4次元チャーン=サイモンズ理論

と　可積分系

山崎雅人　（東大IPMU）

名古屋大学集中講義，2021年4月5日

単位が必要な
履修学生の方のための連絡事項：

レポート：講義内容に関係したことについて考え・調べてNUCTにpdfで提出
（締め切り 8/11 (木曜日)：24時（11日いっぱい））

講義の途中にもいくつか
例題を出す「予定」です

最初のコマは
スライドで四方山話

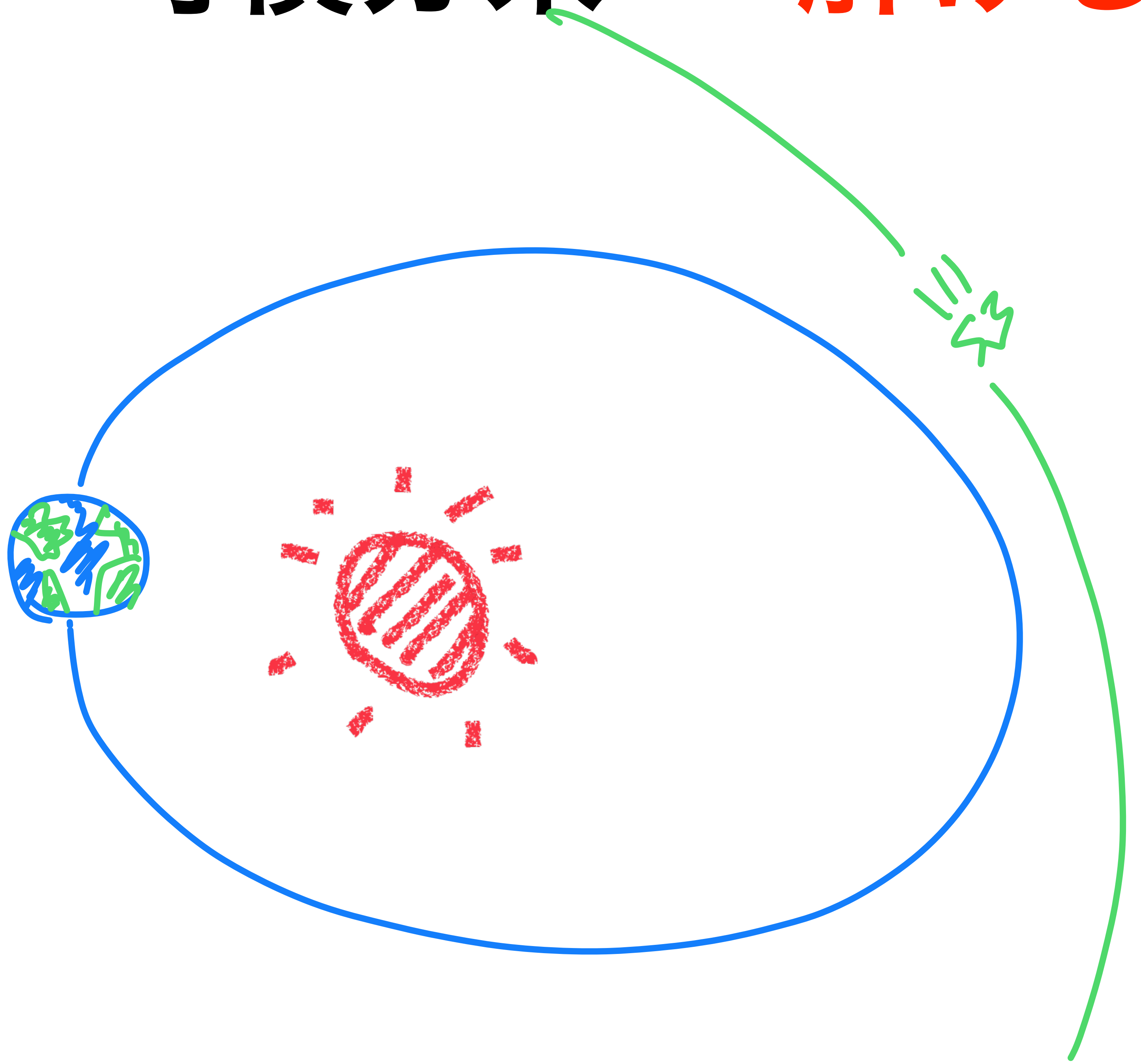
次のコマからは板書です
(blackboards mostly in English from next hour)

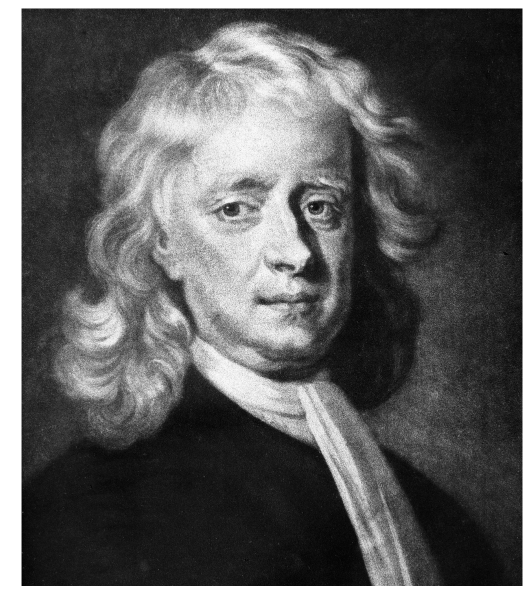
質問は随時受け付けます

可積分系

可積分系：
数学と物理学の交錯する場所

可積分系：”解ける”模型





保存量が自由度の数と同じだけ存在

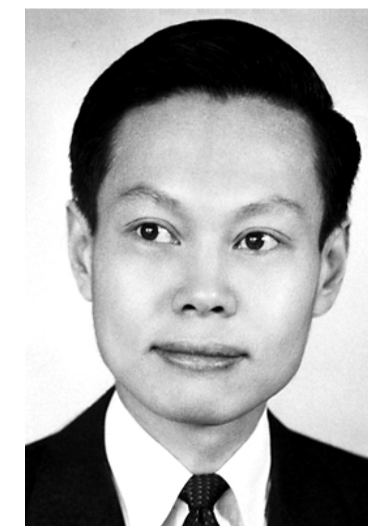
可積分系の一つの特徴づけ：

Yang - Baxter方程式

SOME EXACT RESULTS FOR THE MANY-BODY PROBLEM IN ONE DIMENSION WITH REPULSIVE DELTA-FUNCTION INTERACTION*

C. N. Yang
Institute for Theoretical Physics, State University of New York, Stony Brook, New York

(Received 2 November 1967)



The repulsive δ interaction problem in one dimension for N particles is reduced, through the use of Bethe's hypothesis, to an eigenvalue problem of matrices of the same sizes as the irreducible representations R of the permutation group S_N. For some R's this eigenvalue problem itself is solved by a second use of Bethe's hypothesis, in a generalized form. In particular, the ground-state problem of spin-$\frac{1}{2}$ fermions is reduced to a generalized Fredholm equation.

