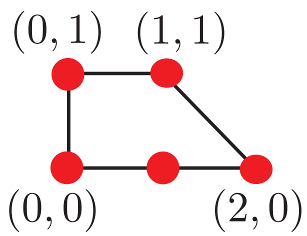
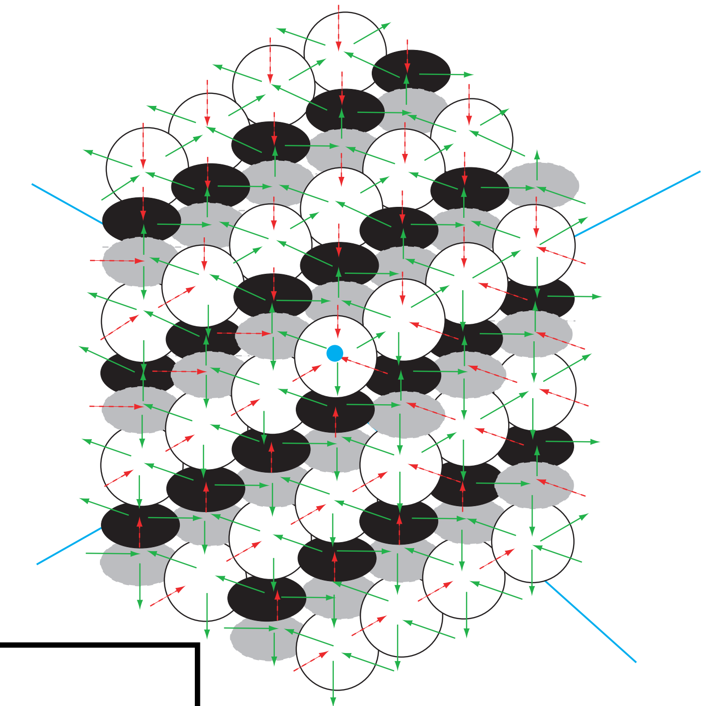


$$
\begin{aligned}
\psi^{(a)}(z)\,\psi^{(b)}(w) &= \psi^{(b)}(w)\,\psi^{(a)}(z)\,,\\
\psi^{(a)}(z)\,e^{(b)}(w) &\simeq \varphi^{b\Rightarrow a}(\Delta)\,e^{(b)}(w)\,\psi^{(a)}(z)\,,\\
e^{(a)}(z)\,e^{(b)}(w) &\sim (-1)^{|a||b|}\varphi^{b\Rightarrow a}(\Delta)\,e^{(b)}(w)\,e^{(a)}(z)\,,\\
\psi^{(a)}(z)\,f^{(b)}(w) &\simeq \varphi^{b\Rightarrow a}(\Delta)^{-1}\,f^{(b)}(w)\,\psi^{(a)}(z)\,,\\
f^{(a)}(z)\,f^{(b)}(w) &\sim (-1)^{|a||b|}\varphi^{b\Rightarrow a}(\Delta)^{-1}\,f^{(b)}(w)\,f^{(a)}(z)\,,\\
\left[e^{(a)}(z), f^{(b)}(w)\right\} &\sim -\delta^{a,b}\frac{\psi^{(a)}(z)-\psi^{(b)}(w)}{z-w}\,,
\end{aligned}
$$



Quiver Yangians and Crystal Melting

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IPMU INSTITUTE FOR THE PHYSICS AND MATHEMATICS OF THE UNIVERSE

Osaka City University / online

March 22, 2021

Based on

Wei Li + MY

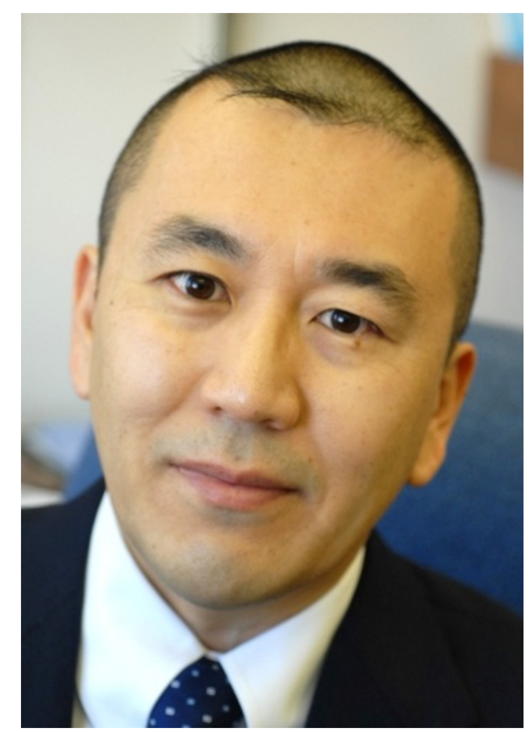
(2003.08909 [hep-th])

Dmitry Galakhov + MY

(2008.07006 [hep-th])

And many works in the literature

··· more work to come! [Galakohov-Li-Yamazaki]



Also earlier works, e.g.

Hirosi Ooguri + MY (0811.2810 [hep-th])

MY (Ph.D. thesis, 1002.1709 [hep-th])

MY (Master thesis, 0803.4474 [hep-th])

Overview

Geometry

string theory

supersymmetric gauge theory

BPS states

BPS degeneracy

Enumerative Invariants

Geometric Representation theory

BPS state algebra

Many papers, e.g. [Nakajima,⋯, Kontsevich-Soibelman, Alday-Gaiotto-Tachikawa, Schiffman-Vasserot, Maulik-Okounkov,⋯]

$CY_3 : X$

type IIA string theory

$\mathbb{R}^{3,1} \times X$

$\cup$

$\mathbb{R} \times \{\text{hol. cycle}\}$

BPS particles wrapping hol. cycle

$$Z^X_{BPS} = \sum_\gamma \underbrace{\Omega^X_\gamma(\cdots)}_{\text{BPS degeneracy}} \, q^\gamma \qquad \gamma \in H^{even}(X)$$

toric CY_3 : X

type IIA string theory

$$\mathbb{R}^{3,1} \times X$$
$$\cup$$
$$\mathbb{R} \times \{\text{hol. cycle}\}$$

BPS particles wrapping hol. cycle

$$Z^X_{BPS} = \sum_\gamma \underbrace{\Omega^X_\gamma(\cdots)}_{\text{BPS degeneracy}} q^\gamma \qquad \gamma \in H^{even}(X)$$

$= Z_{crystal}$ ← fixed point

BPS quiver Yangian

toric CY3 : X

type IIA string theory

$$\mathbb{R}^{3,1} \times X$$
$$\cup$$
$$\mathbb{R} \times \{\text{hol. cycle}\}$$

BPS particles wrapping hol. cycle

$$Z^X_{BPS} = \sum_\gamma \underbrace{\Omega^X_\gamma(\cdots)}_{\text{BPS degeneracy}} \mathfrak{z}^\gamma \qquad \gamma \in H^{even}(X)$$

$= Z_{crystal}$ fixed point

SUSY QM of BPS particles

BPS quiver Yangian

Plan

— Crystal Melting [Ooguri-Y]

— Quiver Yangian: Algebra [Li-Y]

— Quiver Yangian: Representation

— Derivation from Quantum Mechanics

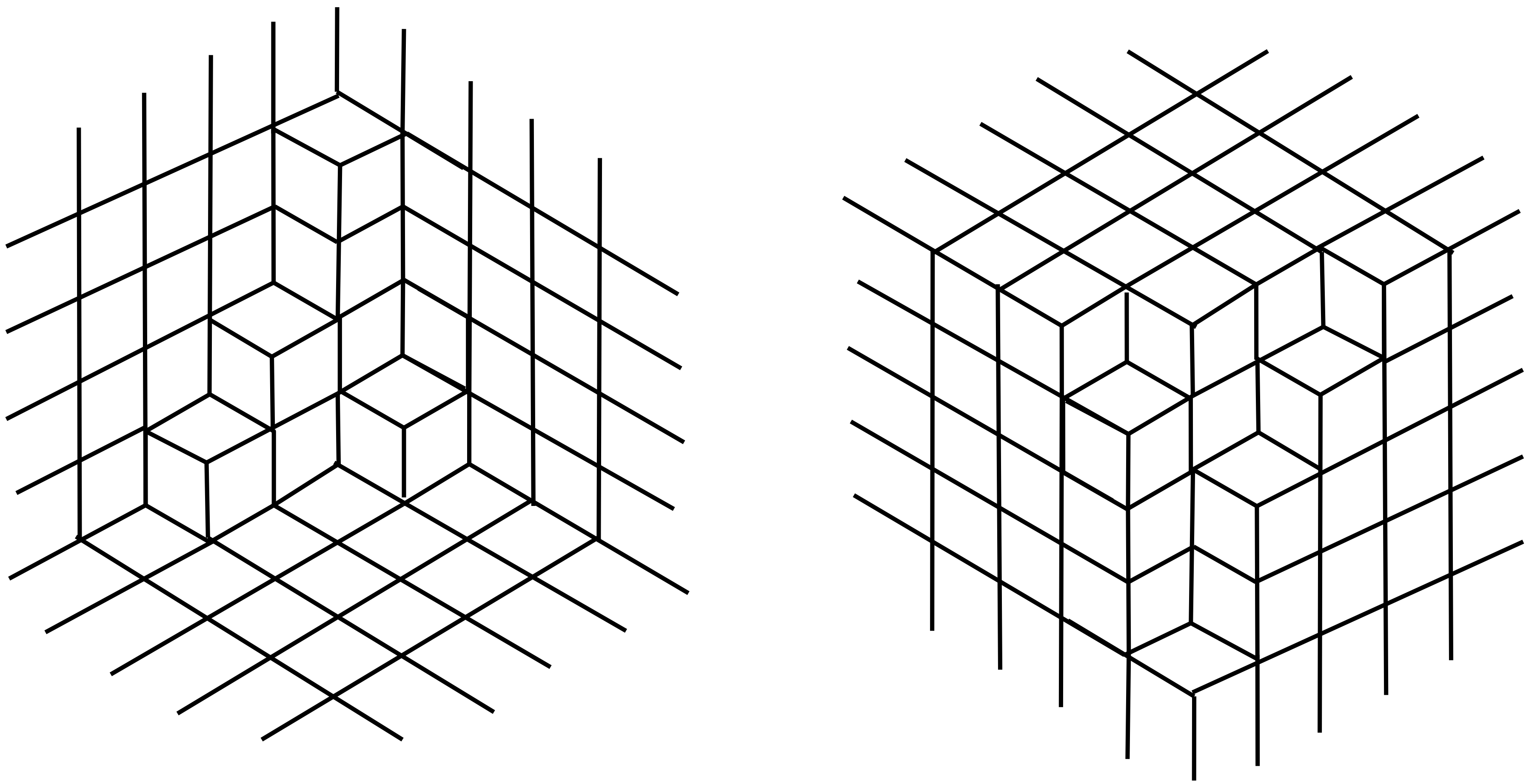
— Summary

[Galakhov-Y]

Crystal Melting

[Szendroi; Mozgovoy, Reineke; Nagao, Nakajima; Ooguri, MY; Jafferis, Chuang, Moore; Sulkowski; Aganagic, Vafa; …]

$\mathbb{C}^3$: crystal melting [Okounkov-Reshetikhin-Vafa; Iqbal, Nekrasov,...]



