrow[r] & E_6 \arrow[r] & D_5 \arrow[r] & A_4 \arrow[r] & A_2+A_1 \arrow[r] & A_1+A_1 \arrow[r] \arrow[dr] & A_1' \arrow[r] & A_0 \\
\sin(x) & & & & & & & & A_1 & \\
 & & & & P_{VI} & P_{V} & P_{III} & & & \\
\text{rational} & E_8 \arrow[r] & E_7 \arrow[r] & E_6 \arrow[r] & D_4 \arrow[r] & A_3 \arrow[r] \arrow[dr] & 2A_1 \arrow[r] \arrow[dr] & A_1' \arrow[r] & A_0 & \\
x & & & & & & A_2 \arrow[r] & A_1 \arrow[r] & A_0 & \\
 & & & & & & P_{IV} & P_{II} & P_{I} &
\end{tikzcd}$$

discrete Painleve eqn [Sakai '01]

$$\begin{tikzcd}
\begin{array}{c}\text{"lens-elliptic"}\\ \theta(x,x;\tau)\\ (x\in\mathbb{Z})\end{array} & \boxed{\boxed{E_8}} \arrow[d] \\
\begin{array}{c}\text{elliptic}\\ \theta(x,\tau)\end{array} & \boxed{E_8} \arrow[d] \\
\begin{array}{c}\text{trigonometric}\\ \sin(x)\end{array} & E_8 \arrow[r] \arrow[ddd] & E_7 \arrow[r] & E_6 \arrow[r] & D_5 \arrow[r] & A_4 \arrow[r] & A_2+A_1 \arrow[r] & A_1+A_1 \arrow[r] \arrow[dr] & A_1' \arrow[r] & A_0 \\
 & & & & & & & & A_1 & \\
 & & & & P_{VI} & P_{V} & P_{III} & & & \\
\begin{array}{c}\text{rational}\\ x\end{array} & E_8 \arrow[r] & E_7 \arrow[r] & E_6 \arrow[r] & D_4 \arrow[r] & A_3 \arrow[r] \arrow[dr] & 2A_1 \arrow[r] \arrow[dr] & A_1' \arrow[r] & A_0 & \\
 & & & & & & A_2 \arrow[r] & A_1 \arrow[r] & A_0 & \\
 & & & & & & P_{IV} & P_{II} & P_{I} &
\end{tikzcd}$$