(1) Consider the one-dimensional N-body problem

$$H = -\sum_{1}^{N} \partial^2/\partial x_i^{\,2} + 2c \sum_{i<j} \delta(x_i - x_j), \quad c > 0, \qquad (1)$$

with no limitation on the symmetry of the wave function ψ. For a given irreducible representation R_ψ of the permutation group S_N of the N coordinates x_i, we want to determine the wave function ψ. Assume Bethe's hypothesis[1] to be valid: Let $p_1, \cdots, p_N$ = a set of unequal

with the aid of the following identities:

$$Y_{ij}^{ab} Y_{ji}^{ab} = 1, \qquad (7)$$

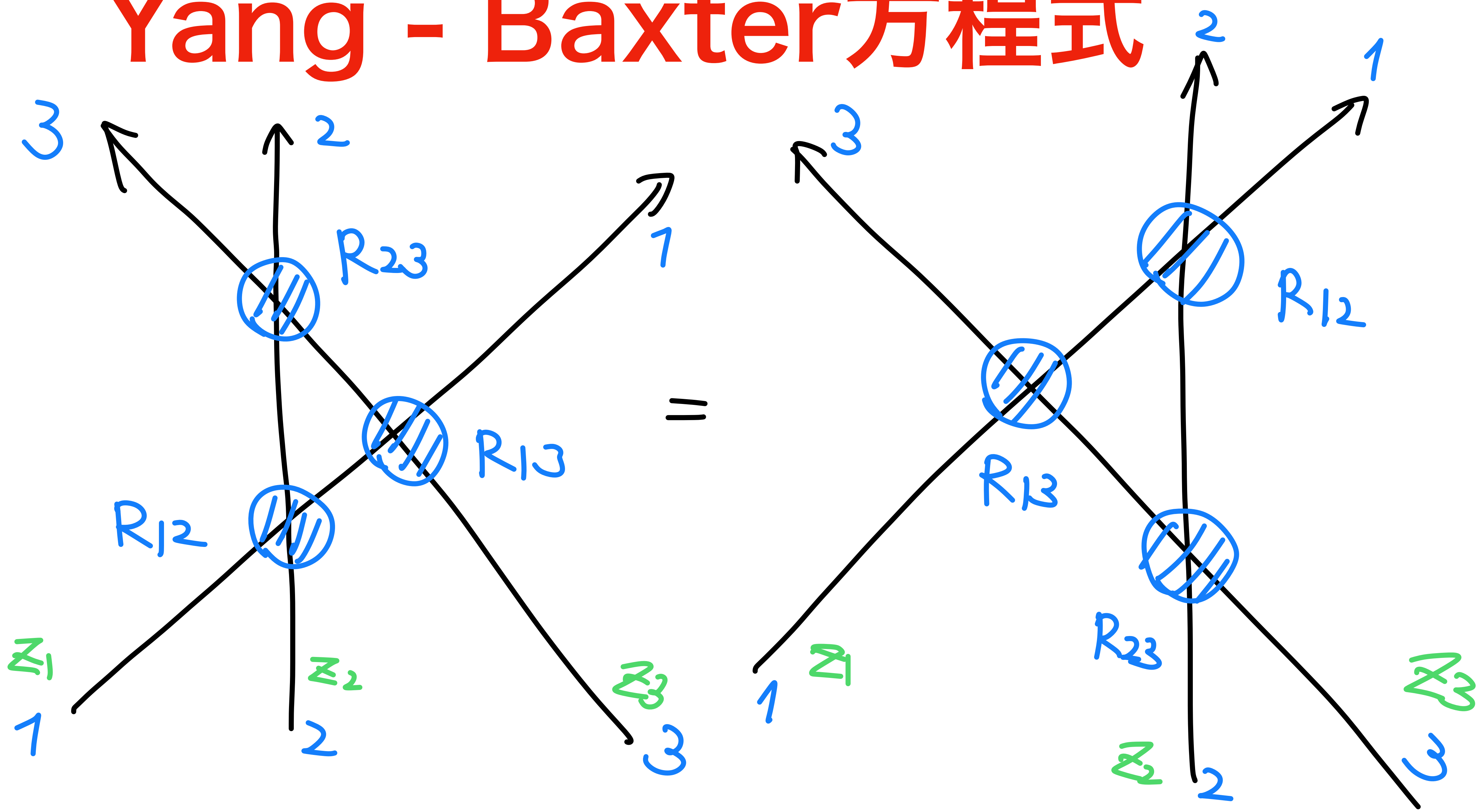
and

$$Y_{jk}^{ab} Y_{ik}^{bc} Y_{ij}^{ab} = Y_{ij}^{bc} Y_{ik}^{ab} Y_{jk}^{bc}, \qquad ($$

which are easily verified. Thus given a set of unequal p's, and $\xi_0 = \xi_P$ for P = identity, all ξ_P's are determined.