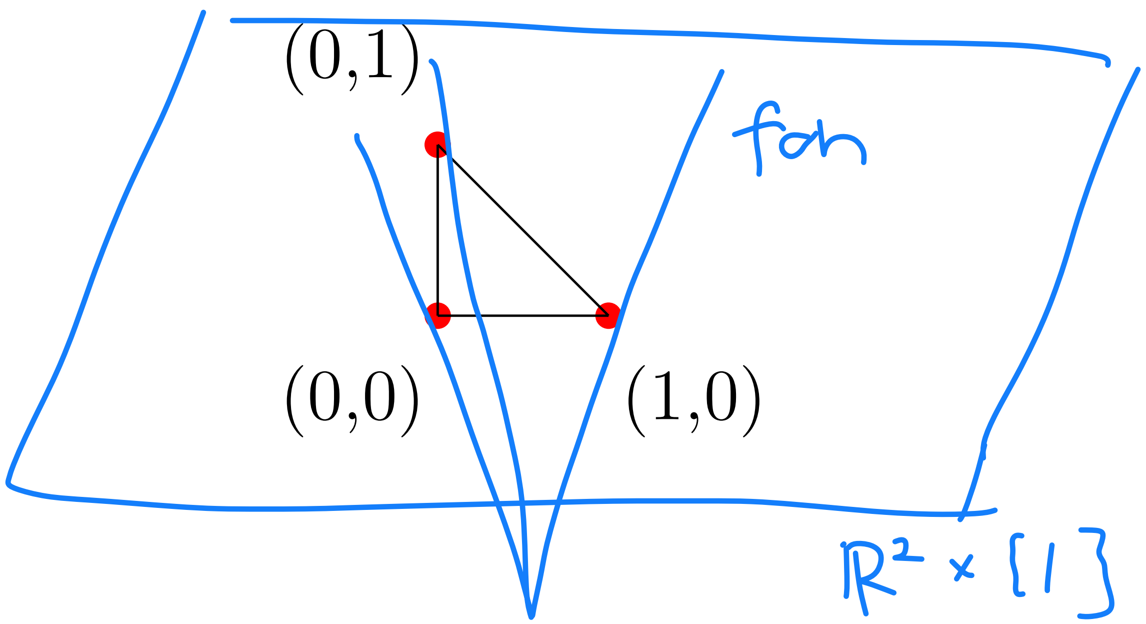
plane partition

$$M(q) \equiv \sum_{\Lambda \in \text{ plane partition}} q^{|\Lambda|} = \prod_{k=1}^{\infty} \frac{1}{(1-q^k)^k}$$

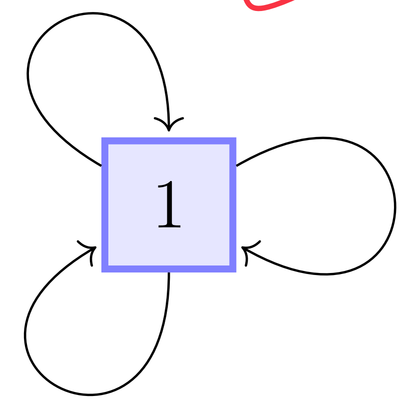
$$= 1 + q + 3q^2 + 6q^3 + 13q^4 + 24q^5 + 48q^6 + \ldots \; ,$$

$$= Z^{\mathbb{C}^3}_{\text{top A-model}}$$

toric diagram $\subset \mathbb{Z}^2$



$\rightsquigarrow$ quiver Q



$+$ superpotential

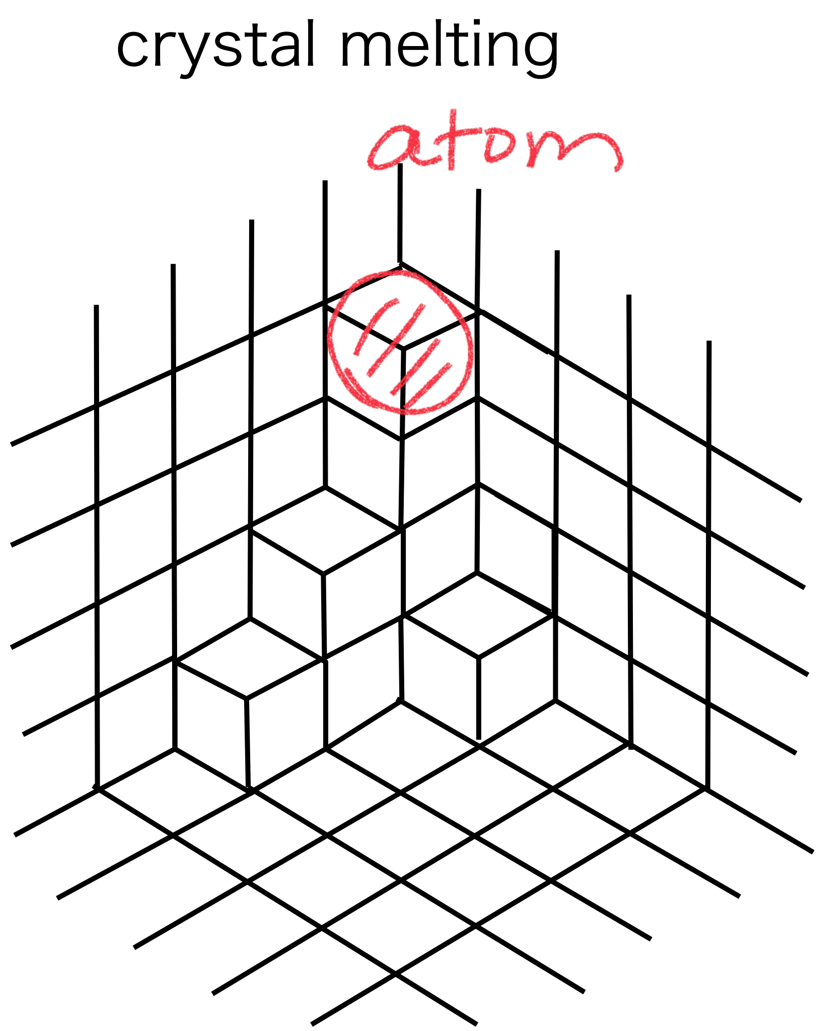
$$\left(W = \mathrm{Tr}\, xyz - \mathrm{Tr}\, xzy \right)$$

$\rightsquigarrow$ path algebra

$$\mathbb{C}\langle x, y, z \rangle / (\partial W)$$

$$(\partial W) = (xy - yx,\ yz - zy,\ zx - xz)$$

$$\mathbb{C}\langle x, y, z \rangle / (\partial W) = \mathbb{C}[x, y, z]$$

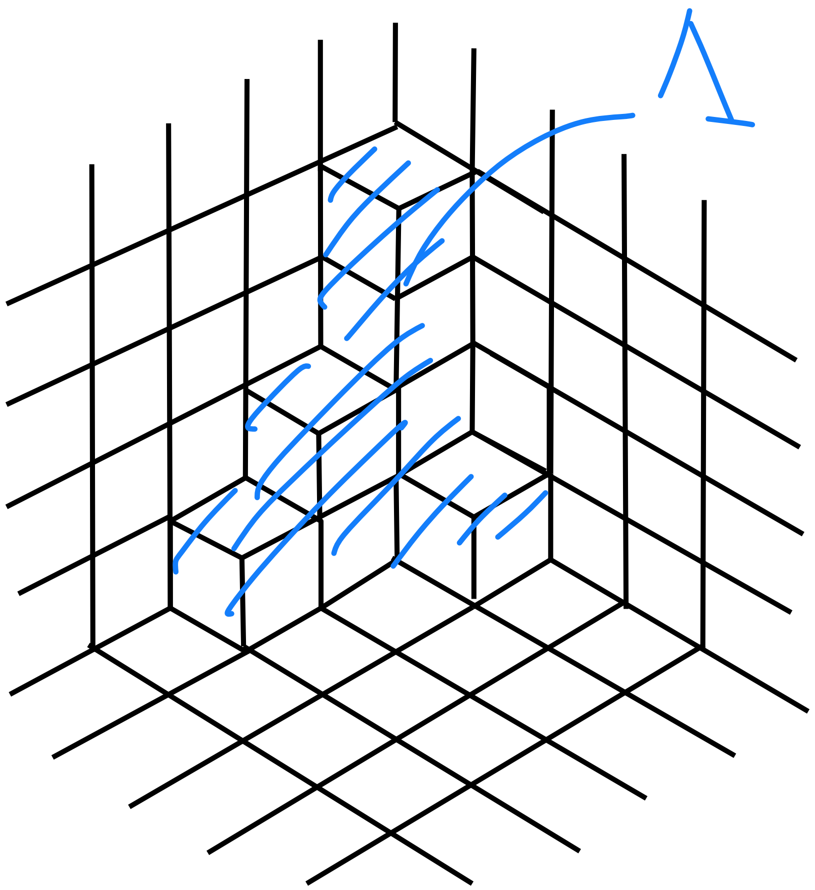


- "atom" at location (i, j, k):

$$x^i y^j z^k \in \mathbb{C}[x, y, z] = \mathbb{C}\langle x, y, z \rangle / (\partial W)$$

- atom = element of $\mathbb{C}Q / \partial W$

- connection between atoms determined naturally crystal melting



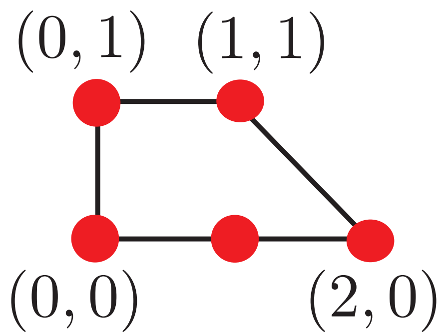
Λ^c the complement: ideal sheaf

(ideal of the path alg.)

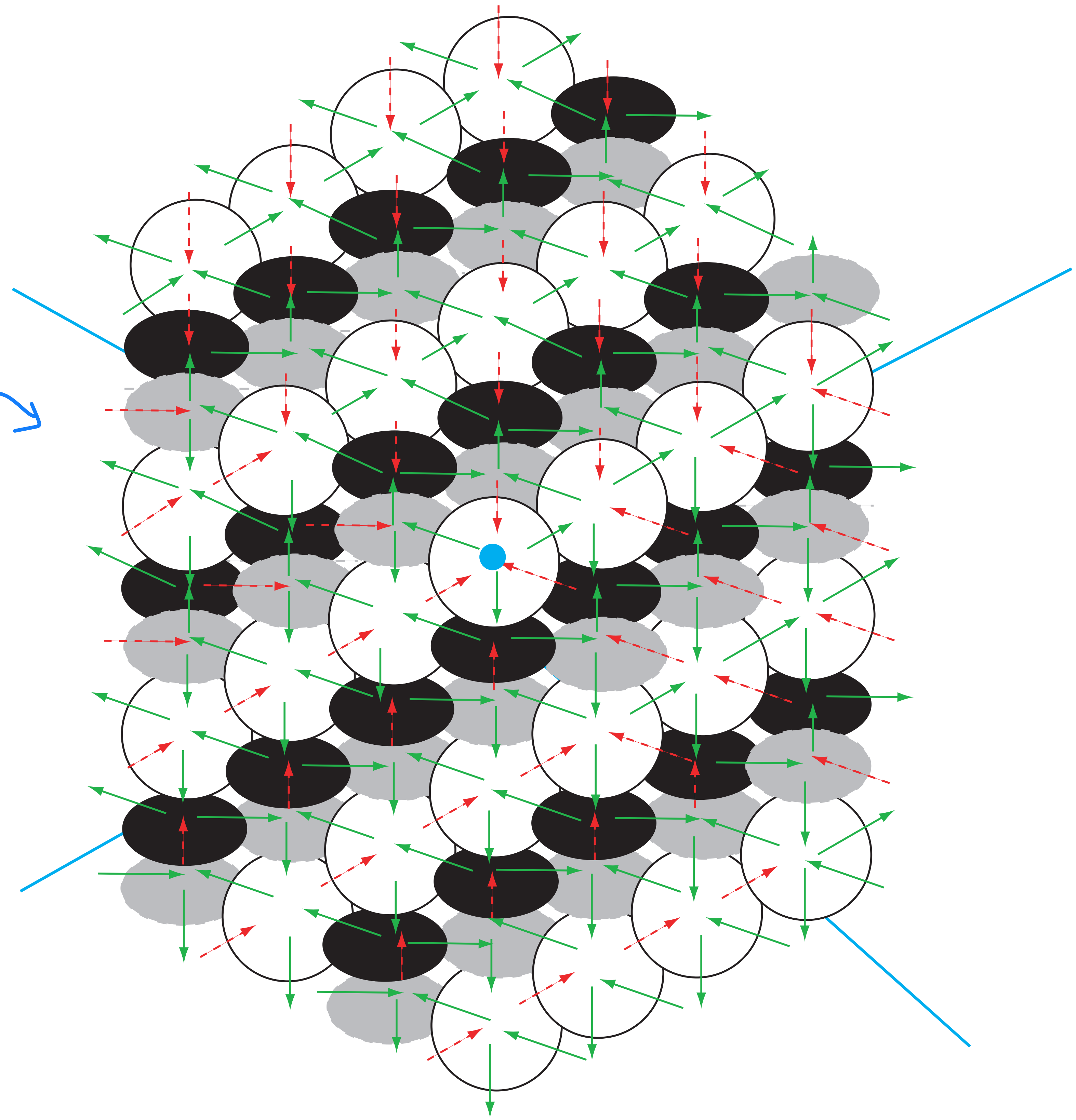
$$I_\Lambda \subset \mathbb{C}[x, y, z]$$
$$\| $$
$$\mathrm{Span}\{x^i y^j z^k \mid (i,j,k) \notin \Lambda\}$$

$$x \cdot I_\Lambda,\ y \cdot I_\Lambda,\ z \cdot I_\Lambda \subset I_\Lambda$$

The story generalizes to an arbitrary toric CY3 toric diagram $\left(\begin{array}{l} \text{SPP} \\ xy = zw^2 \end{array} \right)$