Today

① Hirota bilinear eqn for τ-function

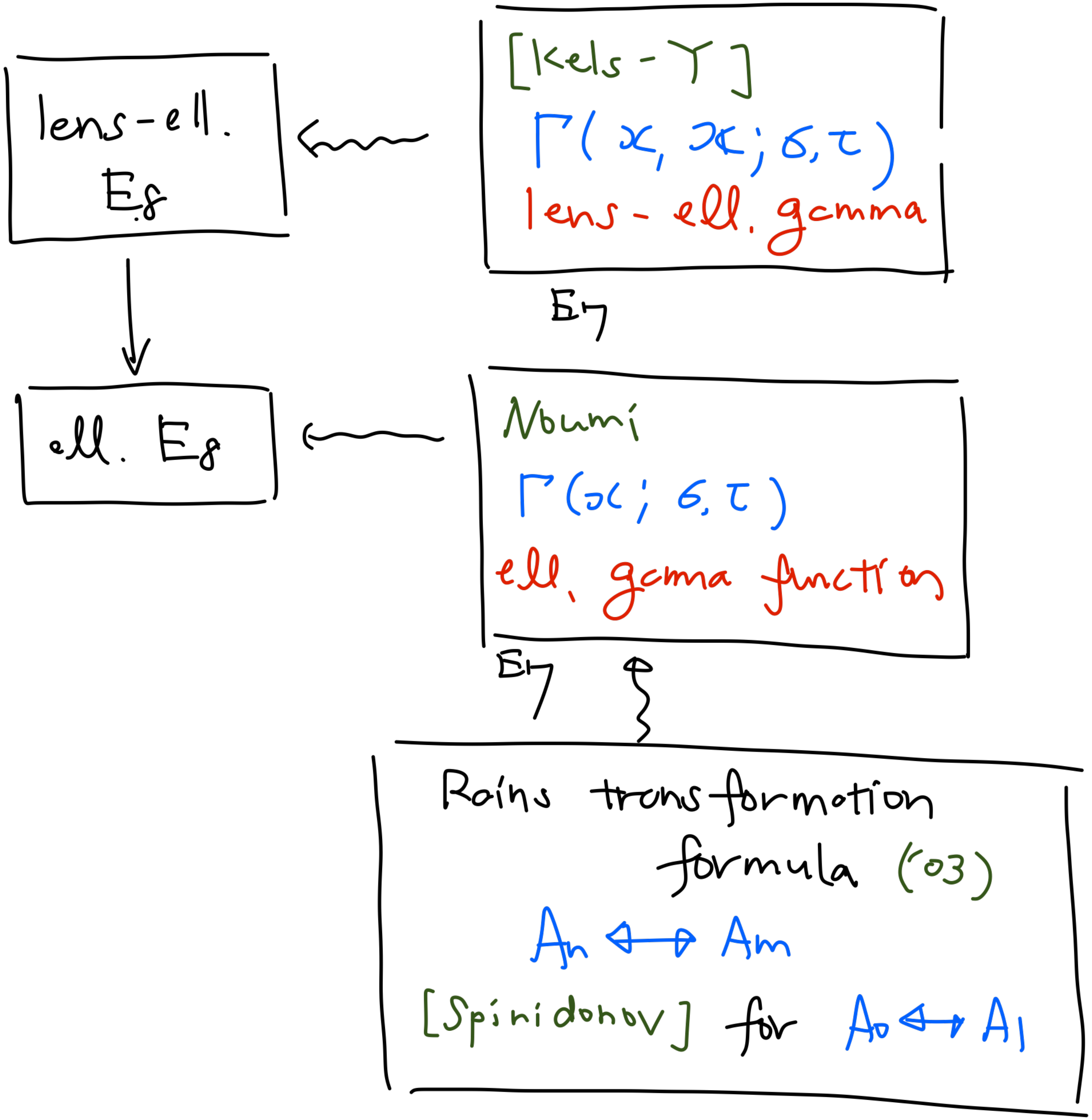
for lens-elliptic discrete Painlevé

(E_8 symmetry)

② special solution by

lens-elliptic gamma function

(E_7 symmetry)



lens-ell. E_8

[Kels-Y]

$\Gamma(x, x; \sigma, \tau)$

lens-ell. gamma

E_7

ell. E_8

Noumi

$\Gamma(x; \sigma, \tau)$

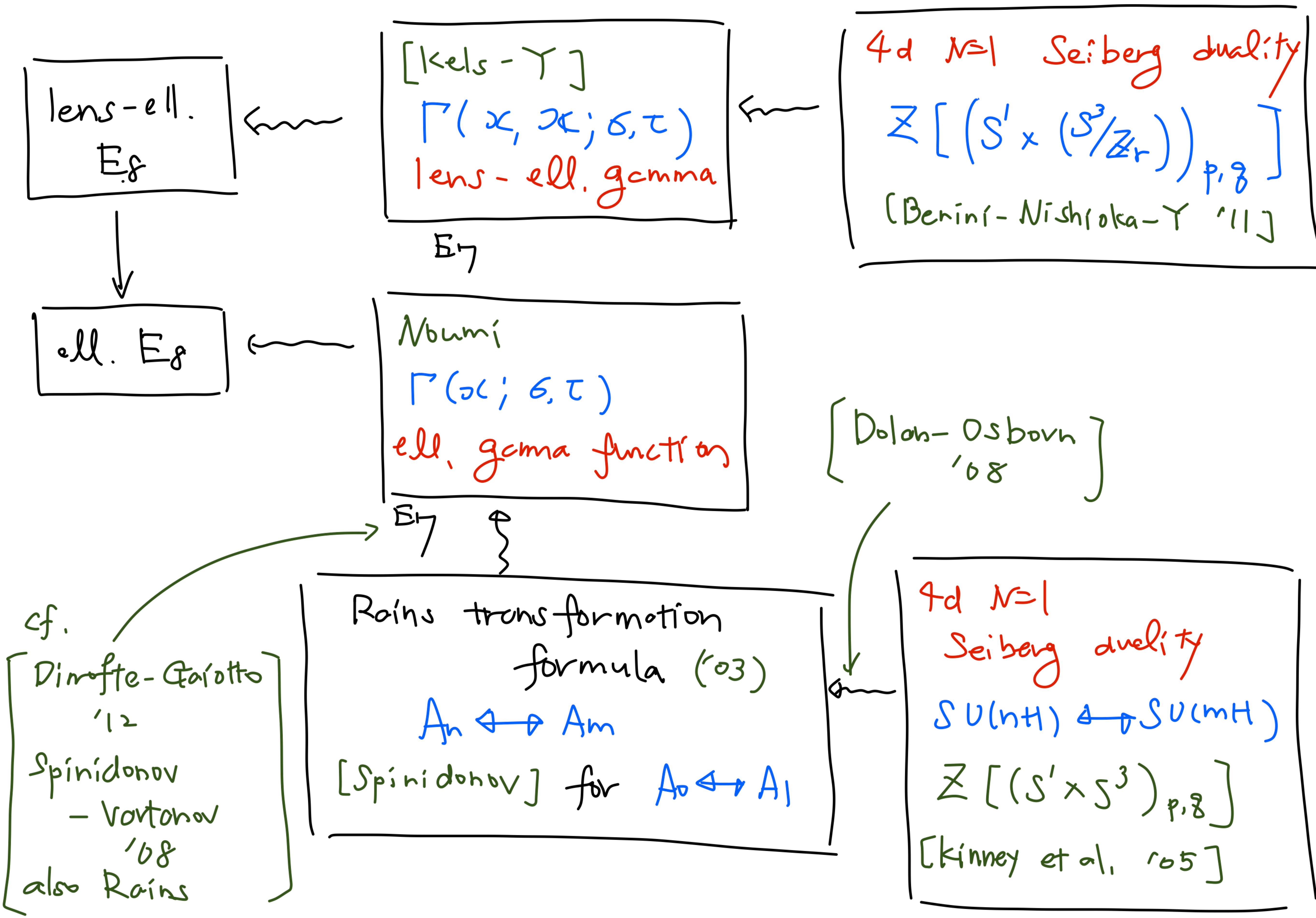
ell. gamma function

E_7

Rains transformation formula ('03)