(2) The imposition of the periodic boundary

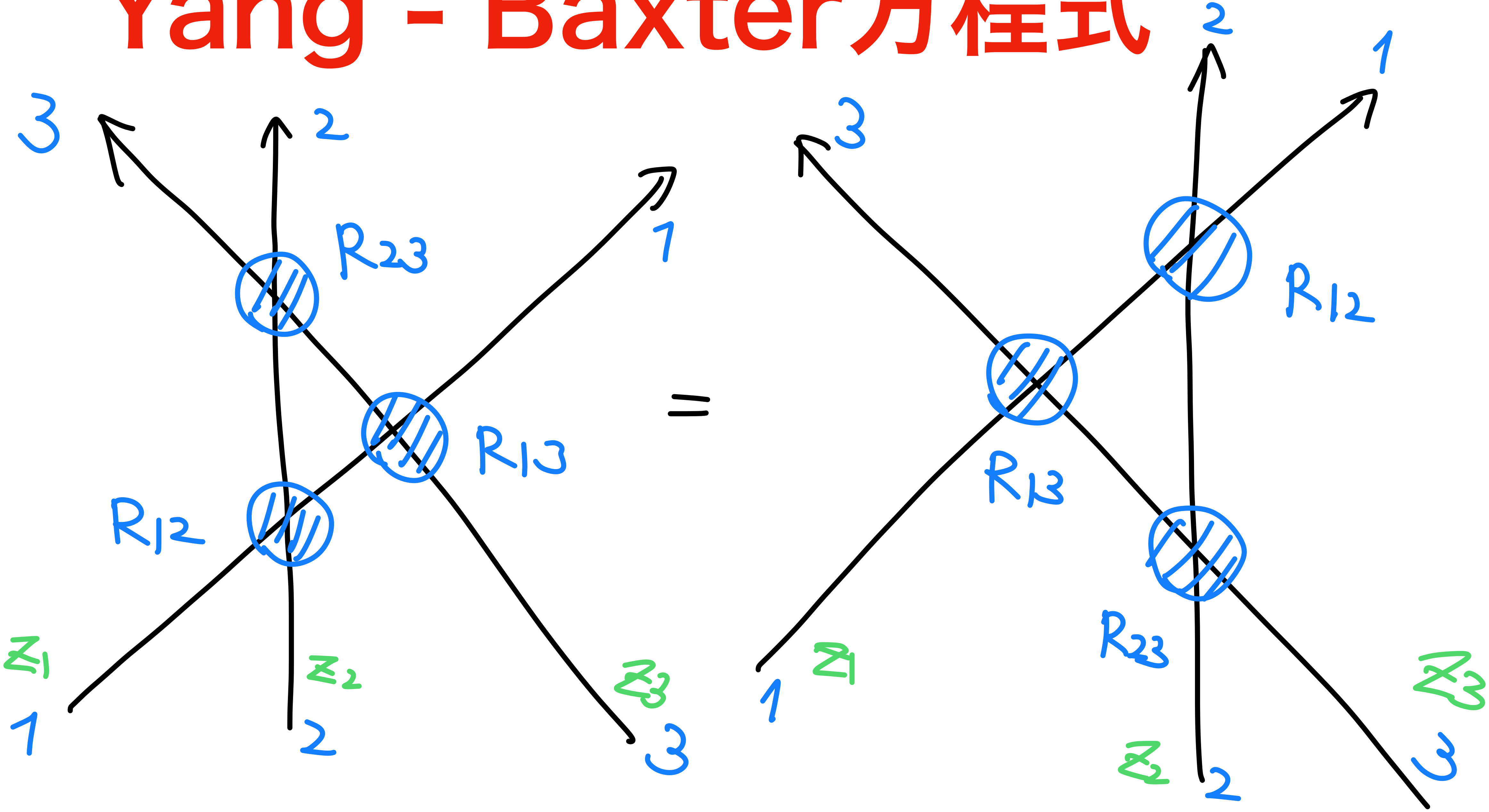
Yang - Baxter方程式



$$R_{ij}(z_i - z_j) \in \mathrm{End}(V_i \otimes V_j)$$

($z_i - z_j$: spectral parameter)

Yang - Baxter方程式



$$R_{23}(z_2-z_3)\,R_{13}(z_1-z_3)\,R_{12}(z_1-z_2)$$

$$= R_{12}(z_1-z_2)\,R_{13}(z_1-z_3)\,R_{23}(z_2-z_3) \in \mathrm{End}(V_1\otimes V_2\otimes V_3)$$

McGuire-Yang：

有理関数解

rational

↑ 極限

6-vertex model：

三角関数解

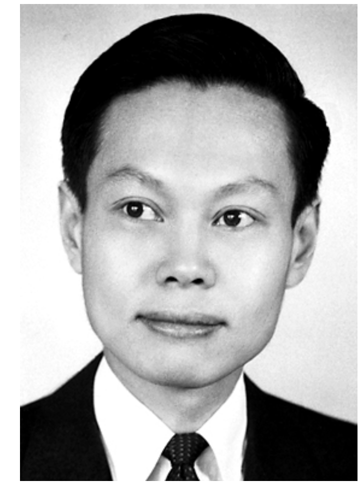
trigonometric

↑ 極限

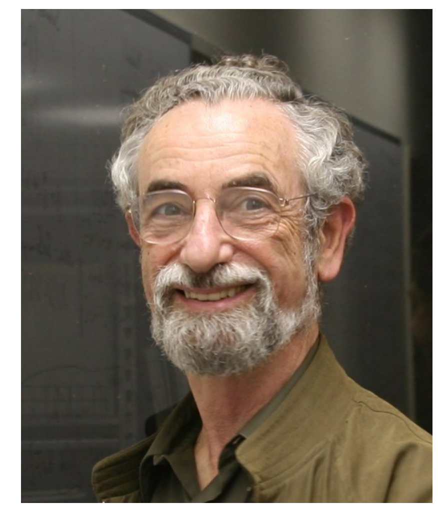
8-vertex model：

楕円関数解

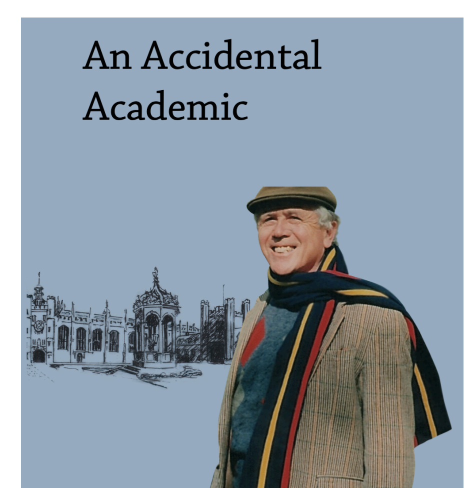
elliptic



E.H. Lieb



R.J. Baxter



可積分系が存在すること
自体が奇跡？

興味深いパターン

例：種数が高いリーマン面に付随した解は

（ほとんど）知られていない

数学 = 論理？

「どうしても自分で証明を丁寧に読んで論証を追跡して見ないと定理が分かった気がしないというのは，証明が単なる論証ではなく，論理以上の何者かが証明の背後に潜んでいるからであろう．」（小平邦彦）

Yang-Baxter方程式の
解であることは直接計算すれば
確認できる

しかし，「自分なりに」
別の方法でスッキリと
理解できないか？

Quantum Physics

[Submitted on 19 Jun 2021 (v1), last revised 25 Jun 2021 (this version, v2)]

Quantum computing 40 years later

John Preskill

Forty years ago, Richard Feynman proposed harnessing quantum physics to build a more powerful kind of computer. Realizing Feynman's vision is one of the grand challenges facing 21st century science and technology. In this article, we'll recall Feynman's contribution that launched the quest for a quantum computer, and assess where the field stands 40 years later.

Comments:	49 pages. To appear in Feynman Lectures on Computation, 2nd edition, published by Taylor & Francis Group, edited by Anthony J. G. Hey. (v2) typos corrected
Subjects:	**Quantum Physics (quant-ph)**
Cite as:	arXiv:2106.10522 **[quant-ph]**
	(or arXiv:2106.10522v2 **[quant-ph]** for this version)

8-vertex model は
ファインマンにも難しかった

In particular, he hoped that tools for solving integrable models might be helpful for treating the soft part of QCD, the physics beyond the reach of renormalization-group improved perturbation theory. He wanted some students to study integrable models with him, to help him learn the subject. Well, he wanted students, and I had students, so we made an arrangement. Feynman and the students met once a week in his office, and those meetings would sometimes last all afternoon; a few times Feynman invited the students to dinner afterward.

Feynman told the students "We gotta know how to solve every problem that has been solved," and he urged them to solve the problems on their own because "What I cannot create I do not understand." To get things started he described the six-vertex model, and told everyone to solve it without looking up any references [85]. That went on for weeks, without notable progress, until Feynman triumphantly unveiled his own solution. The next challenge was the eight-vertex model, but the students never solved that one, and neither did Feynman!