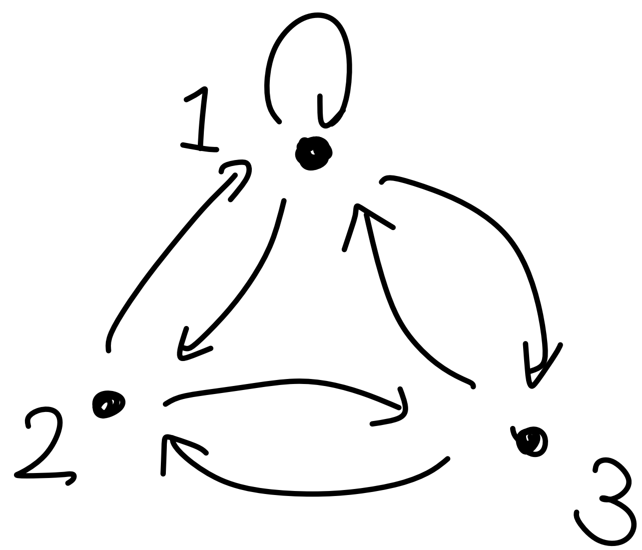


[Ooguri-MY ‘08]



We have an associated SQM

Q

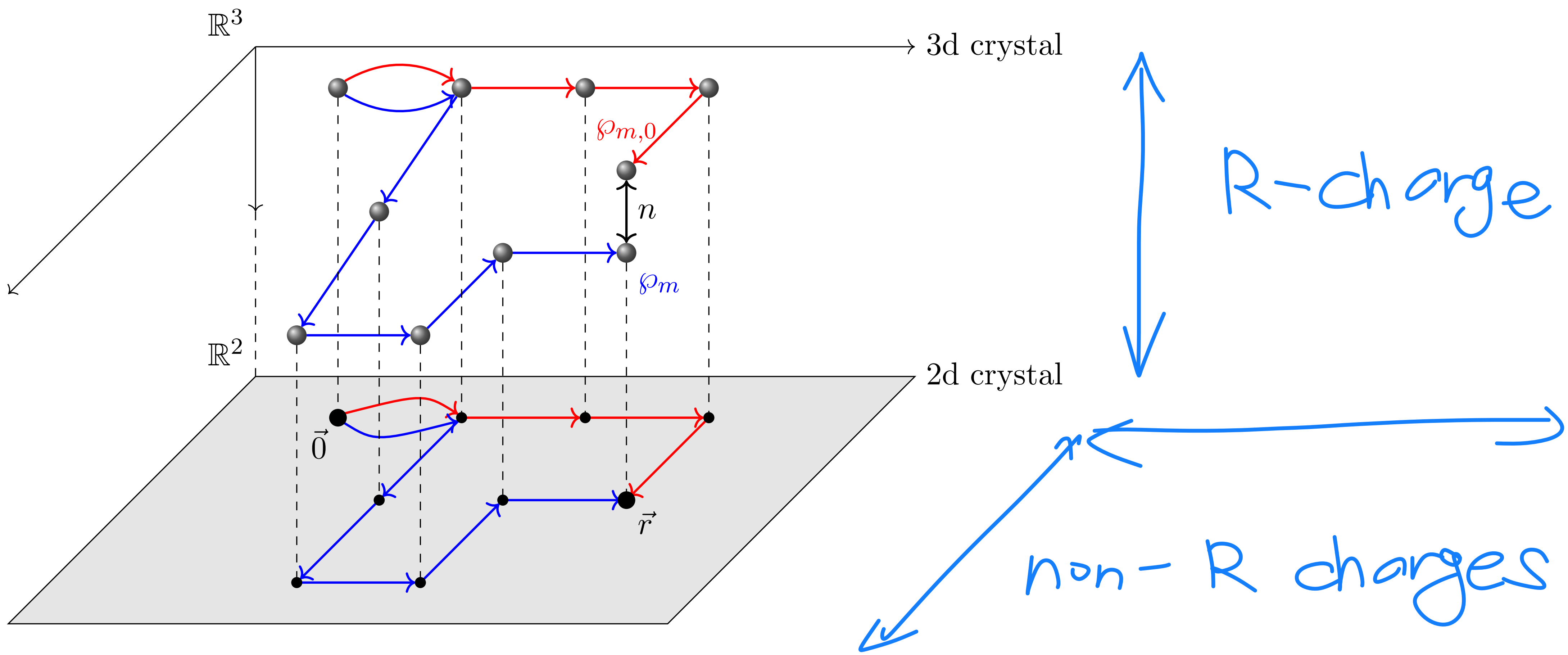


W $\quad W = \mathrm{Tr}\big(X_{11} X_{12} X_{21} - X_{11} X_{13} X_{31} - X_{12} X_{21} X_{23} X_{32} + X_{23} X_{31} X_{13} X_{23} \big)$

$\mathbb{C}Q/(\partial W)$: path algebra (non-commutative in general)

$\rightsquigarrow$ atom in the crystal

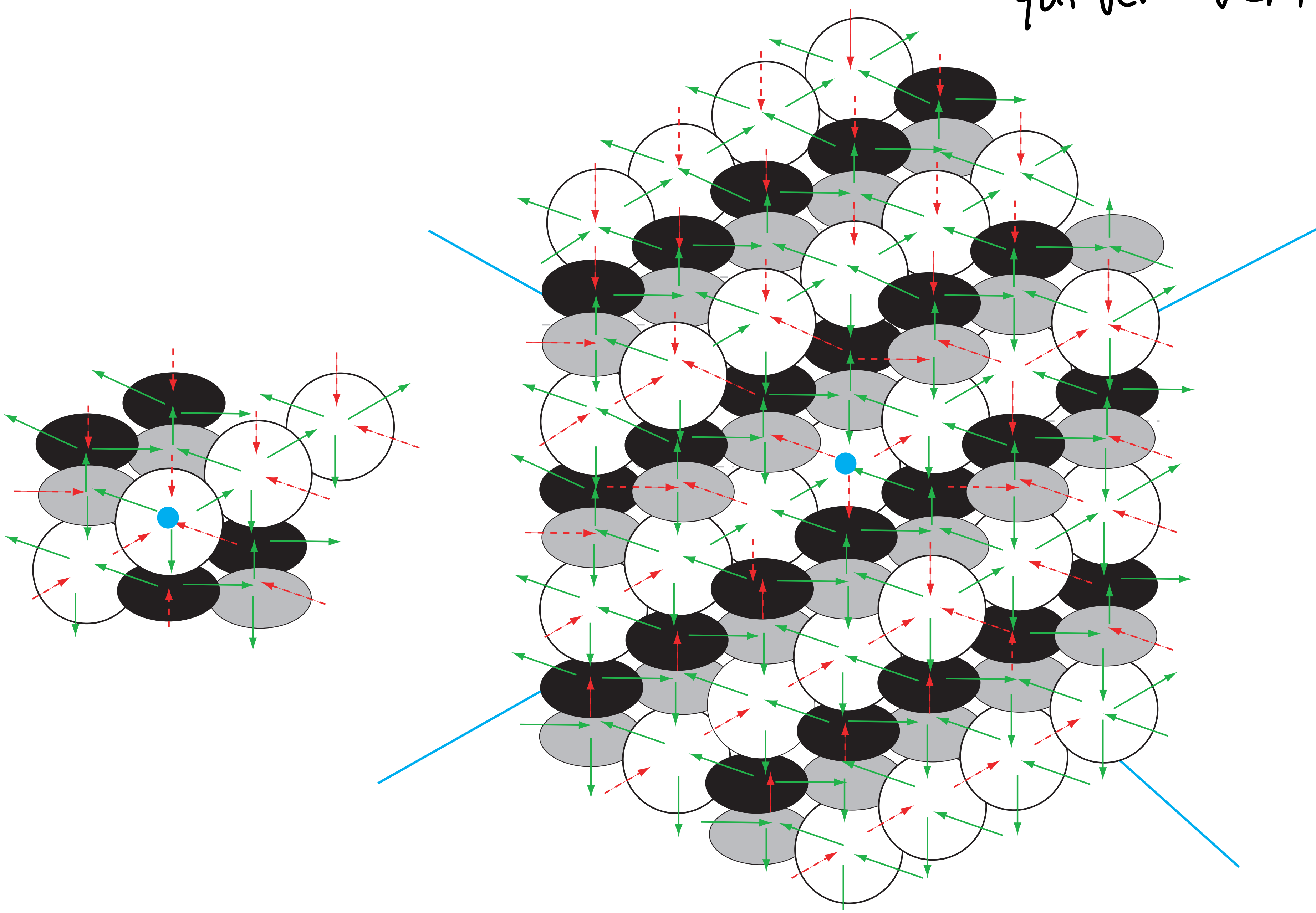
We can place the atoms in 3D according to their symmetry charges
(equivariant parameters corresponding to toric isometries)



BPS partition function

$$Z(q_1, \ldots, q_{|Q_0|}) = \sum_{\mathrm{K}} \prod_{a \in Q_0} q_a^{|\mathrm{K}(a)|}$$

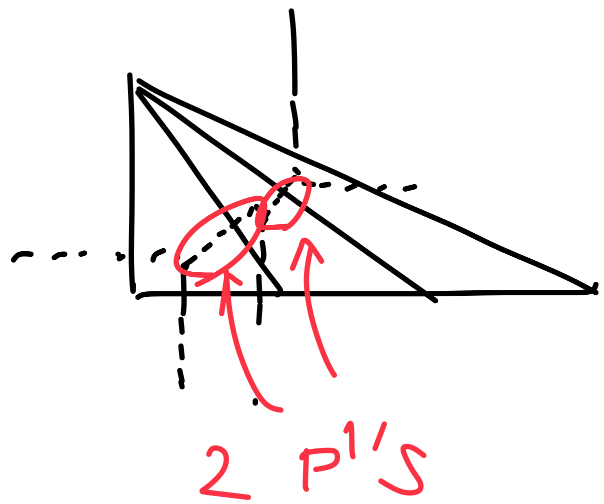
formal variable for each quiver vertex



Infinite-product forms discussed

in [Szendroi, Young, Nagao, Aganagic-Ooguri-Vafa-MY]

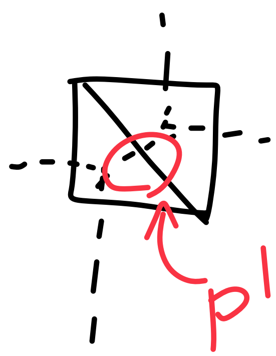
$(\mathbb{C}^2/\mathbb{Z}_3) \times \mathbb{C}$



$$Z \sim \prod_n \frac{1}{1-q^n Q_1} \frac{1}{1-q^n Q_2} \frac{1}{1-q^n Q_1 Q_2}$$

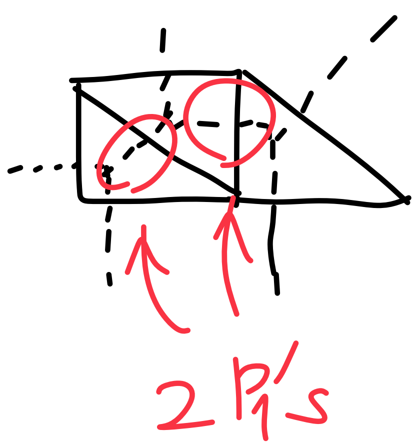
$$\begin{pmatrix} n\delta+\alpha_1, & n\delta+\alpha_2, & n\delta+\alpha_1+\alpha_2 \\ \text{even} & \text{even} & \text{even} \end{pmatrix}$$

conifold



$$Z \sim \prod_n (1-q^n Q) \qquad \begin{pmatrix} n\delta+\alpha \\ \text{odd} \end{pmatrix}$$

SPP



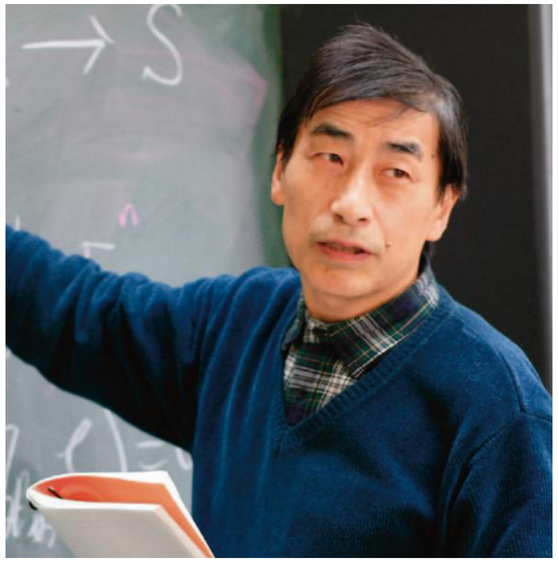
$$Z \sim \prod_n (1-q^n Q_1)(1-q^n Q_1 Q_2) \frac{1}{1-q^n Q_2}$$

$$\begin{pmatrix} n\delta+\alpha_1, & n\delta+\alpha_2, & n\delta+\alpha_1+\alpha_2 \\ \text{odd} & \text{even} & \text{odd} \end{pmatrix}$$

[Nagao-MY] discussed chamber structures in terms of affine Weyl groups]

Lie superalgebra?