$A_n \leftrightarrow A_m$

[Spiridonov] for $A_0 \leftrightarrow A_1$



lens-ell. E_8

[Kels-Y]

$\Gamma(x, x; \sigma, \tau)$

lens-ell. gamma

E_7

4d $\mathcal{N}=1$ Seiberg duality

$Z[(S^1 \times (S^3/\mathbb{Z}_r))_{p,q}]$

[Benini-Nishioka-Y '11]

ell. E_8

Noumi

$\Gamma(x; \sigma, \tau)$

ell. gamma function

E_7

[Dolan-Osborn '08]

Rains transformation formula ('03)

$A_n \leftrightarrow A_m$

[Spiridonov] for $A_0 \leftrightarrow A_1$

4d $\mathcal{N}=1$ Seiberg duality

$SU(n+1) \leftrightarrow SU(m+1)$

$Z[(S^1 \times S^3)_{p,q}]$

[Kinney et al. '05]

cf.

[Dimofte-Gaiotto '12

Spiridonov-Vartanov '08

also Rains]

Lens-elliptic

Discrete Painleve Eqn

- $Q(E_8)$: E_8 root lattic

- $\theta_\tau(z, Z; \sigma, \tau)$
  $\theta_\sigma(z, Z; \sigma, \tau)$: lens theta function

E_8 root lattice

$Q(E_8)$ $\quad \{v_0, v_1, \cdots v_7\}$: basis of $\mathbb{Z}^8$

$$\left\{ \begin{array}{l} \pm v_i \pm v_j \quad \{0 \le i < j \le 7\} \\ \frac{1}{2}(\pm v_0 \pm v_1 \pm \cdots \pm v_7) \quad (\text{even \# of minus}) \end{array} \right\}$$

E_8 root lattice

$Q(E_8)$ $\quad \{v_0, v_1, \cdots \; v_7\}$: basis of $\mathbb{Z}^8$

$$\left\{ \begin{array}{ll} \pm v_i \pm v_j & \{0 \leq i < j \leq 7\} \\ \frac{1}{2}(\pm v_0 \pm v_1 \pm \cdots \; \pm v_7) & \text{(even \# of mins)} \end{array} \right\}$$

Def

$2l$ vectors $\{\pm a_0, \cdots \; \pm a_{l-1}\}$ of $2l$ vectors in $\mathbb{C}^8$ is called C_l-frame iff

(1) $\quad (a_i \mid a_j) = \delta_{ij}$

(2) $\quad a_i \pm a_j \in Q(E_8) \qquad 0 \leq i < j < l$

$\quad 2a_i \in Q(E_8) \qquad 0 \leq i < l$

//

$(\{\pm a_i \pm a_j\} \cup \{\pm 2a_i\}$ forms type C_l root lattice$)$

lens theta function [Kels-Y '17]

$$\theta_\tau(z, \mathbb{z}; \sigma, \tau) = e^{\phi_\tau(z, \mathbb{z}; \sigma, \tau)} \theta\left(e^{-2\pi i z} e^{2\pi i \mathbb{z} \tau} \middle| e^{2\pi i \tau r}\right)$$

$$\theta_\sigma(z, \mathbb{z}; \sigma, \tau) = e^{\phi_\sigma(z, \mathbb{z}; \sigma, \tau)} \theta\left(e^{2\pi i z} e^{2\pi i \mathbb{z} \sigma} \middle| e^{2\pi i \sigma r}\right)$$

$z \in \mathbb{C}$

$\mathbb{z} \in \mathbb{Z}$

σ, τ : elliptic nome $\mathrm{Im}\,\sigma, \mathrm{Im}\,\tau > 0$

$r \in \mathbb{Z}_{>0}$: "lensing parameter"

$$\theta(z \mid q) := \underbrace{(z;q)_\infty}_{\substack{\shortparallel \\ \prod_{j=0}^{\infty}(1 - z q^{j})}} \left(\frac{q}{z};q\right)_\infty \qquad q = e^{2\pi i \tau}$$

$$\phi_\tau(z, Z; \sigma, \tau) := \frac{\pi i}{6r}\Big[3(r+1-2Z)(2Z+1) - (r^2-1)(\sigma-\tau-1) - 6Z(r-Z)(\tau+1)\Big]$$

$$\phi_\sigma(z, Z; \sigma, \tau) := -\frac{\pi i}{6}\Big[3(r-1-2Z)(2Z-1) - (r^2-1)(\sigma-\tau-1) + 6Z(r-Z)(\sigma-1)\Big]$$

$$[\mathbb{Z}] := [(\underset{\substack{\Uparrow \\ \mathbb{C}}}{z}, \underset{\substack{\uparrow \\ \mathbb{Z}_1}}{\mathbb{Z}})] = e^{\frac{\pi i}{r}(-z_j + \mathbb{Z}_j)} \theta_\sigma(z_j, \mathbb{Z})$$