無限次元の対称性：
Yangian

V.G. Drinfeld

Докл. Акад. Наук СССР
Том 283 (1985), № 5

Soviet Math. Dokl.
Vol. 32 (1985), No. 1

HOPF ALGEBRAS AND THE QUANTUM YANG-BAXTER EQUATION

UDC 512.554.3+512.667.7

V. G. DRINFEL'D

Let $\mathfrak{a}$ be a given finite-dimensional simple Lie algebra over $\mathbf{C}$ with a fixed invariant inner product. According to [1], the function $r(u) = u^{-1} I_\mu \otimes I_\mu$, where $\{I_\mu\}$ is an orthonormal basis in $\mathfrak{a}$ (summation over repeated indices is always assumed to be carried out), satisfies the classical Yang-Baxter equation (CYBE). If, in addition, a representation $\rho\colon \mathfrak{a} \to \operatorname{End} V$ is given, the question arises (see [4]) whether the quantum Yang-Baxter equation (QYBE) has a solution which can be written as a formal series

$$R(u, h) = 1 + h \cdot (\rho \otimes \rho)(r(u)) + \sum_{k=2}^{\infty} A_k(u) h^k, \qquad A_k(u) \in \operatorname{End}(V \otimes V).$$

無限次元の対称性：
Yangian

V.G. Drinfeld

THEOREM 2. *The bialgebra* $(\mathfrak{a}[u], \varphi)$ *admits a unique homogeneous quantization* (A, Δ). *The algebra* A *regarded as an associative topological algebra with unity is generated by elements* I_λ *and* J_λ *with defining relations*

$$(2)\qquad [I_\lambda, I + \mu] = c_{\lambda\mu\nu} I_\nu, \qquad [I_\lambda, J_\mu] = c_{\lambda\mu\nu} J_\nu;$$

$$(3)\qquad [J_\lambda, [J_\mu, I_\nu]] - [I_\lambda, [J_\mu, J_\nu]] = h^2 a_{\lambda\mu\nu\alpha\beta\gamma}\{I_\alpha, I_\beta, I_\gamma\};$$

$$(4)\qquad [[J_\lambda, J_\mu], [I_r, J_s]] + [[J_r, J_s], [I_\lambda, J_\mu]] = h^2(a_{\lambda\mu\nu\alpha\beta\gamma} c_{rs\nu} + a_{rs\nu\alpha\beta\gamma} c_{\lambda\mu\nu})\{I_\alpha, I_\beta, J_\gamma\},$$

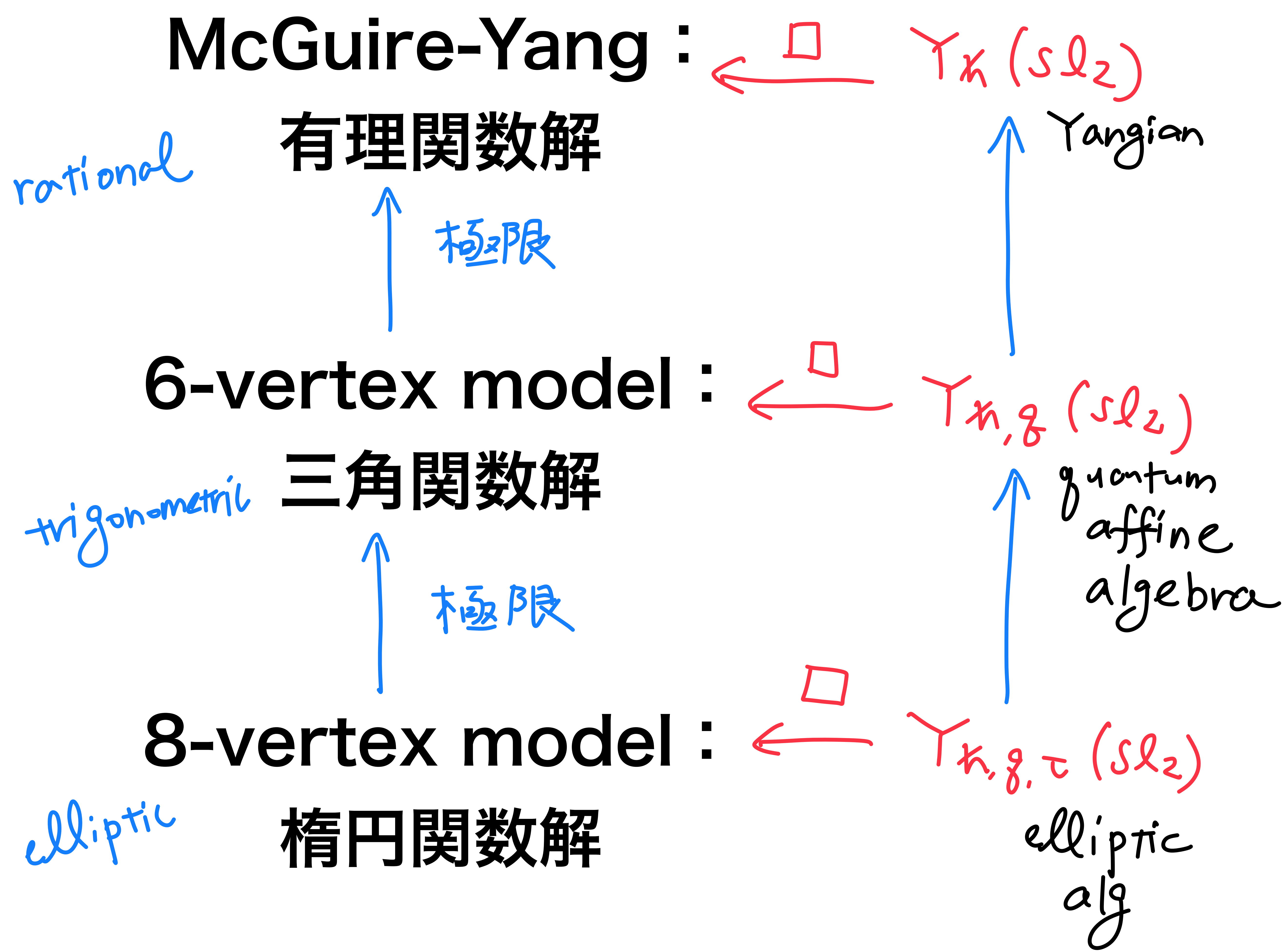
$Y_\hbar(\mathfrak{g})$

where the $c_{\lambda\mu\nu}$ *are the structure constants of* $\mathfrak{a}$, *and*

$$a_{\lambda\mu\nu\alpha\beta\gamma} = \frac{1}{24} c_{\lambda\alpha i} c_{\mu\beta j} c_{\nu\gamma k} c_{ijk}, \qquad \{x_1, x_2, x_3\} = \sum_{i \neq j \neq k} x_i x_j x_k.$$

Here, $\deg I_\lambda = 0$ *and* $\det J_\lambda = 1$. *Moreover,*

$$(5)\qquad \Delta(I_\lambda) = I_\lambda \otimes 1 + 1 \otimes I_\lambda, \qquad \Delta(J_\lambda) = J_\lambda \otimes 1 + 1 \otimes J_\lambda + \tfrac{1}{2} h c_{\lambda\mu\nu} I_\nu \otimes I_\mu.$$


McGuire-Yang：
有理関数解
rational
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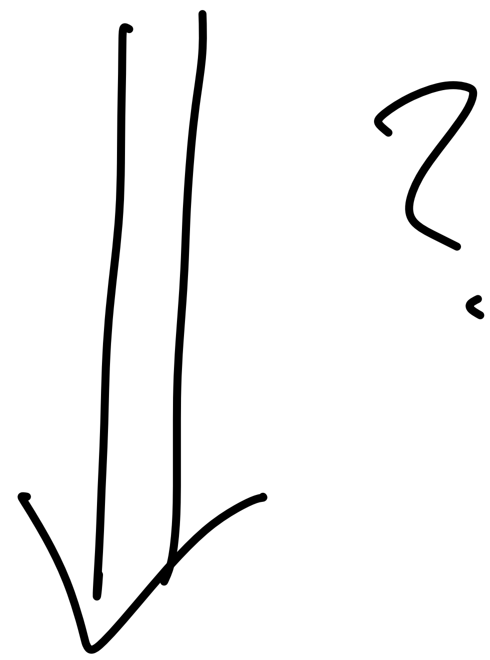
6-vertex model：
三角関数解
trigonometric
極限
8-vertex model：
楕円関数解
elliptic
□
$Y_\hbar(\mathfrak{sl}_2)$
Yangian
□
$Y_{\hbar,q}(\mathfrak{sl}_2)$
quantum affine algebra
□
$Y_{\hbar,q,\tau}(\mathfrak{sl}_2)$
elliptic alg