Circa 2009-2010



Elliptic !!

Quantum toroidal !!

Later important developments on quantum toroidal algebras (Ding-Iohara-Miki) and affine Yangians by [B. Feigin, E. Feigin, Jimbo, Miwa, Mukhin; Tsymbaulik; Prochazka, ⋯]

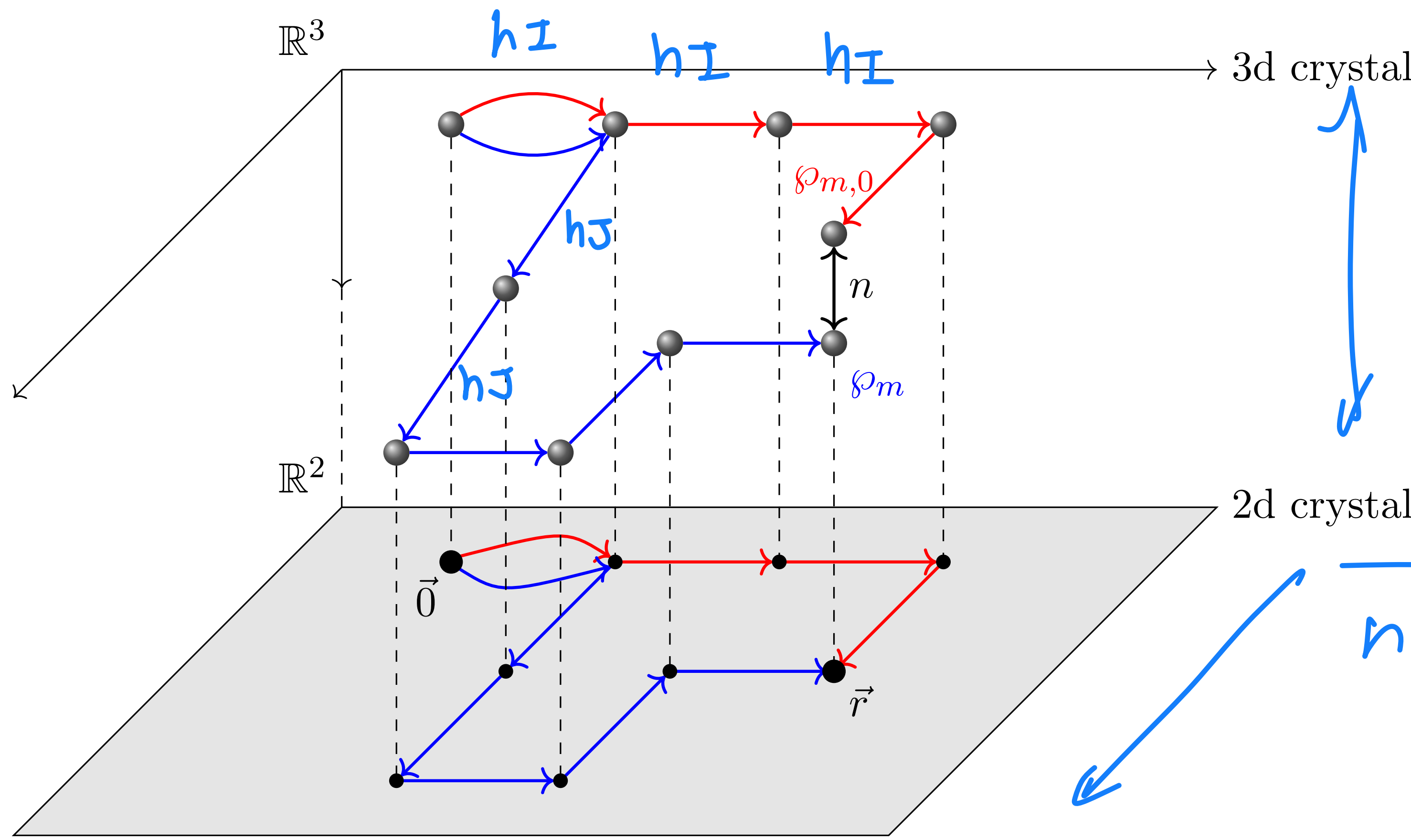
also in higher spin algebras [Gaberdiel, Gopakumar; Li, Peng,⋯]

Quiver Yangian

: Algebra

[Li-MY '20]

A. equivariant parameters



R-sym

non-R sym.

$$h(\boxed{a}) \equiv \sum_{I \in \text{path}[\mathfrak{o} \to \boxed{a}]} h_I \; .$$

loop constraint: $\displaystyle\sum_{I \in L} h_I = 0 \;,$

(W has charge 0)

B. Chevally-type generators

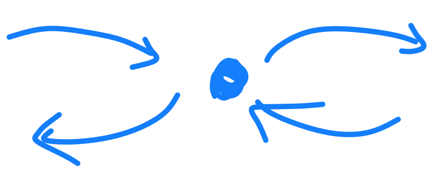
(z: spectral parameter)

$$e^{(a)}(z) \equiv \sum_{n=0}^{+\infty} \frac{e_n^{(a)}}{z^{n+1}}\,, \qquad \psi^{(a)}(z) \equiv \sum_{n=-\infty}^{+\infty} \frac{\psi_n^{(a)}}{z^{n+1}}\,, \qquad f^{(a)}(z) \equiv \sum_{n=0}^{+\infty} \frac{f_n^{(a)}}{z^{n+1}}\,,$$

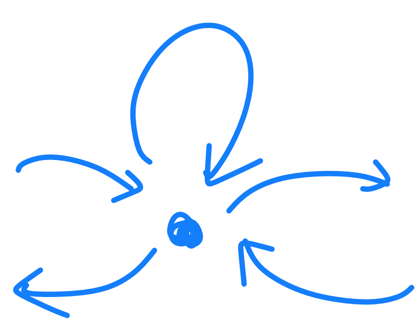
$$e^{(a)}(u) : \ \text{creation}\,, \qquad \psi^{(a)}(u) : \ \text{charge}\,, \qquad f^{(a)}(u) : \ \text{annihilation}$$

Z2-grading (super algebra)

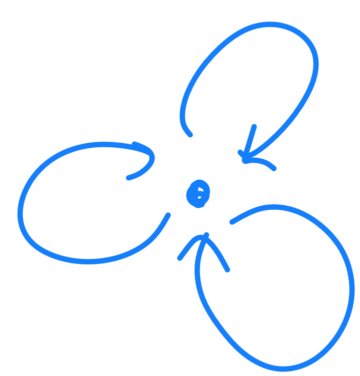
$$|a| = \begin{cases} 0 & (\exists I \in Q_1 \quad \text{such that} \quad s(I) = t(I) = a)\,, \\ 1 & (\text{otherwise})\,, \end{cases}$$



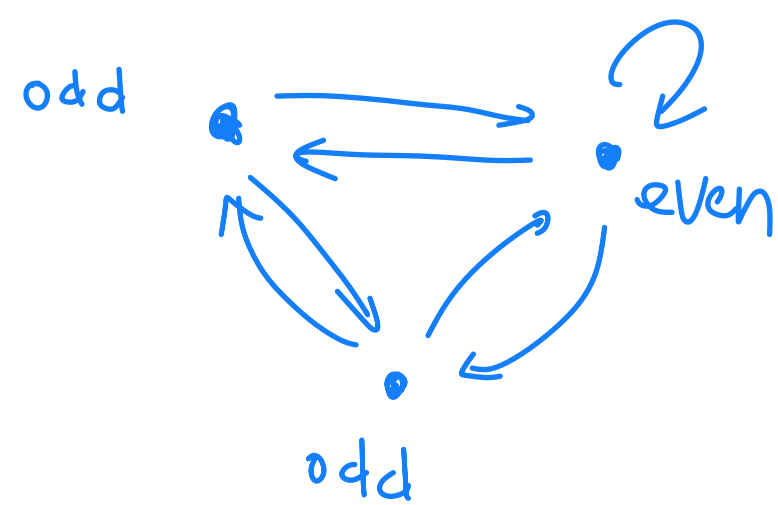
odd



even



even



C. “OPE relations”

$$
\begin{aligned}
\psi^{(a)}(z)\,\psi^{(b)}(w) &= \psi^{(b)}(w)\,\psi^{(a)}(z)\ ,\\
\psi^{(a)}(z)\,e^{(b)}(w) &\simeq \varphi^{b\Rightarrow a}(\Delta)\,e^{(b)}(w)\,\psi^{(a)}(z)\ ,\\
e^{(a)}(z)\,e^{(b)}(w) &\sim (-1)^{|a||b|}\varphi^{b\Rightarrow a}(\Delta)\,e^{(b)}(w)\,e^{(a)}(z)\ ,\\
\psi^{(a)}(z)\,f^{(b)}(w) &\simeq \varphi^{b\Rightarrow a}(\Delta)^{-1}\,f^{(b)}(w)\,\psi^{(a)}(z)\ ,\\
f^{(a)}(z)\,f^{(b)}(w) &\sim (-1)^{|a||b|}\varphi^{b\Rightarrow a}(\Delta)^{-1}\,f^{(b)}(w)\,f^{(a)}(z)\ ,\\
\left[e^{(a)}(z), f^{(b)}(w)\right\} &\sim -\delta^{a,b}\,\frac{\psi^{(a)}(z)-\psi^{(b)}(w)}{z-w}\ ,
\end{aligned}
$$

“$\simeq$” means equality up to $z^n w^{m\geq 0}$ terms

“$\sim$” means equality up to $z^{n\geq 0} w^m$ and $z^n w^{m\geq 0}$ terms

$$
\varphi^{a\Rightarrow b}(u) \equiv \frac{\prod_{I\in\{b\to a\}}(u+h_I)}{\prod_{I\in\{a\to b\}}(u-h_I)}
$$

Example

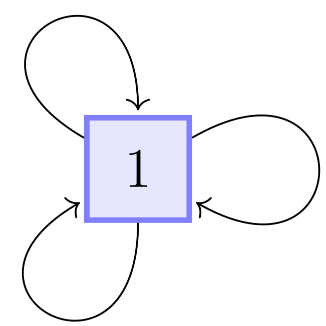
OPE relation

$$
\begin{aligned}
\psi(z)\,\psi(w) &\sim \psi(w)\,\psi(z)\ , \\
\psi(z)\,e(w) &\sim \varphi_3(\Delta)\,e(w)\,\psi(z)\ , \\
\psi(z)\,f(w) &\sim \varphi_3^{-1}(\Delta)\,f(w)\,\psi(z)\ , \\
e(z)\,e(w) &\sim \varphi_3(\Delta)\,e(w)\,e(z)\ , \\
f(z)\,f(w) &\sim \varphi_3^{-1}(\Delta)\,f(w)\,f(z)\ , \\
[e(z)\,,f(w)] &\sim -\frac{1}{\sigma_3}\,\frac{\psi(z)-\psi(w)}{z-w}\ ,
\end{aligned}
$$

$$
\varphi_3(z) \equiv \frac{(z+h_1)(z+h_2)(z+h_3)}{(z-h_1)(z-h_2)(z-h_3)}\ .
$$

$$
h_1+h_2+h_3 = 0\ ,
$$

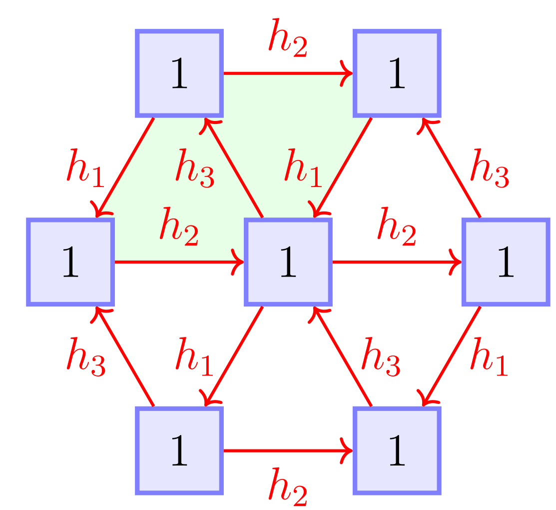
$$
\sigma_3 \equiv h_1\,h_2\,h_3\ .
$$



Serre relation

$$
\mathrm{Sym}_{z_1,z_2,z_3}(z_2-z_3)[\,e(z_1)\,,[\,e(z_2)\,,e(z_3)]] = 0\,;
$$

$$
\mathrm{Sym}_{z_1,z_2,z_3}(z_2-z_3)[f(z_1)\,,[f(z_2)\,,f(z_3)]] = 0\,.
$$



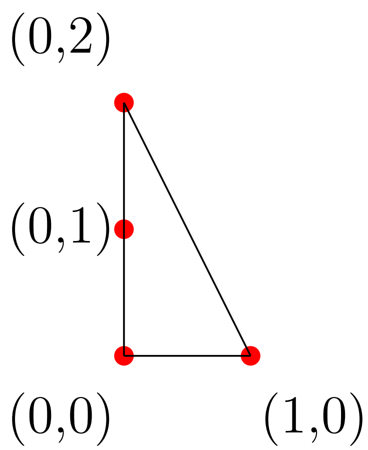
This gives $Y(\widehat{gl}_1)$: affine Yangian