Prop

- $[(z, \mathbb{Z} + 2r)] = [(z, \mathbb{Z})]$
- $[-\mathbb{Z}] = -[\mathbb{Z}]$
- $[z_j + z_k][z_j - z_k][z_i + \mathbb{Z}][z_i - \mathbb{Z}]$

  $+ (i, j, k \text{ cyclic}) = 0$

$$D \subset \underset{\{Z\}}{V} := \underset{\{z\}}{\mathbb{C}^8} \times \underset{\{\not{z}\}}{Q(E_8)}$$

s.t. $D = D + Q(E_8)\underbrace{T}_{(\tau,\,1)}$ "invariant under time translation"

"discrete time"

$$D \subset \underset{\{Z\}}{V} := \underset{\{z\}}{\mathbb{C}^8} \times \underset{\{\cancel{z}\}}{Q(E_8)}$$

s.t. $\mathbb{D} = D + Q(E_8)\underbrace{T}_{(\tau,\, 1)}$ "invariant under time translation"

"discrete time"

<u>Def</u> $\tau(X)$ defined over $\mathbb{D}$ is called τ-function

if for any C_3-frame $(\pm a_0, \pm a_1, \pm a_2)$

any $X \in \mathbb{D}$

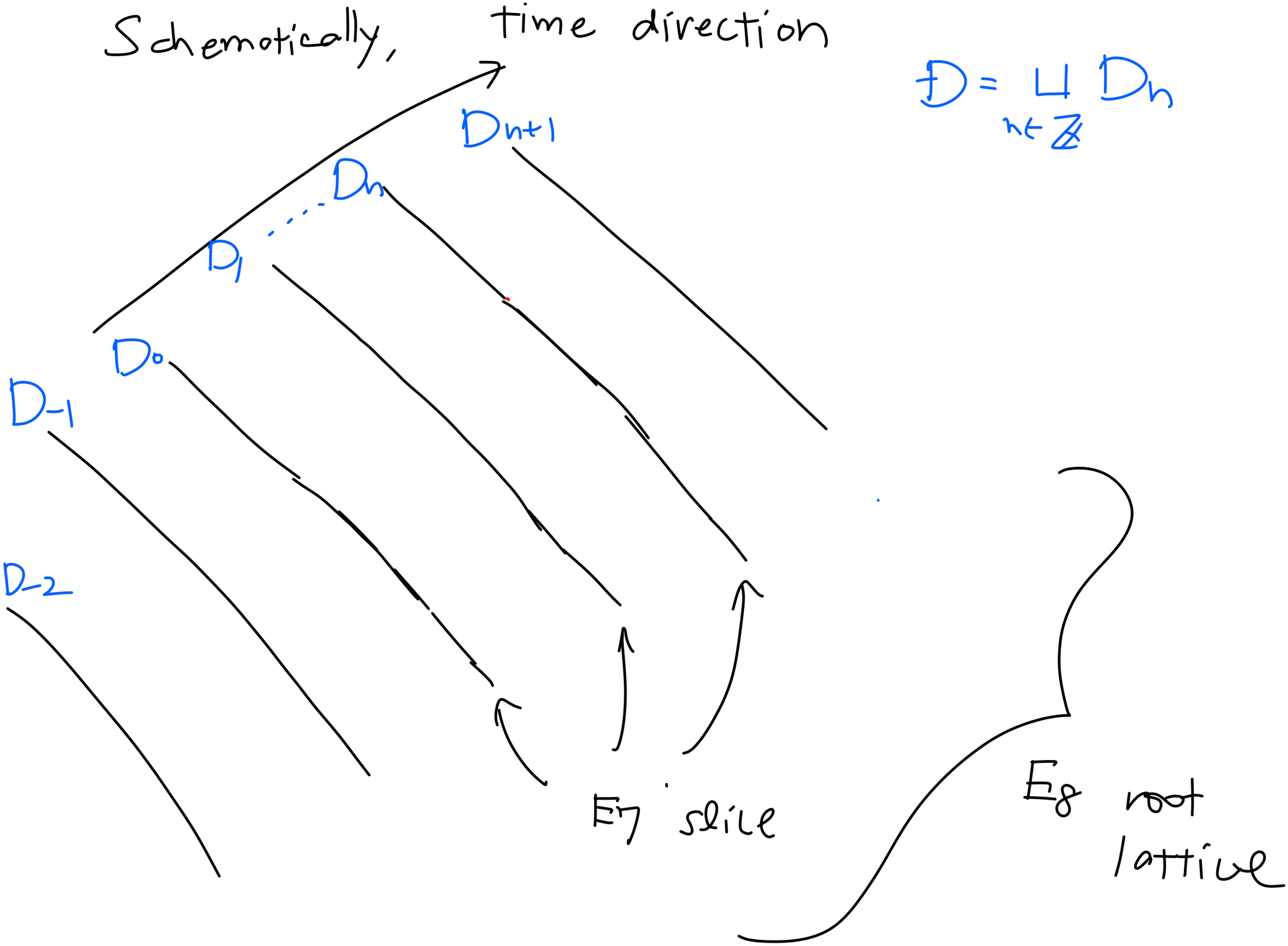
$$[(a_1 \pm a_2 | X)]\, \tau(X \pm a_0 T)$$
$$+ [(a_2 \pm a_0 | X)]\, \tau(X \pm a_1 T)$$
$$+ [(a_0 \pm a_2 | x)]\, \tau(X \pm a_2 T) = 0$$

$$\boxed{\begin{array}{c} \mathbb{C}f \\ \cup \\ * \quad (a|X) \in \mathbb{D} \\ \shortparallel \\ \left[\underbrace{(a|X)}_{\substack{\cap \\ \mathbb{C}}},\ \underbrace{(a|\cancel{X})}_{\substack{\cap \\ \cancel{\mathbb{Z}},}}\right] \end{array}}$$