場の理論



可積分系

Commun. Math. Phys. 121, 351–399 (1989)

Communications in
Mathematical
Physics

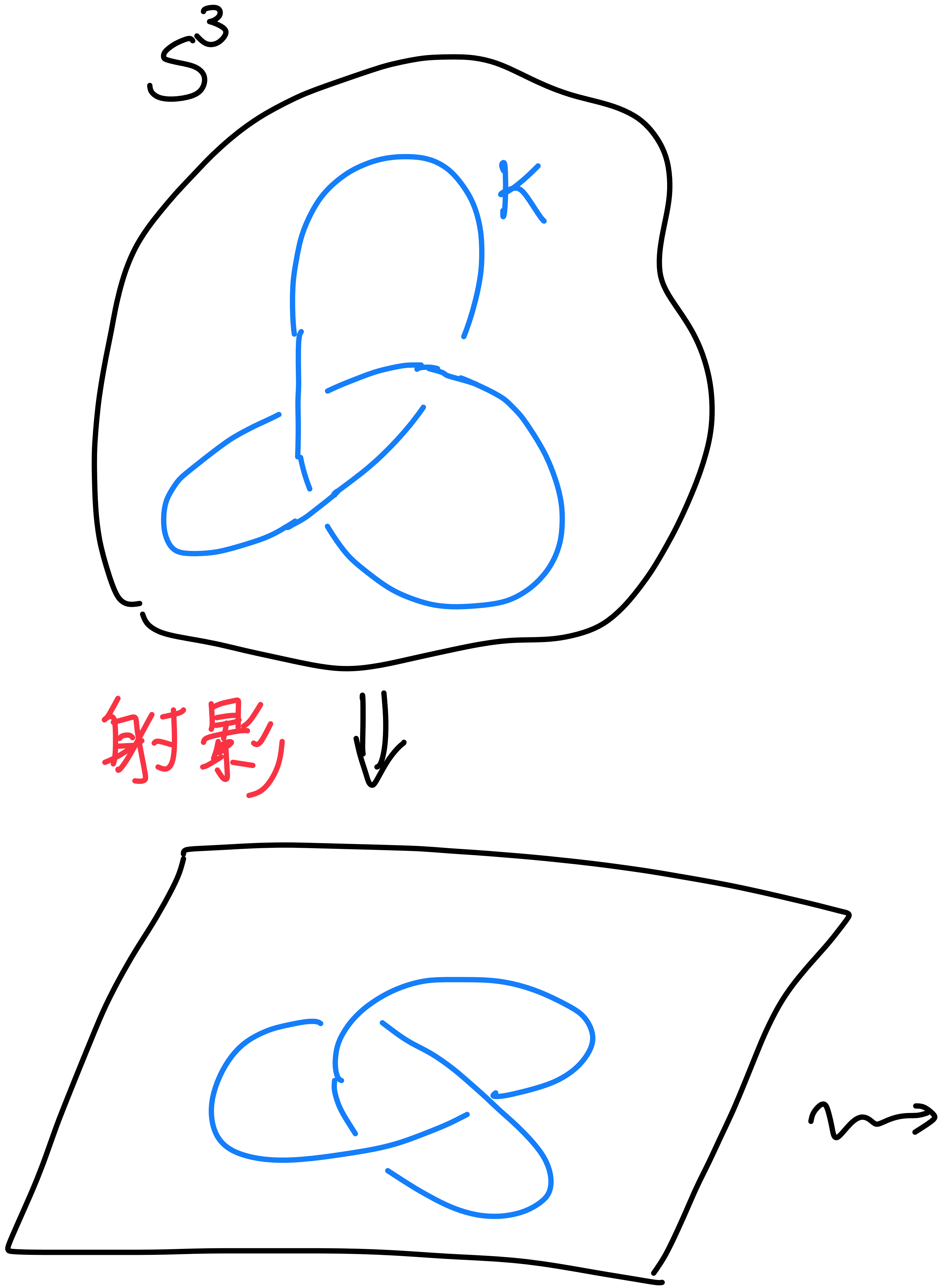
お手本：Wittenによる
Jones多項式の説明

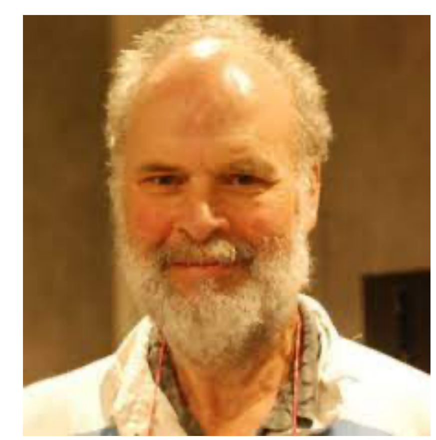
Quantum Field Theory and the Jones Polynomial *

Edward Witten **

School of Natural Sciences, Institute for Advanced Study, Olden Lane, Princeton,
NJ 08540, USA