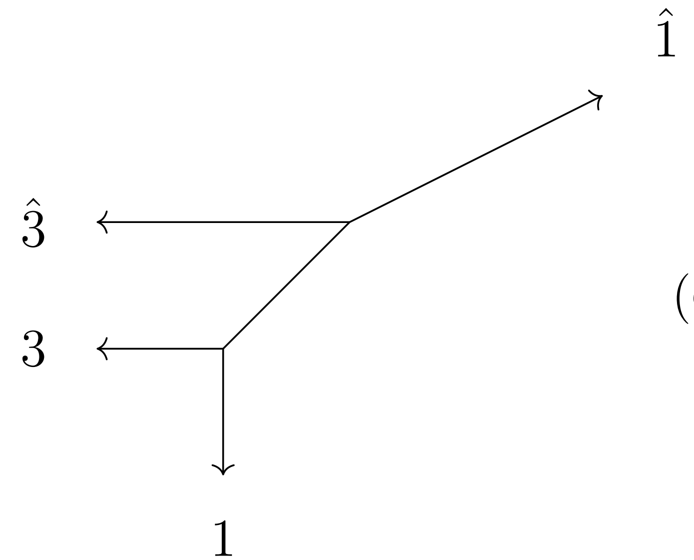
$\cong$

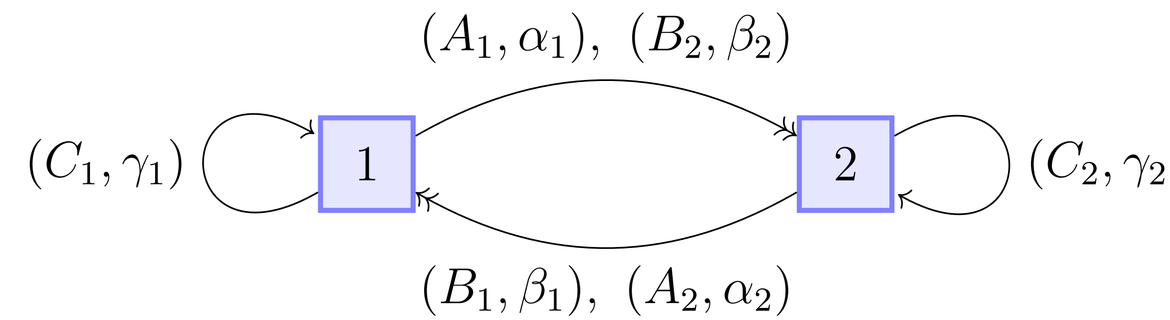
$\mathcal{U}(W_{1+\infty})$

[Schiffmann-Vasserot; Tsymbaulik; Prochazka; Gaberdiel-Gopakumar-Li-Peng,...]

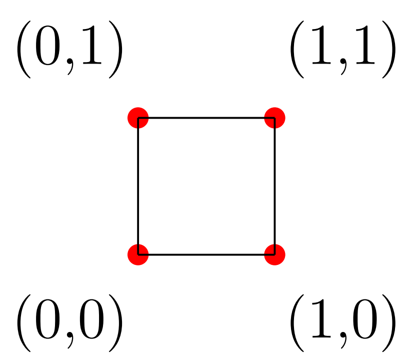
* $(\mathbb{C}^2/\mathbb{Z}_2)\times\mathbb{C} \rightsquigarrow Y(\widehat{\mathfrak{gl}}_2)$

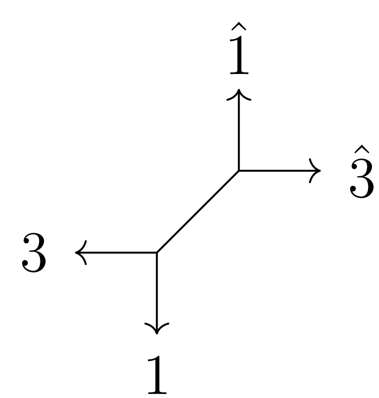


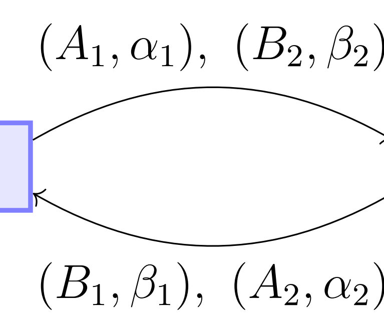




* conifold $\rightsquigarrow Y(\widehat{\mathfrak{gl}}_{1|1})$







* more generally,

$xy = z^n w^m \rightsquigarrow Y(\widehat{\mathfrak{gl}}_{m|n})$ [Rapcak; Bezerra-Mukhin]

Some Properties of Quiver Yangians [Li-MY]

a. triangular decomposition

$$\mathsf{Y}_{(Q,W)} = \mathsf{Y}^{+}_{(Q,W)} \oplus \mathsf{B}_{(Q,W)} \oplus \mathsf{Y}^{-}_{(Q,W)} \,, \qquad e^{(a)}(z) \leftrightarrow f^{(a)}(z) \,, \quad \psi^{(a)}(z) \leftrightarrow \psi^{(a)}(z)^{-1} \,,$$

$\{e_a\}$ $\{\psi_a\}$ $\{f_a\}$ order 2 involution

b. grading

$$\deg_a(e_n^{(b)}) = \delta_{a,b} \,, \quad \deg_a(\psi_n^{(b)}) = 0 \,, \quad \deg_a(f_n^{(b)}) = -\delta_{a,b} \,.$$

$$\deg_{\text{level}}(e_n^{(b)}) = \deg_{\text{level}}(f_n^{(b)}) = n + \tfrac{1}{2} \,, \quad \deg_{\text{level}}(\psi_n^{(b)}) = n + 1 \,,$$

grading when $\deg(\hbar z) = 1$

c. spectral shift

$$e^{(a)}(z) \equiv \sum_{n=0}^{+\infty} \frac{e_n^{(a)}}{z^{n+1}} \,, \qquad \psi^{(a)}(z) \equiv \sum_{n=-\infty}^{+\infty} \frac{\psi_n^{(a)}}{z^{n+1}} \,, \qquad f^{(a)}(z) \equiv \sum_{n=0}^{+\infty} \frac{f_n^{(a)}}{z^{n+1}} \,,$$

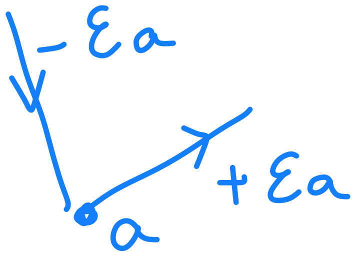
$z \to z - \varepsilon$ causes

$$e'_l = \sum_{k=0}^{l} \binom{l}{k} \varepsilon^k e_{l-k} \,, \quad f'_l = \sum_{k=0}^{l} \binom{l}{k} \varepsilon^k f_{l-k} \,, \quad \psi'_l = \sum_{k=0}^{l} \binom{l}{k} \varepsilon^k \psi_{l-k} \quad (l = 0, 1, \dots) \,,$$

$$\psi'_{-l-1} = \sum_{k=l}^{\infty} \binom{k}{l} (-\varepsilon)^{k-l} \psi_{-k-1} \quad (l = 0, 1, \dots,) \,.$$

Some Properties of Quiver Yangians [Li-MY]

d. gauge shift



$$h_I \to h'_I = h_I + \varepsilon_a \, \mathrm{sign}_a(I) \ , \qquad \mathrm{sign}_a(I) \equiv \begin{cases} +1 & (s(I)=a \ , \quad t(I) \neq a) \ , \\ -1 & (s(I) \neq a \ , \quad t(I) = a) \ , \\ 0 & (\text{otherwise}) \ , \end{cases}$$

consistent with **loop constraint:** $\displaystyle\sum_{I\in L} h_I = 0 \ ,$

$E + 2I - 1$ parameters

$$\varphi^{a\Rightarrow b}(u) \to \varphi^{a\Rightarrow b\prime}(u) = \frac{\prod_{I\in\{b\to a\}} (u + h_I + \varepsilon_a \, \mathrm{sign}_a(I))}{\prod_{I\in\{a\to b\}} (u - h_I - \varepsilon_a \, \mathrm{sign}_a(I))} \ .$$

which reshuffles generators

e^a_m mixes w/ e^a_n $(n<m)$

To eliminate this ambiguity,

vertex constraint: $\displaystyle\sum_{I\in a} \mathrm{sign}_a(I) \, h_I = 0$

$\hookrightarrow$ 2 parameters

Quiver Yangian :

Representation

Representation by crystal melting [Li-MY '20], inspired by [FFJMM] and [Prochazka]

$$\psi^{(a)}(z)|\mathrm{K}\rangle = \Psi^{(a)}_{\mathrm{K}}(z)|\mathrm{K}\rangle \; ,$$

$$e^{(a)}(z)|\mathrm{K}\rangle = \sum_{\boxed{a} \in \mathrm{Add}(\mathrm{K})} \frac{E^{(a)}(\mathrm{K} \to \mathrm{K} + \boxed{a})}{z - h(\boxed{a})} |\mathrm{K} + \boxed{a}\rangle \; ,$$

$$f^{(a)}(z)|\mathrm{K}\rangle = \sum_{\boxed{a} \in \mathrm{Rem}(\mathrm{K})} \frac{F^{(a)}(\mathrm{K} \to \mathrm{K} - \boxed{a})}{z - h(\boxed{a})} |\mathrm{K} - \boxed{a}\rangle \; ,$$

poles in z

add/remove an atom

$$h(\boxed{a}) \equiv \sum_{I \in \mathrm{path}[\mathfrak{o} \to \boxed{a}]} h_I \; .$$

$$\Psi^{(a)}_{\mathrm{K}}(u) = \psi^{(a)}_0(z) \prod_{b \in Q_0} \prod_{\boxed{b} \in \mathrm{K}} \varphi^{b \Rightarrow a}(u - h(\boxed{b})) \; ,$$

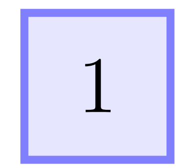
$$\varphi^{a \Rightarrow b}(u) \equiv \frac{\prod_{I \in \{b \to a\}} (u + h_I)}{\prod_{I \in \{a \to b\}} (u - h_I)}$$