//

Decomposition into E_7-orbit

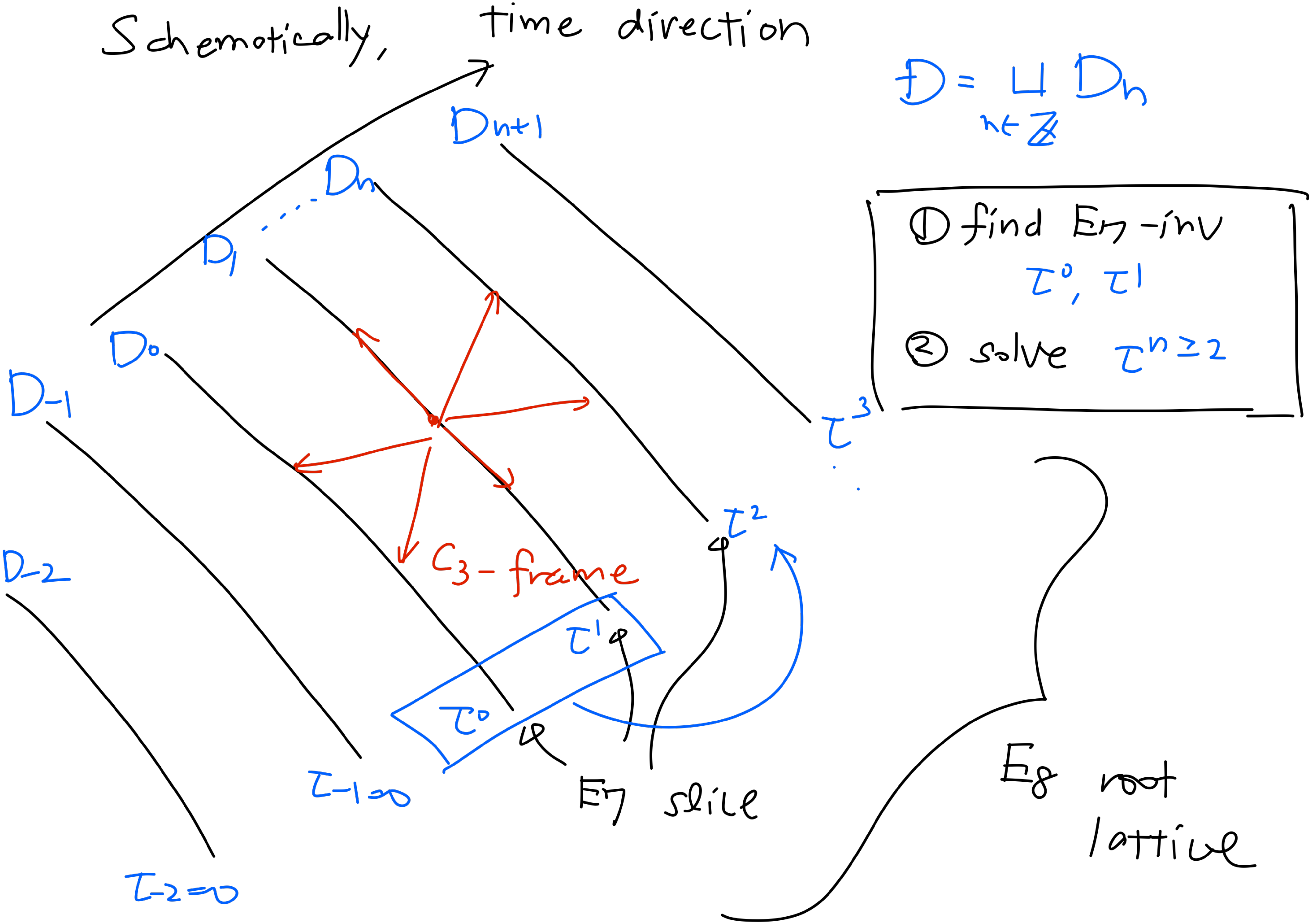
Schematically,



$$\mathcal{D} = \bigsqcup_{n\in\mathbb{Z}} D_n$$

Schematically,

$$\mathfrak{D} = \bigsqcup_{n\in\mathbb{Z}} D_n$$



① find E_7-inv τ^0, τ^1

② solve $\tau^{n\geq 2}$

$$D = \bigsqcup_{n \in \mathbb{Z}} D_n$$

$$D_n = H_{n\tau+6} \times \left(H_{n+r-1} \cap Q(E_7) \right)$$

$$H_k = \left\{ x \in \mathbb{C}^8 \;\middle|\; \left(x \;\middle|\; \frac{v_0 + \cdots + v_7}{2} \right) = k \right\}$$

$$\sum x_i = 2(n\tau + 6), \quad \sum x_i = 2(n + r - 1)$$

τ on D

$$\leftrightsquigarrow \left\{ \tau^{(n)} = \tau|_{D_n} \right\}_{n \in \mathbb{Z}}$$