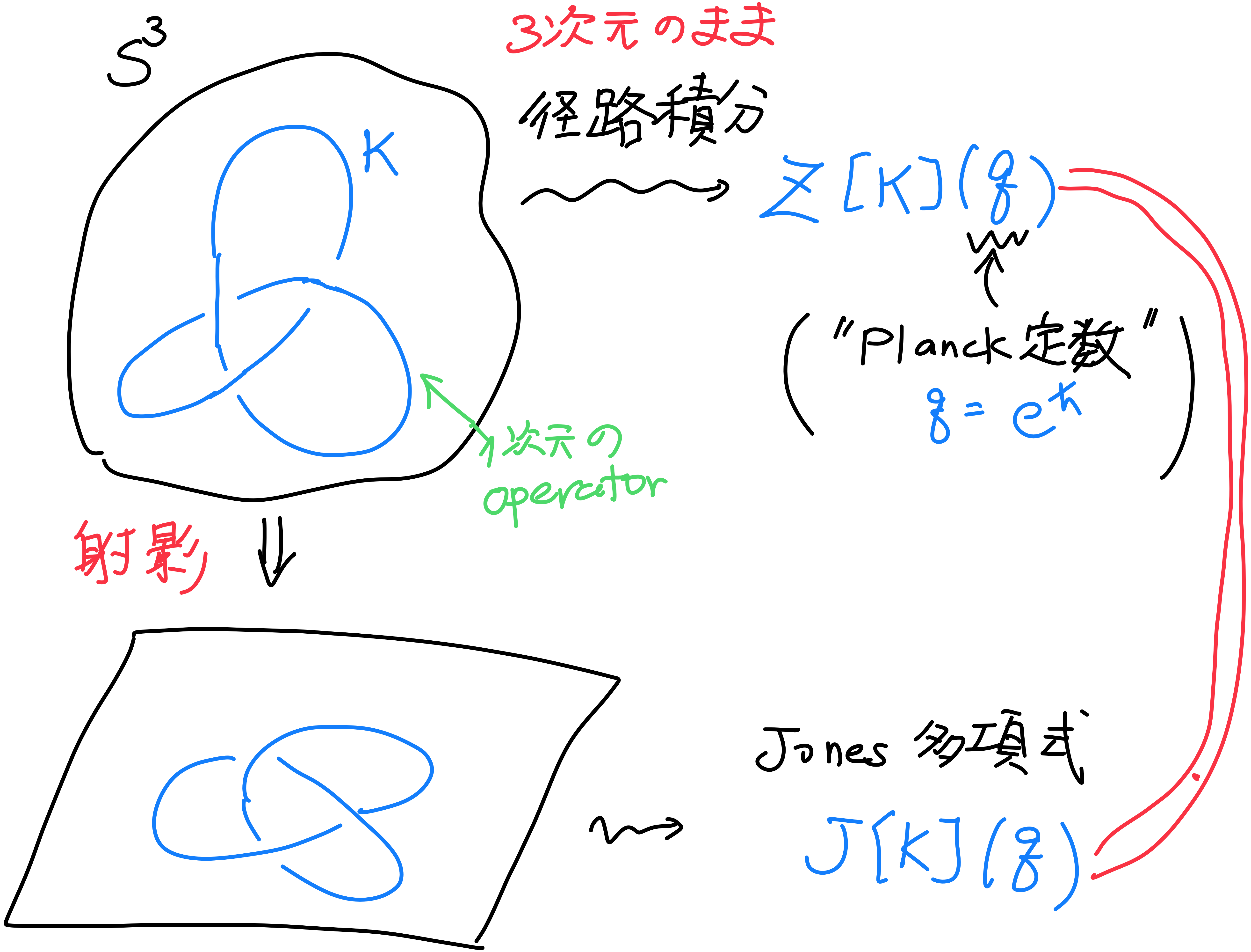
Abstract. It is shown that 2 + 1 dimensional quantum Yang-Mills theory, with an action consisting purely of the Chern-Simons term, is exactly soluble and gives a natural framework for understanding the Jones polynomial of knot theory in three dimensional terms. In this version, the Jones polynomial can be generalized from S^3 to arbitrary three manifolds, giving invariants of three manifolds that are computable from a surgery presentation. These results shed a surprising new light on conformal field theory in 1 + 1 dimensions.





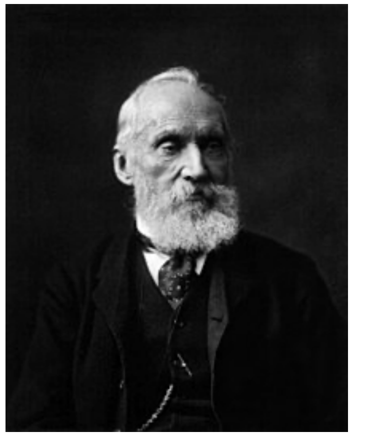
Jones 多項式

$J[K](q)$



S^3
K
3次元のまま
経路積分
Z[K](q)
("Planck定数"
q = e^ħ)
次元の
operator
射影
Jones 多項式
J[K](q)

場の理論と結び目



山 崎　雅 人

1. 結び目理論のおこり

本特集号の取りまとめ役である村上先生からは，「場の理論と結び目理論」についての記事を依頼された．**場の理論**とは（少なくとも完全には）まだ厳密な数学にはなっていない物理学の理論のことである．結び目理論の様々な側面を扱ってきた本特集号ではあるが，純粋数学としての結び目理論に親しんできた読者にとっては，結び目理論が物理学の理論に応用を持つことは驚きかもしれない．

しかし，実は歴史を辿ると，そもそも結び目理論が生まれてきた背景には，物理学からの強い動機があったのだ．

時代は遡ること 19 世紀の後半，1860 年頃のことである．当時スコットランドでそして世界で最も著名な物理学者の一人がウィリアム＝トムソン（1824–1907，図 1 左），後のケルビン卿であった．

当時の物理学においては光は波であるとの波動説が有力であったが，それならば光が伝搬するための媒質が必要となると考えられる．この媒質をエーテルと呼び，その流体としての性質が活発に研究されていた．

一方，当時の科学では既に 19 世紀初頭にドルトン（1766–1844）によって原子論が提唱されており，それ以上分解できない最小単位としての原子の存在の証拠が集まっていた．しかし仮に原子が存在するとしても，なぜそれが安定なのかは謎であった．

そこでケルビン卿は，エーテルという流体の中にある渦の作る結び目が原子でないかと考えた[1)]．結び目は連続変形で保たれるので，原子の安定性をうまく説明できる可能性がある．エーテルという波動論から原子という粒子を導く点において，この理論はいわば波動と粒子の統一理論とでもいう

http://member.ipmu.jp/masahito.yamazaki/files/2020/2020_mathphys_EM.pdf

对称性：量子群

Ribbon Graphs and Their Invariants Derived from Quantum Groups

N. Yu. Reshetikhin and V. G. Turaev
L.O.M.I., Fontanka 27, SU-191011 Leningrad, USSR

Abstract. The generalization of Jones polynomial of links to the case of graphs in R^3 is presented. It is constructed as the functor from the category of graphs to the category of representations of the quantum group.

1. Introduction

The present paper is intended to generalize the Jones polynomial of links and the related Jones-Conway and Kauffman polynomials to the case of graphs in R^3.

Originally the Jones polynomial was defined for links of circles in R^3 via an astonishing use of von Neumann algebras (see [Jo]). Later on it was understood that this and related polynomials may be constructed using the quantum R-matrices (see, for instance, [Tu_1]). This approach enables one to construct similar invariants for coloured links, i.e. links each of whose components is