In fact, we can "bootstrap" the algebra from this Ansatz

Crucial ingredient: poles keep track of the crystal structure

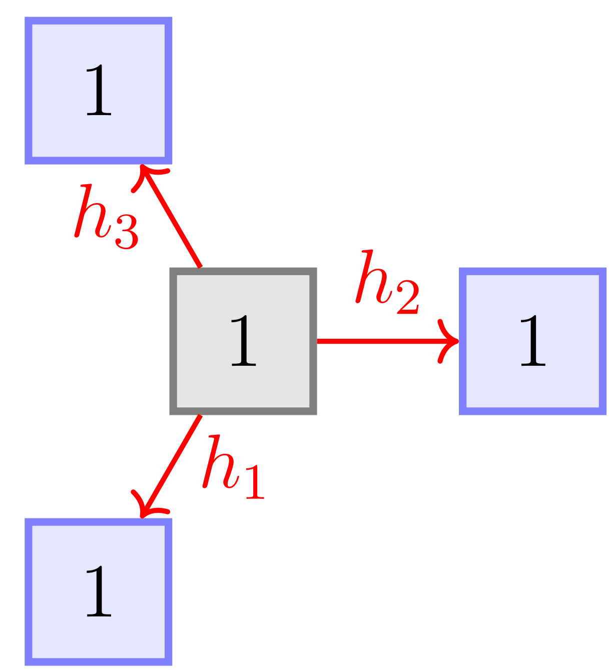
$\sum_a \psi_0^a$: centeral element

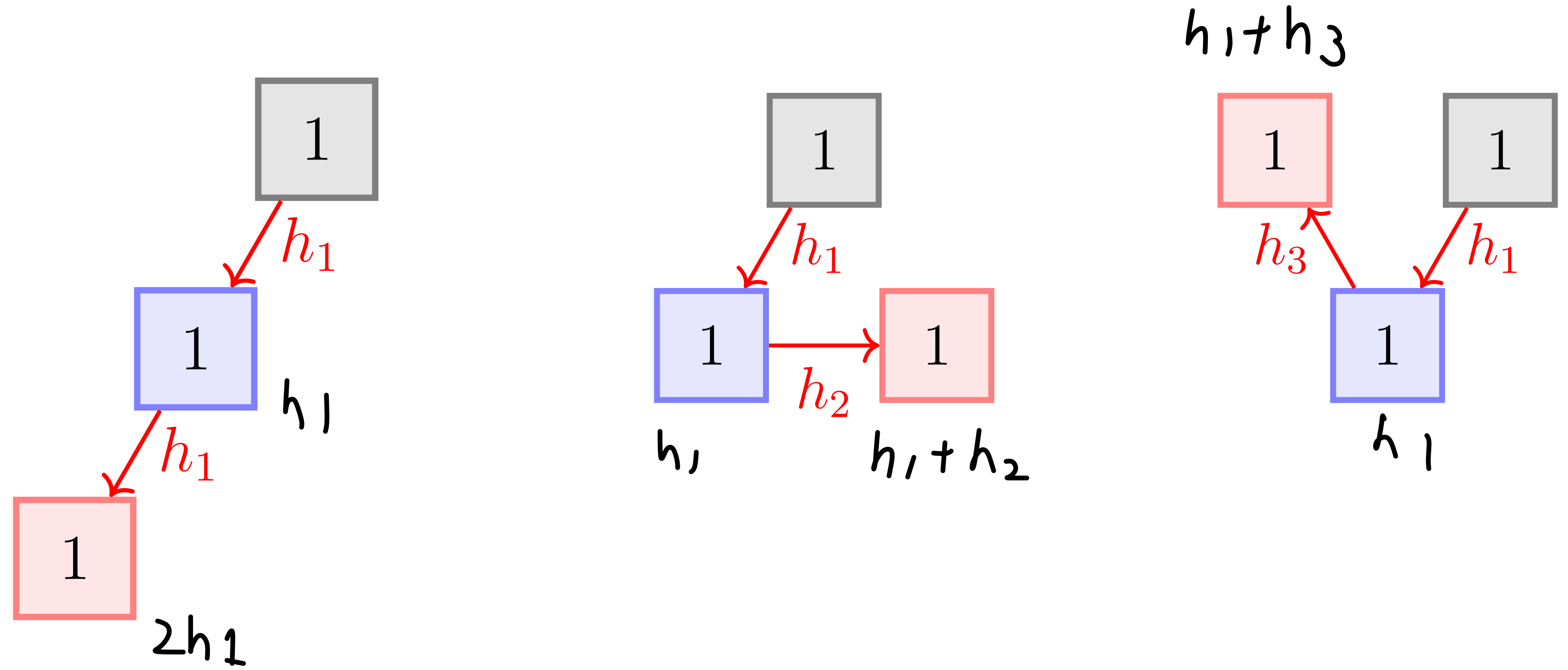
$$\Psi_\Lambda(z) = \psi_0(z) = \frac{z+C}{z}$$



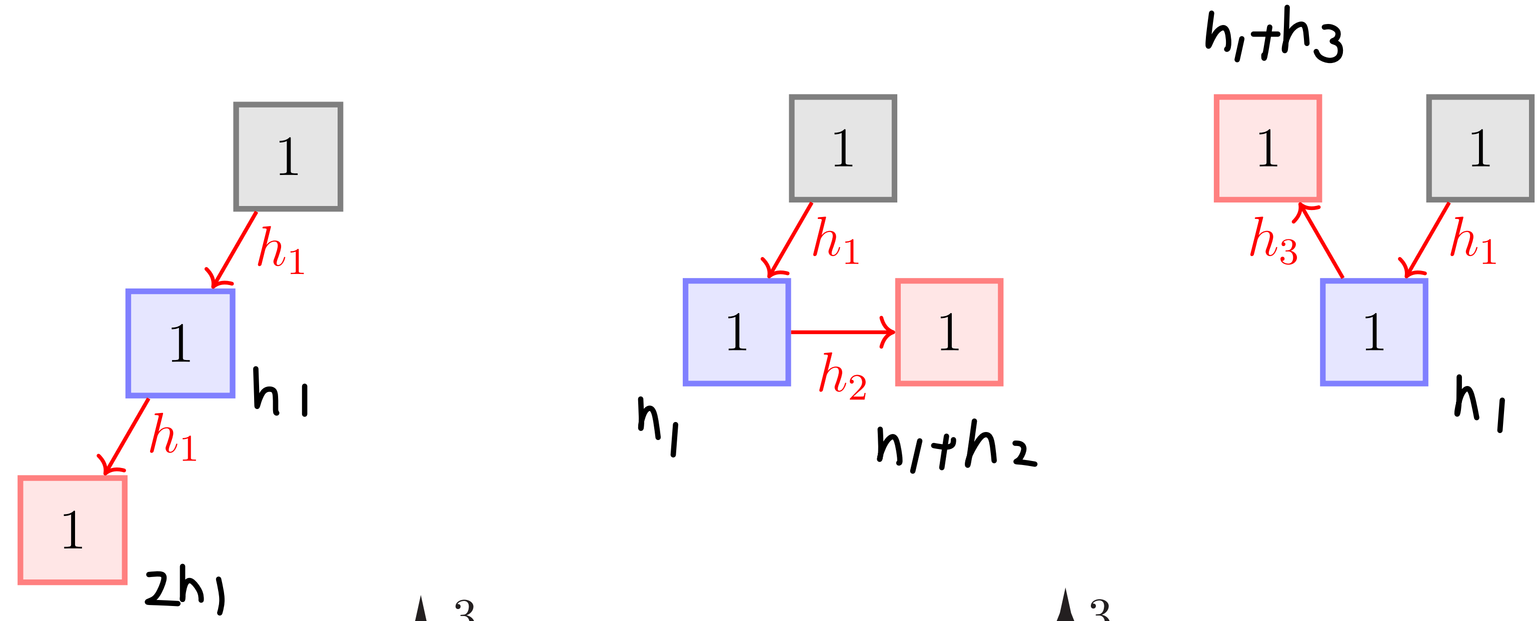
$$\Psi_\Lambda(z) = \psi_0(z)\psi_{\Box_0}(z)$$

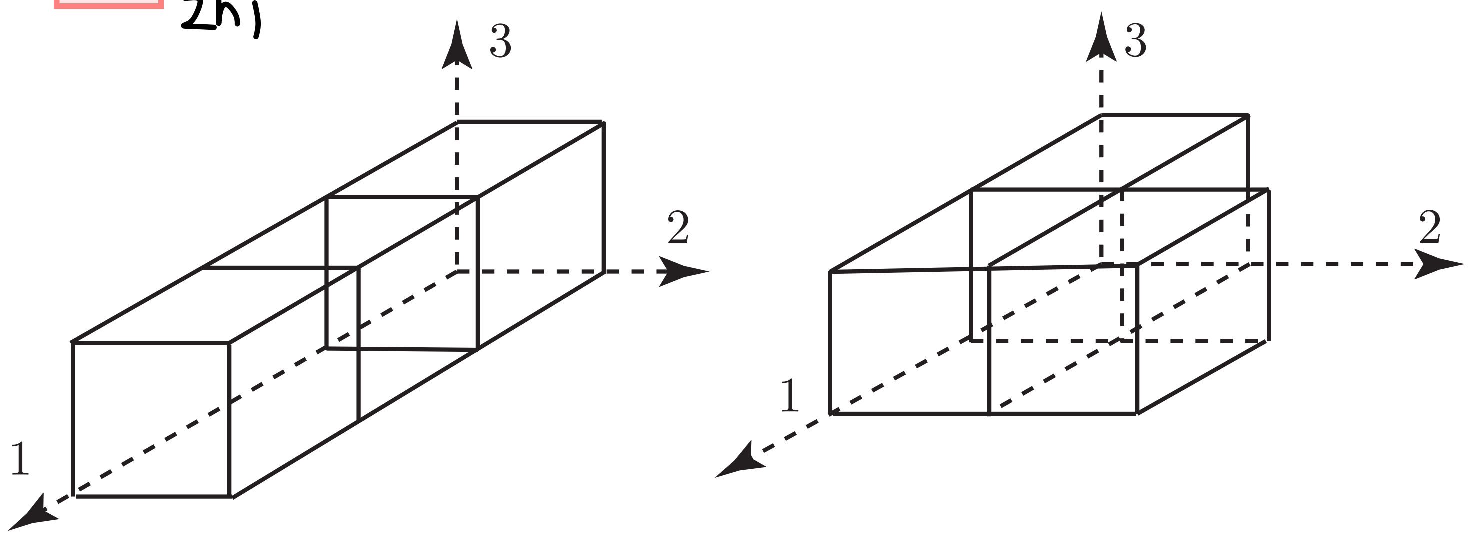
$$= \frac{z+C}{z} \cdot \frac{(z+h_1)(z+h_2)(z+h_3)}{(z-h_1)(z-h_2)(z-h_3)}$$





$$\Psi_\Lambda(z) = \psi_0(z)\psi_{\square_0}(z)\psi_{\square_1}(z)$$
$$= \frac{z+C}{z} \cdot \frac{(z+h_1)(z+h_2)(z+h_3)}{(z-h_1)(z-h_2)(z-h_3)} \cdot \frac{z(z+h_2-h_1)(z+h_3-h_1)}{(z-2h_1)(z-h_2-h_1)(z-h_3-h_1)}$$





(h1+h2+h3=0)

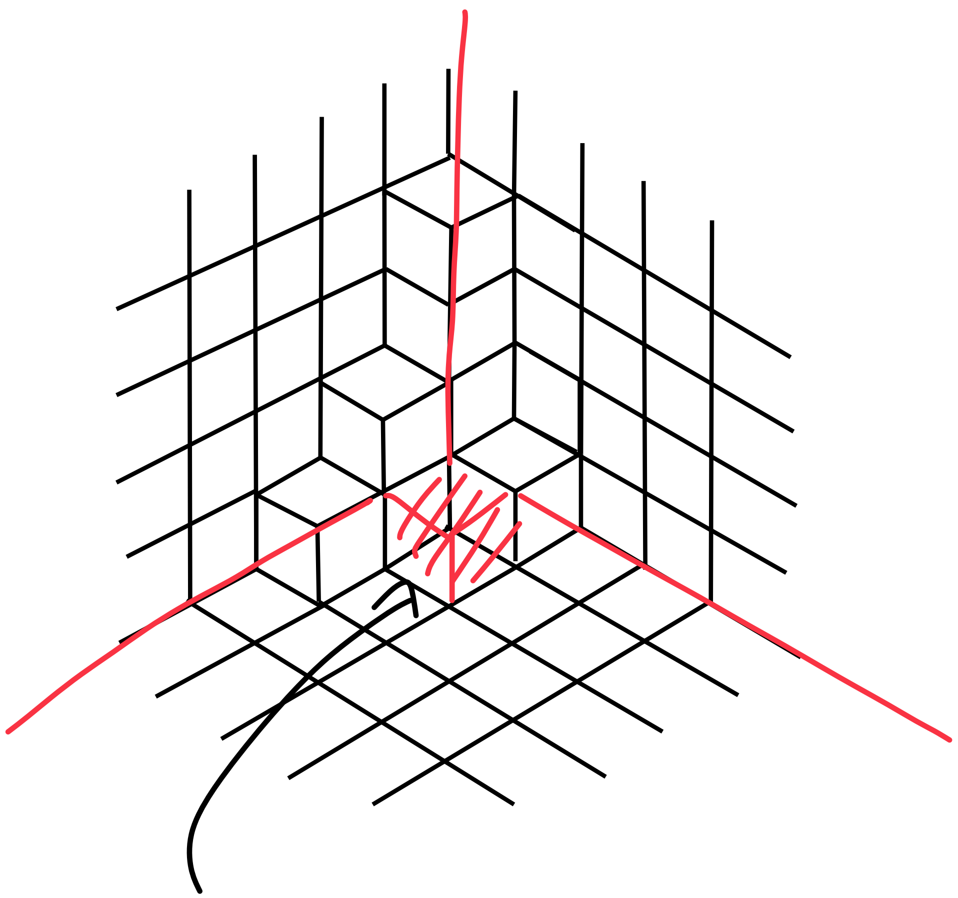
$$
\begin{aligned}
\Psi_\Lambda(z) &= \psi_0(z)\psi_{\square_0}(z)\psi_{\square_1}(z) \\
&= \frac{z+C}{z} \cdot \frac{(z+h_1)(z+h_2)(z+h_3)}{(z-h_1)(z-h_2)(z-h_3)} \cdot \frac{z(z+h_2-h_1)(z+h_3-h_1)}{(z-2h_1)(z-h_2-h_1)(z-h_3-h_1)}
\end{aligned}
$$