(※ we choose $\tau_{\mathbb{Z}<0} = 0$)

$$
\begin{aligned}
&[(a_1 \pm a_2 | X)]\, \tau(X \pm a_0 T) \\
&\quad + [(a_2 \pm a_0 | X)]\, \tau(x \pm a_1 T) \\
&\quad + [(a_0 \pm a_1 | x)]\, \tau(X \pm a_2 T) = 0
\end{aligned}
$$

We can decompose this into $\tau^{(n)}$

depends on $\left(a_i \,\middle|\, \frac{v_0 + \cdots + v_7}{2}\right)$

for C_3- frame $(\pm a_0, \pm a_1, \pm a_2)$

Prop [Noumi '16]

$W(E_7)$-orbit of C_3-frame $(\pm a_0, \pm a_1, \pm a_2)$

(I) $(\phi | a_0) = (\phi | a_1) = (\phi | a_2) = \frac{1}{2}$

$(\mathrm{II})_0$ $(\phi | a_0) = (\phi | a_1) = (\phi | a_2) = 0$

$(\mathrm{II})_1$ $(\phi | a_0) = 1$ $\quad (\phi | a_1) = (\phi | a_2) = 0$

$(\mathrm{II})_2$ $(\phi | a_0) = (\phi | a_1) = 1$ $\quad (\phi | a_2) = 0$

equations for (I), $(\mathrm{II})_0$, $(\mathrm{II})_1$, $(\mathrm{II})_2$

$$(\mathrm{II}_0)_n : \ [(a_1 \pm a_2 | X)]\, \tau^{(n)}(X \pm a_0 T) + [(a_2 \pm a_0 | X)]\, \tau^{(n)}(X \pm a_1 T) + [(a_0 \pm a_1 | X)]\, \tau^{(n)}(X \pm a_2 T) = 0$$

$$(\mathrm{II}_1)_n : \ [(a_1 \pm a_2 | X)]\, \tau^{(n-1)}(X - a_0 T)\, \tau^{(n+1)}(X + a_0 T) + [(a_2 \pm a_0 | X)]\, \tau^{(n)}(X \pm a_1 T) + [(a_0 \pm a_1 | X)]\, \tau^{(n)}(X \pm a_2 T) = 0$$

$\vdots$

$$\tau^{n-1},\ \tau^{n} \rightarrow \tau^{n+1}$$

$$\begin{tikzcd}
(\mathrm{II}_1)_n \arrow[r] \arrow[dr, bend right] \arrow[ddr, bend right] & (\mathrm{I})_{n+\frac{1}{2}} \\
& (\mathrm{II}_0)_{n+1} \\
& (\mathrm{II}_2)_n
\end{tikzcd}$$

recursive structure, mutually consistent

Special Solution from

Lens-Elliptic Gamma function

elliptic gamma function [Ruijsenaars '97]

$$\Gamma_{r=1}(z, \cancel{z}; \sigma, \tau) = \prod_{j,k=0}^{\infty} \frac{1 - e^{-2\pi i z} e^{2\pi i \sigma (j+1)} e^{2\pi i \tau (k+1)}}{1 - e^{2\pi i z} e^{2\pi i \sigma j} e^{2\pi i \tau k}}$$

lens-elliptic gamma function

[Benini-Nishioka-Y ('11)
Kels ('15) Kels-Y ('17)]

$$\Gamma(\underbrace{z, \mathbb{z}}_{\mathbb{Z}}; \sigma, \tau) = e^{\phi_e(z,\mathbb{z};\sigma,\tau)} \gamma_\tau(z,\mathbb{z};\sigma,\tau)\, \gamma_\sigma(z,\mathbb{z};\sigma,\tau)$$

$$\gamma_\sigma = \prod_{j,k=0}^{\infty} \frac{1 - e^{-2\pi i z} e^{-2\pi i \sigma \mathbb{z}} e^{2\pi i (\sigma+\tau)(j+1)} e^{2\pi i r\sigma (k+1)}}{1 - e^{2\pi i \mathbb{z}} e^{2\pi i \sigma \mathbb{z}} e^{2\pi i (\sigma+\tau)(j+1)} e^{2\pi i r\sigma k}}$$

$$\gamma_\tau = \prod_{j,k=0}^{\infty} \frac{1 - e^{-2\pi i z} e^{-2\pi i (r-\mathbb{z})\tau} e^{2\pi i (\sigma+\tau)(j+1)} e^{2\pi i r\tau (k+1)}}{1 - e^{2\pi i z} e^{2\pi i (r-\mathbb{z})\tau} e^{2\pi i (\sigma+\tau) j} e^{2\pi i r\tau k}}$$

$$\phi_e = \pi i \frac{\mathbb{z}(\mathbb{z}-r)(6x - 3\sigma - 3\tau + (1-\sigma+\tau)(r-2\mathbb{z}))}{6r}$$