Nuclear Physics B330 (1990) 285–346
North-Holland

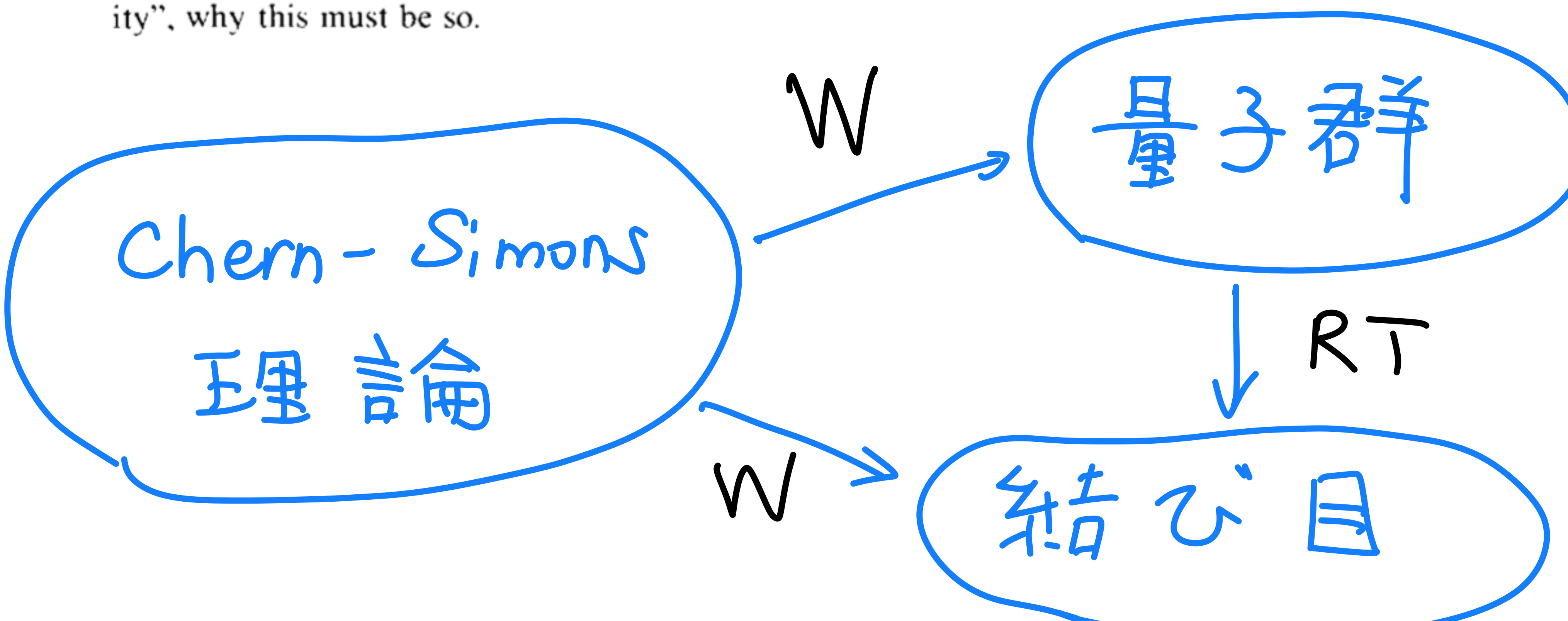
GAUGE THEORIES, VERTEX MODELS, AND QUANTUM GROUPS

Edward WITTEN*

School of Natural Sciences, Institute for Advanced Study, Olden Lane, Princeton, NJ 08540, USA

Received 7 June 1989

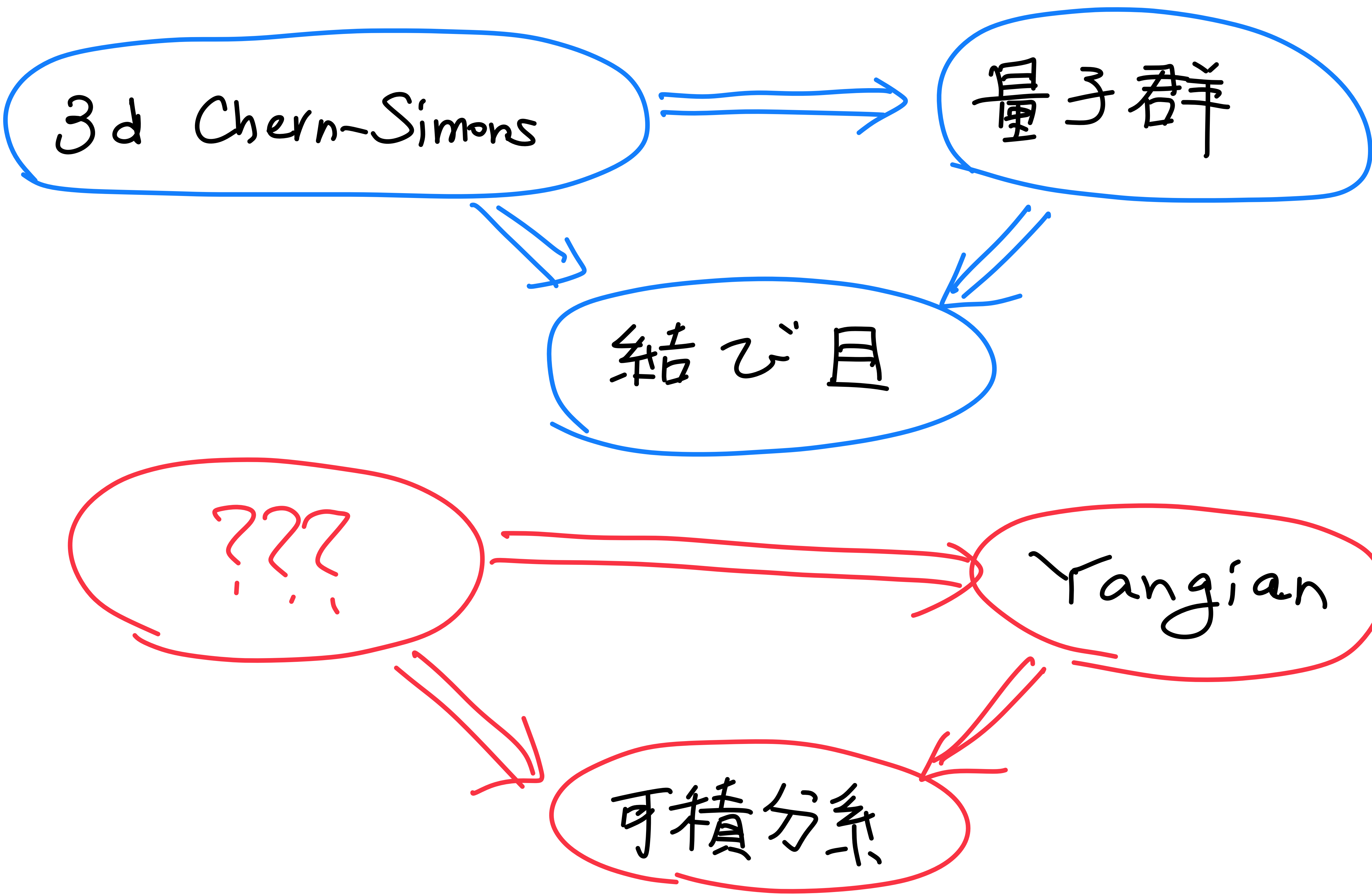
It is known that the Jones polynomial of knot theory, and its generalizations, are closely related to the integrable "vertex models" of two-dimensional statistical mechanics, and to quantum groups. In this paper, an attempt is made to show on a priori grounds, starting only from general covariance of three-dimensional Chern–Simons gauge theory and two-dimensional "duality", why this must be so.