In general, loop constraint ensures that poles are in correct positions as dictated by the melting rule of the crystal

Truncations and D4-branes

For non-generic equivariant parameters, we have null states, so that the crystal truncates at the “pit”

$$N_1 h_1 + N_2 h_2 + N_3 h_3 + C = 0$$

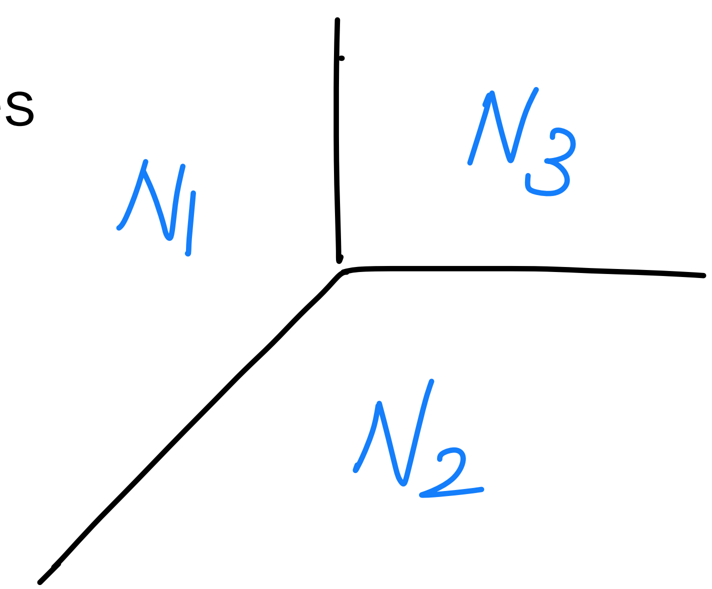


pit: location of null state

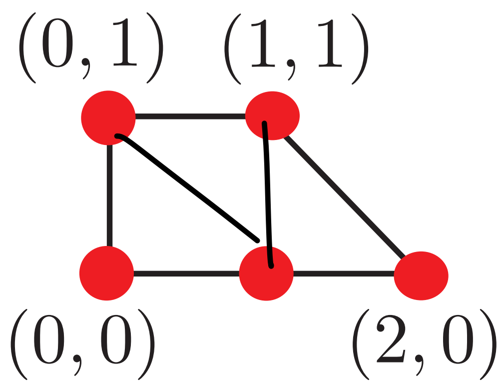
There is a corresponding truncation of the algebra studied by [Gaiotto-Rapcak] (also [Bershtein, Feigin, Merzon])

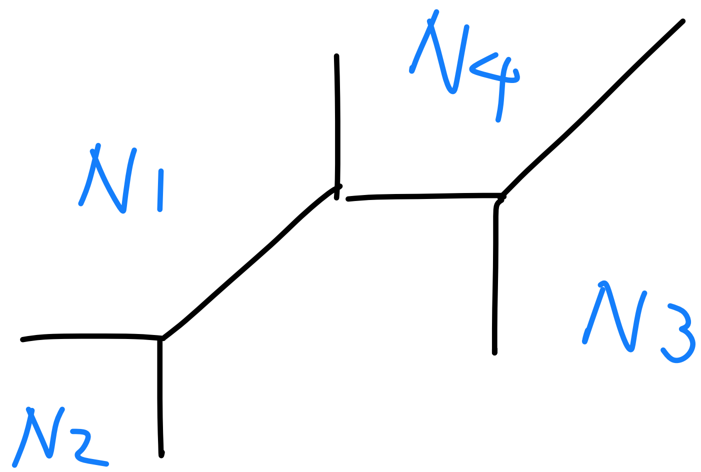
$$Y(\hat{gl}_1) \to Y_{N_1, N_2, N_3}$$

Physically: D4-branes



Generalization



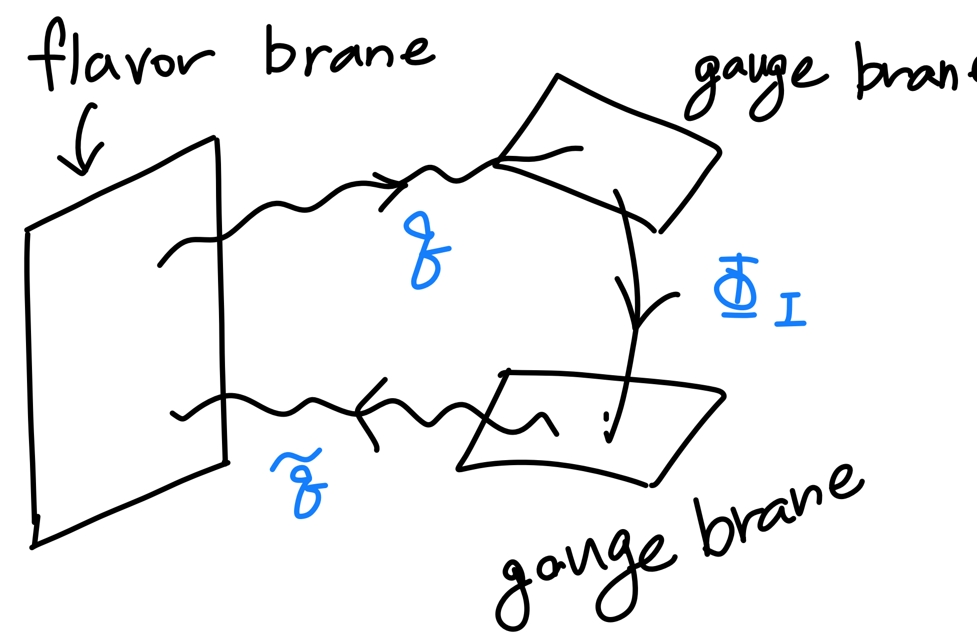


null state happens at $\sum_I M_I h_I + C = 0$

Answer in terms of perfect matchings [Li-MY]

Physically D4-brane = divisor [Imamura-Kimura-Y]

$$W = \tilde{q}\,\Phi_I\,q\,. \qquad \underset{\displaystyle 0}{\underset{\shortparallel}{\Phi_I}} = \prod_{p \ni I} \tilde{\Phi}_p\,.$$



Derivation from

Quantum Mechanics

[Galakhov-MY]

toric CY_3 : X

type IIA string theory

$$\mathbb{R}^{3,1} \times X$$
$$\cup$$
$$\mathbb{R} \times \{\text{hol. cycle}\}$$

BPS particles wrapping hol. cycle

$$Z^X_{BPS} = \sum_\gamma \underbrace{\Omega^X_\gamma(\cdots)}_{\text{BPS degeneracy}} q^\gamma \qquad \gamma \in H^{even}(X)$$

$$= Z_{crystal}$$ fixed point

SUSY QM of BPS particles

BPS quiver Yangian

Step 1: SQM and its equivariant cohomology

Vect mult at vertex

(A_μ, X^3_v, Φ_v)

$\Phi_v = X^1_v + i X^2_v$

We have the vacuum moduli space from supersymmetric quiver quantum mechanics (e.g. [Denef])

$$\mathcal{M}_{\mathrm{SQM}}: \quad \begin{array}{l} X^3_v \in \mathfrak{u}(n_v),\ \Phi_v \in \mathfrak{gl}(n_v,\mathbb{C})\ , \\ q_{(a:\ v\to w)} \in \mathrm{Hom}(\mathbb{C}^{n_v}, \mathbb{C}^{n_w})\ , \end{array}$$

chiral mult. at edge

Supercharge [Galakhov-MY]

$$\bar{Q}_{\mathrm{i}} = e^{-\mathfrak{H}}\left(d_{X^3} + \bar{\partial}_{\Phi,q} + \iota_V + dW\wedge\right)e^{\mathfrak{H}}\ ,$$

$$\mathfrak{H} := \sum_{v\in\mathcal{V}} \mathrm{Tr}\, X^3_v \left(\frac{1}{2}\left[\Phi_v, \bar{\Phi}_v\right] - \mu_{\mathbb{R},v}\right)\ ,$$

$$V := \sum_{(a:\ v\to w)\in\mathcal{A}} \left(\Phi_w q_a - q_a \Phi_v\right)\frac{\partial}{\partial q_a}\ ,$$

$$\mu_{\mathbb{R},v} := \theta_v \mathbb{1}_{n_v\times n_v} - \sum_{x\in\mathcal{V}}\sum_{(a:v\to x)\in\mathcal{A}} q_a q_a^\dagger + \sum_{y\in\mathcal{V}}\sum_{(b:y\to v)\in\mathcal{A}} q_b^\dagger q_b\ .$$

stability param.

Step 2: Omega-deformation

We introduce Omega-deformation [Nekrasov,⋯]

to "smooth out" the singular geometry

$$V := \sum_{(a\colon v \to w) \in \mathcal{A}} (\Phi_w q_a - q_a \Phi_v) \frac{\partial}{\partial q_a} ,$$

$$V(q_a) = \sum_{(a\colon v \to w) \in \mathcal{A}} (\Phi_w q_a - q_a \Phi_v - \epsilon_a q_a) \frac{\partial}{\partial q_a} .$$

ϵ_a

The equivariant parameters should be consistent with W,
and hence can be identified with h_I introduced previously

Step 3: Higgs branch localization

Ω-bgd   UV scale   FI/stability

$$|\epsilon| \ll \Lambda_{\mathrm{cf}} \ll |\theta|^{\frac{1}{2}} \, .$$

1-parameter ($\mathbf{s}$) deformation of supercharge

$$\bar{Q}^{(\mathbf{s})}_{\mathrm{i}} = e^{-\mathbf{s}\mathfrak{H}} \left(d_{X^3} + \bar{\partial}_{\Phi,q} + \iota_{\mathbf{s}V} + \mathbf{s}\, dW \wedge \right) e^{\mathbf{s}\mathfrak{H}} \, .$$

$$H_0 \sim \sum_i \left(-\partial^2_{x_i} + \omega_i^2 x_i^2 \right) + \sum_i \omega_i \left(\psi_i \psi_i^\dagger - \psi_i^\dagger \psi_i \right) , \quad \bar{Q}^{(0)}_{\mathrm{i}} \sim \sum_i \psi_i \left(\partial_{x_i} + \omega_i x_i \right) \, .$$

leading piece

Wilsonian decomposition of wave function

$$\Psi = \Psi_{|\omega|<\Lambda_{\mathrm{cf}}} \left(x_{|\omega|<\Lambda_{\mathrm{cf}}} \right) \Psi_{|\omega|>\Lambda_{\mathrm{cf}}} \left(x_{|\omega|<\Lambda_{\mathrm{cf}}}, x_{|\omega|>\Lambda_{\mathrm{cf}}} \right) + O(\mathbf{s}^{-1}) \, .$$

$$Q^\dagger_{\mathrm{eff}} \Psi_{|\omega|<\Lambda_{\mathrm{cf}}} = 0, \quad Q^\dagger_{\mathrm{eff}} := \left\langle \Psi_{|\omega|>\Lambda_{\mathrm{cf}}} \middle| \bar{Q}^{(1)}_{\mathrm{i}} \middle| \Psi_{|\omega|>\Lambda_{\mathrm{cf}}} \right\rangle \, .$$

to find the Euler class

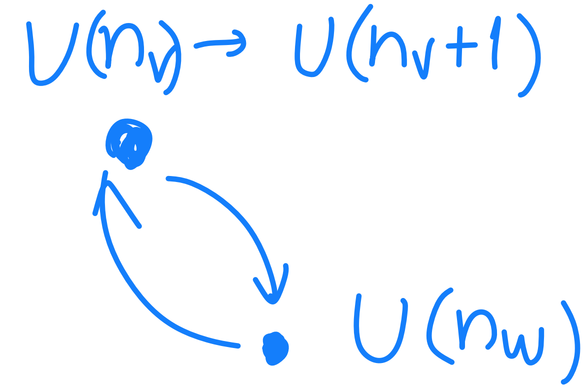
$$\boxed{\Psi_\Lambda \sim \mathrm{Eul}_\Lambda := \prod_i w_i \, .}$$

$$\int \Psi_\Lambda = 1 \, , \quad \int \Psi_\Lambda \wedge \Psi_{\Lambda'} = \mathrm{Eul}_\Lambda \, \delta_{\Lambda,\Lambda'} \, .$$

Step 4: Hecke modification

Raising/lowering operators of the algebra obtained by "Hecke modification"

$\hat{e}$ $\hat{f}$ shifting the dimension vectors at the quiver nodes:



$$n'_v = n_v \pm 1, \quad \text{and} \quad n'_w = n_w, \ \text{for } w \neq v \ .$$