$\hookleftarrow$ some combination of $B_{3,3}$

Prop

(periodicity)

$$\Gamma(z+2kr, \cancel{z}; \sigma, \tau) = \Gamma(z, \cancel{z}; \sigma, \tau)$$

$$\Gamma(z, \cancel{z}+kr; \sigma, \tau) = \Gamma(z, \cancel{z}; \sigma, \tau)$$

(inversion)

$$\Gamma(\sigma+\tau-z, -\cancel{z})^{-1} = \Gamma(z, \cancel{z})$$

(recurrence relation)

$$\Gamma(z+n\sigma, \cancel{z}+n(-1)) = \Gamma(z, \cancel{z}) \prod_{j=0}^{n-1} \theta_\tau(z+j\sigma; \cancel{z}+j(-1))$$

$$\Gamma(z+n\tau, \cancel{z}+n\cdot 1) = \Gamma(z, \cancel{z}) \prod_{j=0}^{n-1} \theta_\sigma(z+j\tau, \cancel{z}+j\cdot 1)$$

elliptic hypergeometric sum/integral

$$I(x, \mathbb{x}; \sigma, \tau) := \frac{\lambda}{2} \sum_{\mathbb{z}=0}^{r-1} \int_{[0,1]} dz \, \frac{\prod_{j=0}^{7} \Gamma(x_j \pm z ; \mathbb{x}_j \pm \tilde{\mathbb{z}})}{\Gamma(\pm 2z, \pm 2\tilde{\mathbb{z}})}$$

$$(e^{2\pi i r\sigma}; e^{2\pi i r\sigma})_\infty$$

$$\times$$

$$(e^{2\pi i r\tau}; e^{2\pi i r\tau})_\infty$$

$$\tilde{\mathbb{z}} = \mathbb{z} + (r + (r+1) \bmod 2) \times (\mathbb{x}_1 \bmod 1)$$

(A$_1$ integral)

Prop [Kels-Y, also Spridonov]

$$\vec{x} \in \mathbb{C}^8 , \quad \vec{\mathbb{x}} \in \mathbb{Z}^8 \sqcup \left(\mathbb{Z} + \frac{1}{2}\right)^8$$

then

$$I(\vec{x}, \vec{\mathbb{x}}) = I(\vec{\tilde{x}}, \vec{\tilde{\mathbb{x}}}) \prod_{\substack{0 \le i < j \le 3 \\ 0 \le i < j \le 7}} \Gamma(x_i + x_j, \mathbb{x}_i + \mathbb{x}_j)$$

where

$$\tilde{x}_i = \begin{cases} x_i + \frac{\sigma+\tau}{2} - \frac{1}{2}\sum_{j=0}^{3} x_j & (i=0, \ 3) \\ x_i + \frac{\sigma+\tau}{2} - \frac{1}{2}\sum_{j=4}^{7} x_j & (i=4, \cdot 7) \end{cases}$$

$$\tilde{\mathbb{x}}_i = \begin{cases} \mathbb{x}_i + \frac{r}{2} - \frac{1}{2}\sum_{j=0}^{3} \mathbb{x}_j & (i=0, \cdot 3) \\ \mathbb{x}_i + \frac{r}{2} - \frac{1}{2}\sum_{j=4}^{7} \mathbb{x}_j & (i=4 \cdot\cdot 7) \end{cases}$$

$$\tilde{x}_i = \begin{cases} x_i + \frac{\sigma+\tau}{2} - \frac{1}{2}\sum_{j=0}^{3} x_j & (i=0,\ 3) \\ x_i + \frac{\sigma+\tau}{2} - \frac{1}{2}\sum_{j=4}^{7} x_j & (i=4,\cdot 7) \end{cases}$$

$$\tilde{\mathbb{X}}_i = \begin{cases} \mathbb{X}_i + \frac{r}{2} - \frac{1}{2}\sum_{j=0}^{3} \mathbb{X}_j & (i=0,\ \cdot\ 3) \\ \mathbb{X}_i + \frac{r}{2} - \frac{1}{2}\sum_{j=4}^{7} \mathbb{X}_j & (i=4\cdot\cdot\ 7) \end{cases}$$

↓

$w_0 \in W(E_7)$

w_0 & $\mathfrak{S}_8$ generate the whole $W(E_7)$

Thm

$$\tau^{(n)}(x) = e^{(n)}(x)\, g^{(n)}(X)\, I_n(x)$$

is a $W(E_7)$-inv. τ-function

in $D = \bigsqcup_{n \in \mathbb{Z}} D_n$

$$D_n = H_{n\tau+6} \times \left(H_{n+r-1} \cap Q(E_7) \right)$$

$$I_{n}(X) = \frac{x^n}{n!} \sum_{z_0, \dots, z_7 = 0}^{r-1} \int \prod_{0 \le i < j \le 7} \theta\left(\pm z_i \pm z_j ; \pm \hat{z}_i \pm \hat{z}_j\right)$$

$$\times \prod_{k=1}^{n} \Delta\left(z_k , \underbrace{\hat{z}_k}_{\varphi} ; \tilde{x} , \tilde{X}\right) dz_k$$

$$z_i + (r + (r+1 \bmod 2))(\tilde{x}_1 \bmod 1)$$

$$\Delta(z, z ; x, x) = \frac{\prod_{j=0}^{7} \Gamma(x_j \pm z ; x_j + \tilde{z}_j)}{\Gamma(\pm 2z , \pm 2\tilde{z})}$$

$$e^{(n)}(X) = e^{\frac{2\pi i}{r}(\sigma+1)\binom{n}{2}} e^{-n\frac{2\pi i}{r}\left[\frac{1}{2\tau}(x|x) - \frac{1}{2}(\mathbb{x}|\mathbb{x})\right]}$$

$$g^{(n)}(X) = \prod_{\substack{0\le i\le 3\\ 4\le j\le 7}} \left\{ \Gamma_\sigma^+((1-n)T + X_i + X_j\,;\sigma, \tau, \tau) \times \gamma_\tau^+(X_i+X_j\,;\sigma, \tau, \mu)^n \right\}$$

$$\times \prod_{\substack{0\le i<j\le 3\\ 4\le i<j\le 7}} \left\{ \Gamma_\sigma^+(T+X_i+X_j\,;\sigma,\tau,\tau)\; \gamma_\tau^+(X_i+X_j+(\mu,0)\,;\sigma,\tau,\mu)^n \right\}$$

$$\Gamma^{+}_{\sigma}(z, \bar{z}; \sigma, \tau, w) = e^{\phi^{+}(z, \bar{z}; \sigma, \tau, w)} \gamma^{+}_{\sigma}(z, \bar{z}; \sigma, \tau, w)$$

$$\gamma^{+}_{\sigma} = g_{\sigma}(z, \bar{z}; \sigma, \tau, w)\, g_{\sigma}(\sigma+\tau+w-z, r+1-\bar{z}; \sigma, \tau, w)$$

$$\gamma^{+}_{\tau} = g_{\tau}(z, \bar{z}; \sigma, \tau, w)\, g_{\tau}(\sigma+\tau+w-z, r-\bar{z}; \sigma, \tau, w)$$

$$g_{\sigma} = \prod_{k_1, k_2, k_3 = 0}^{\infty} \left(1 - e^{2\pi i z} e^{2\pi i \sigma \bar{z}} e^{2\pi i \sigma k_1} e^{2\pi i (\sigma+\tau) k_2} e^{2\pi i (\sigma+w) k_3}\right)$$

$$g_{\tau} = \prod_{k_1, k_2, k_3 = 0}^{\infty} \left(1 - e^{2\pi i z} e^{2\pi i \sigma (r-\bar{z})} e^{2\pi i \tau k_1} e^{2\pi i (\sigma+\tau) k_2} e^{2\pi i (\tau+w) k_3}\right)$$

Gauge theory / Geometry ?

Schematically .

$$\tau_{\ell\text{-}eP_{E_8}} = \{ \tau_n \}_{n=0}^{\infty}$$

$$\parallel$$

$$Z \left[\begin{array}{l} 4d \ \mathcal{N}=2 \\ USp(2N) + 8 \text{ chiral} + 1 \text{ adjoint} \end{array} \right]$$

$$Z[S^1 \times S^3/\mathbb{Z}_r]$$

Schematically.

$$\tau_{q\text{-}eP_{E_8}} = \{ \tau_n \}_{n=\infty}^{\infty}$$

$$Z[S^1 \times S^3/\mathbb{Z}_r]$$

$$\parallel$$

$$Z\left[\begin{array}{l} \text{4d } \mathcal{N}=2 \\ USp(2N) + 8 \text{ chiral} + 1 \text{ adjoint} \end{array}\right]$$

---

compare this w/ [Gamayun-Iorgov-Lisovyy '13]

$$\tau^{(t)}_{VI_{D_4}} \sim \sqrt{} \; \text{4d } \mathcal{N}=2 : \; Z_{N_f=4} = \sum_{n \in \mathbb{Z}} e^{in\rho} Z_{N_f=4}(\sigma+n, t, \{\vec{\theta}\})$$

Z_{Nek}

mass

for discrete Painlevé e.g. [Jimbo-Nagoya-Sakai '17] this similarity is likely not a coincidence

6d (0,1)

$$\text{E-string } (E_8) \xrightarrow{T^2 \text{ w/ flux}} \text{4d } \mathcal{N}=1 \longrightarrow \text{3d } \mathcal{N}=2 \longrightarrow \text{2d } \mathcal{N}=(2,2)$$

elliptic (lens) trigonometric rational

$\uparrow$

F-theory on ell. CY_3

$$\text{F-theory on ell. } CY_3 \longrightarrow \text{6d } (0,1) \xrightarrow{S^1} \text{5d } \mathcal{N}=1 \xrightarrow{S^1} \text{4d } \mathcal{N}=2$$

elliptic trigonometric rational

(del Pezzo
Hirzebruch)

Summary

- bilinear identities

  for "lens-elliptic" E_8

  discrete Painleve equation

- special $W(E_7)$-inv. solution in terms of

  lens-elliptic gamma function

- motivated by supersymmetric gauge theory

Questions

- write down the lens-ell. Painleve eqn
- geometrical reformulation?
  [along the lines of Sakai]
- lens-trigonometric degeneration?
- connection w/ Gamayun-Iorgov-Lisovyy?

⋮