Define generators

$$\hat{e}^{(v)}(z) := \left[\mathrm{Tr}\,(z-\Phi_v)^{-1}, \hat{\mathbf{e}}\right] \ ,$$

$$\hat{f}^{(v)}(z) := -\left[\mathrm{Tr}\,(z-\Phi_v)^{-1}, \hat{\mathbf{f}}\right] \ .$$

and its action on crystal configurations is

$$\hat{e}^{(v)}(z)|\Lambda\rangle = \sum_{\substack{\square\in\Lambda^+ \\ \hat{\square}=v}} \frac{1}{z-\phi_\square} \times \hat{E}(\Lambda\to\Lambda+\square)|\Lambda+\square\rangle \ ,$$

$$\hat{f}^{(v)}(z)|\Lambda\rangle = \sum_{\substack{\square\in\Lambda^- \\ \hat{\square}=v}} \frac{1}{z-\phi_\square} \times \hat{F}(\Lambda\to\Lambda-\square)|\Lambda-\square\rangle \ .$$

$$\hat{\psi}^{(v)}(z)|\Lambda\rangle = \hat{\psi}^{(v)}_\Lambda(z) \times |\Lambda\rangle \ .$$

Need

$$\hat{E}(\Lambda\to\Lambda+\square) := \frac{\langle\Psi_{\Lambda+\square}|\hat{\mathbf{e}}|\Psi_\Lambda\rangle}{\langle\Psi_{\Lambda+\square}|\Psi_{\Lambda+\square}\rangle}$$

$$\hat{F}(\Lambda\to\Lambda-\square) := \frac{\langle\Psi_{\Lambda-\square}|\hat{\mathbf{f}}|\Psi_\Lambda\rangle}{\langle\Psi_{\Lambda-\square}|\Psi_{\Lambda-\square}\rangle}$$

The correct formula:

$$\hat{\mathbf{e}}\,\Psi_\Lambda = \sum_{\square\in\Lambda^+} \frac{\mathrm{Eul}_\Lambda}{\mathrm{Eul}_{\Lambda,\Lambda+\square}}\Psi_{\Lambda+\square}\ .$$

$$\hat{\mathbf{f}}\,\Psi_\Lambda = \sum_{\square\in\Lambda^-} \frac{\mathrm{Eul}_\Lambda}{\mathrm{Eul}_{\Lambda-\square,\Lambda}}\Psi_{\Lambda-\square}\ .$$

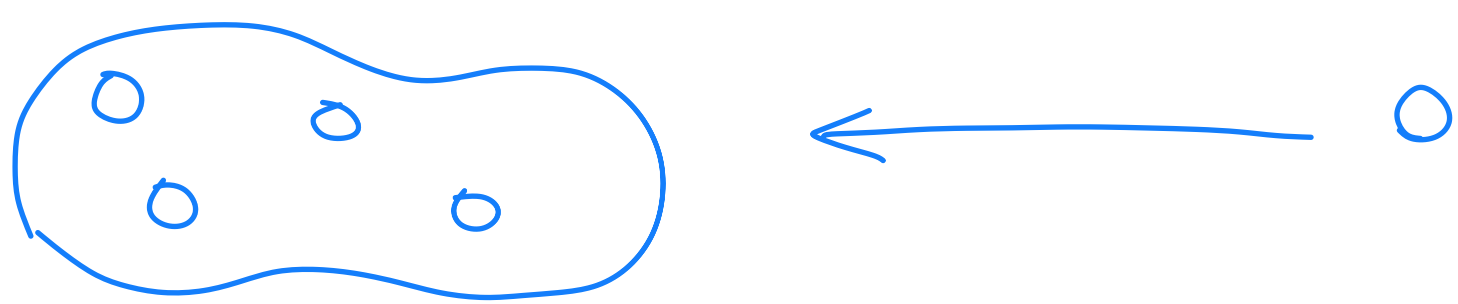
$\mathcal{M}_\Sigma \times \mathcal{M}_{\Sigma+\square}$

$I_1 \supset I_2$

Mathematically, this is derived by the Fourier-Mukai transform with the incident locus as a kernel [Nakajima,⋯]

Physically, we need to bring in particles from infinity. Along the process Some low-frequency modes get exchanged with high-frequency modes

BPS bound state



$$\Psi = \Psi_{|\omega|<\Lambda_{\rm cf}}\left(x_{|\omega|<\Lambda_{\rm cf}}\right)\Psi_{|\omega|>\Lambda_{\rm cf}}\left(x_{|\omega|<\Lambda_{\rm cf}}, x_{|\omega|>\Lambda_{\rm cf}}\right) + O(\mathbf{s}^{-1})\ .$$

Highly non-trivial cancellations!

For example, for one of the Serre relations of $Y(\widehat{\mathfrak{gl}}_{3|1})$

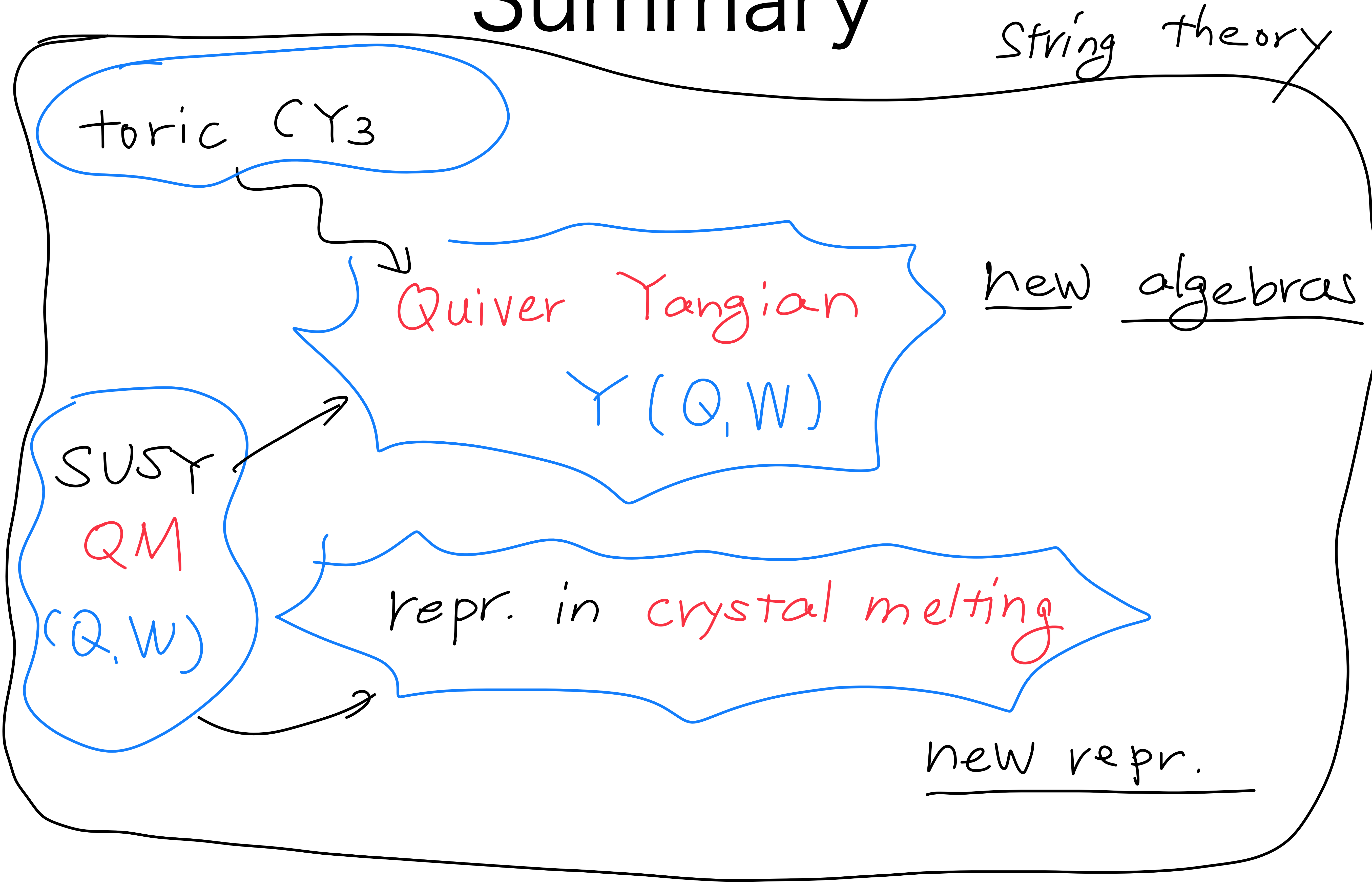
$$\mathrm{Sym}_{z_1,z_2}\Big[e^{(2)}(z_1),\Big[e^{(3)}(w_1),\Big[e^{(2)}(z_2),e^{(1)}(w_2)\Big\}\Big\}\Big\}$$

$$A_2 := \underset{z_1,z_2,w_1,w_2}{\mathrm{Res}} \langle\Lambda|A_1|\Lambda_0\rangle =$$

$$= [1,2,4,3]+[1,3,4,2]-[2,1,3,4]+[2,1,4,3]-[2,3,1,4]+[2,4,1,3]+$$
$$+[2,4,3,1]-[3,1,2,4]+[3,1,4,2]-[3,2,1,4]+[3,4,1,2]+[3,4,2,1]-$$
$$-[4,1,2,3]-[4,1,3,2]-[4,2,1,3]-[4,3,1,2] = 0\ !$$

$$[2,4,1,3]=-\frac{1}{48}\,,\quad [4,2,1,3]=-\frac{1}{96}\,,\quad [2,1,4,3]=-\frac{1}{48}\,,\quad [1,2,4,3]=\frac{1}{32}\,,$$
$$[4,1,2,3]=\frac{1}{64}\,,\quad [1,4,2,3]=\frac{1}{64}\,,\quad [4,1,3,2]=-\frac{1}{64}\,,\quad [1,4,3,2]=-\frac{1}{64}\,,$$
$$[2,4,3,1]=\frac{2\hbar_1+\hbar_2}{24\,(4\hbar_1+\hbar_2)}\,,\quad [4,2,3,1]=\frac{2\hbar_1+\hbar_2}{48\,(4\hbar_1+\hbar_2)}\,,$$
$$[2,3,4,1]=\frac{(2\hbar_1+\hbar_2)^2}{12\,(4\hbar_1+\hbar_2)\,(4\hbar_1+3\hbar_2)}\,,\quad [3,2,4,1]=-\frac{(2\hbar_1+\hbar_2)^2}{12\,(4\hbar_1+\hbar_2)\,(4\hbar_1+3\hbar_2)}\,,$$
$$[4,3,2,1]=-\frac{2\hbar_1+\hbar_2}{48\,(4\hbar_1+\hbar_2)}\,,\quad [3,4,2,1]=-\frac{(2\hbar_1+\hbar_2)^2}{24\,(4\hbar_1+\hbar_2)\,(4\hbar_1+3\hbar_2)}\,,$$
$$[2,1,3,4]=-\frac{2\hbar_1+\hbar_2}{24\,(4\hbar_1+3\hbar_2)}\,,\quad [1,2,3,4]=\frac{2\hbar_1+\hbar_2}{16\,(4\hbar_1+3\hbar_2)}\,,$$
$$[2,3,1,4]=\frac{(2\hbar_1+\hbar_2)^2}{12\,(4\hbar_1+\hbar_2)\,(4\hbar_1+3\hbar_2)}\,,\quad [3,2,1,4]=-\frac{(2\hbar_1+\hbar_2)^2}{12\,(4\hbar_1+\hbar_2)\,(4\hbar_1+3\hbar_2)}\,,$$
$$[1,3,2,4]=-\frac{2\hbar_1+\hbar_2}{16\,(4\hbar_1+3\hbar_2)}\,,\ [3,1,2,4]=\frac{(2\hbar_1+\hbar_2)^2}{8\,(4\hbar_1+\hbar_2)\,(4\hbar_1+3\hbar_2)}\,,$$
$$[4,3,1,2]=\frac{2\hbar_1+\hbar_2}{32\,(4\hbar_1+\hbar_2)}\,,\quad [3,4,1,2]=\frac{(2\hbar_1+\hbar_2)^2}{16\,(4\hbar_1+\hbar_2)\,(4\hbar_1+3\hbar_2)}\,,$$
$$[1,3,4,2]=-\frac{2\hbar_1+\hbar_2}{32\,(4\hbar_1+3\hbar_2)}\,,\quad [3,1,4,2]=\frac{(2\hbar_1+\hbar_2)^2}{16\,(4\hbar_1+\hbar_2)\,(4\hbar_1+3\hbar_2)}.$$

Summary



String theory
toric CY3
Quiver Yangian
Y(Q,W)
new algebras
SUSY
QM
(Q,W)
repr. in crystal melting
new repr.