There are several obvious areas for further investigation. In terms of statistical mechanics, one compelling question is to understand the origin of the spectral parameter (and the elliptic modulus) in IRF and vertex models; this is essential for explaining the origin of integrability. Another question, which may or may not be related, is to understand the spin models formulated only rather recently [24] in which the spectral parameter is not an abelian variable (as in previous construc-

Wittenは可積分系の説明には成功しなかった

長年の謎



EXACTLY SOLVABLE MODELS AND KNOT THEORY

Miki WADATI and Tetsuo DEGUCHI

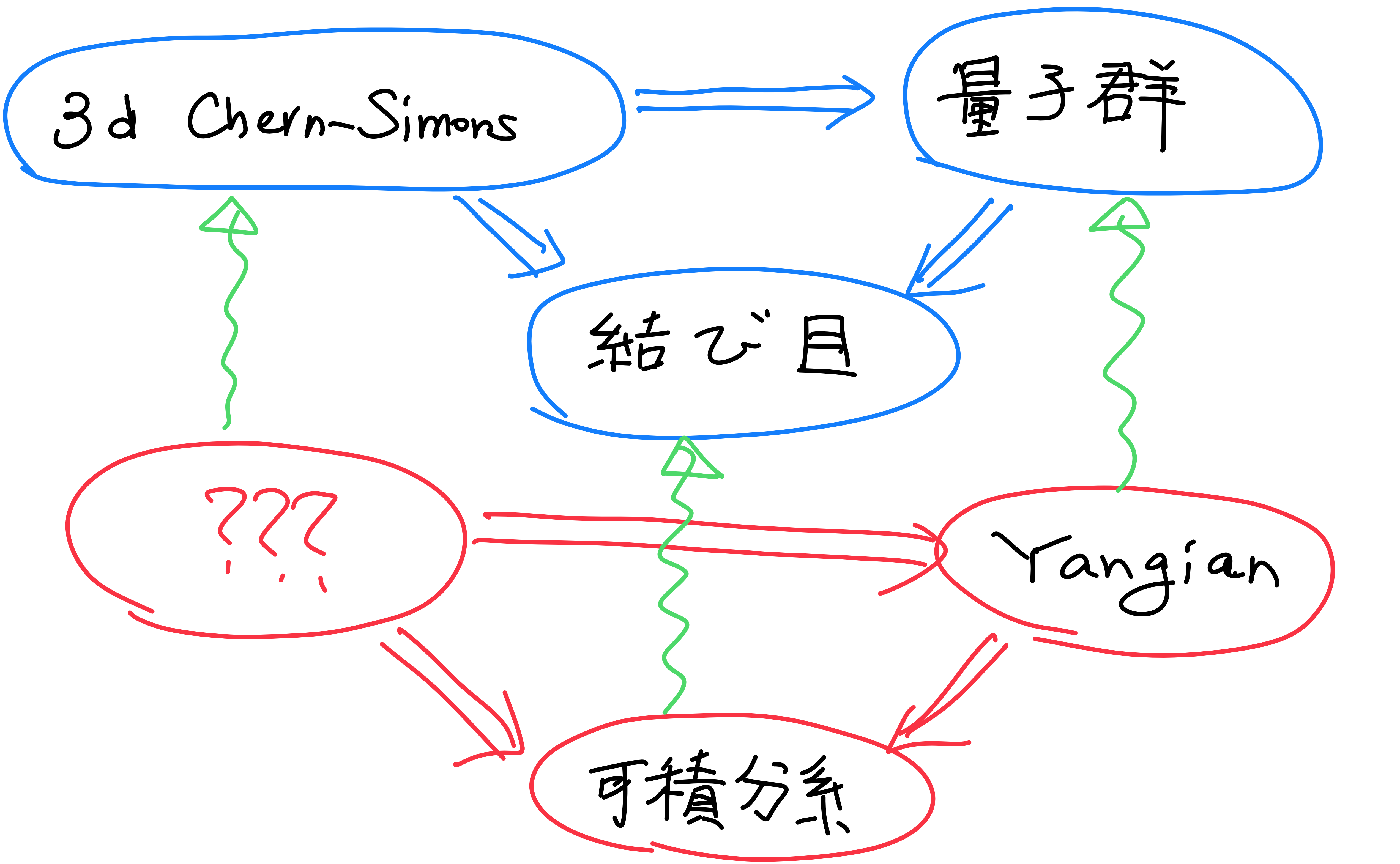
Institute of Physics, College of Arts and Sciences, University of Tokyo, Komaba, Meguro-ku, Tokyo, Japan 153

and

Yasuhiro AKUTSU

Institute of Physics, Kanagawa University, Rokkakubashi, Kanagawa-ku, Yokohama, Kanagawa, Japan 221

長年の謎



1990年数学セミナーでのインタビュー

うです(笑)．勇気づけられています．また，ここ2~3年私がとりつかれている問題は可解格子模型におけるいわゆる spectral parameter の問題です．Chern-Simons 理論における Wilson line の期待値はいろいろな仕方で計算できますが，可解格子模型を用いる方法——Jones が初めて試みたわけですが——では spectral parameter を無限大にとばしてしまいます．しかし可解模型をより深く理解するには spectral parameter が本質的に重要です．したがって，もし我々が Chern-Simons 理論のアプローチを信ずるならば，spectral parameter を含むような一般化が必要です．また，次のような問題——解がないかも知れませ

2014

Round Table Talk：
エドワード・ウィッテン博士に聞く

エドワード・ウィッテン Edward Witten
プリンストン高等研究所教授

大栗 博司 おおぐり・ひろし
Kavli IPMU 主任研究員

戸田 幸伸 とだ・ゆきのぶ
Kavli IPMU准教授

山崎 雅人 やまざき・まさひと
Kavli IPMU助教


右から左へ：ウィッテンさん、大栗さん、山崎さん、戸田さん

京都賞と4度目の京都訪問

大栗　京都賞受賞おめでとうございます。京都賞基礎科学部門では、4年に1回、数理科学の分野に授賞されますが、この分野で物理学者への授賞は今回が初めてでした。

ウィッテン　この賞をいただいたことを、私は本当に光栄に思っています。

私自身も高校生のときに愛読し、今でも定期購読しています。立川裕二さんは、1994年のあなたのインタビューを読んだことが、この分野に進む動機のひとつとなったと言っています。

ウィッテン　立川さんから親切にそう言ったいただいて、とても嬉しいです。

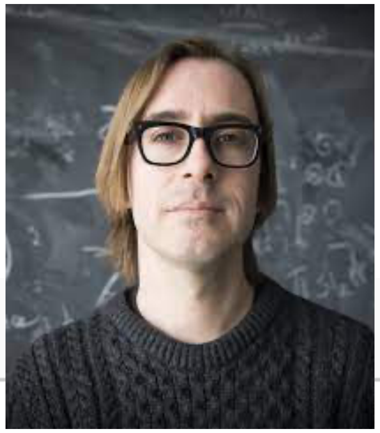
大栗　今日の座談会についての記事も、次世代の若い学生を刺激し、数学に限らず、科学や工学の分野に興味を持ってもらう役に立てばと思っています。この分野の現在の状況と将来の展望をするよい機会としたいです。

あなたは、すでに『数学セミナー』誌上で、2回インタビューを受けられていますね。1990年に京都で国際数学者会議が開かれ、フィールズ賞を受賞なさった際には、江口 徹さんがインタビューをなさっています。同じ号には、同じくフィールズ賞を受賞されたボン・ジョーンズさんとの対談も掲載されています。その対談では、あなたのチャーン・サイモンズ理論に、スペクトル変数を持たせる拡張に興味があるとおっしゃっていました。これは、可解模型の見地からは自然なことですね。

ウィッテン　はい、イジング模型のような2次元格子模型の厳密解を得るときに使う可積分性と同じような筋の説明を見つけ出したいということでした。私は全く成功しなかったのですが、ついこの2、3年の間に、ケヴィン・コステロが、私がこうしたいと思っていた趣旨に沿ったことをなし遂げてくれました。

大栗　ちょうどいらっしゃる前に、コステロの仕事についてお話をしていたところでした。この仕事は、あなたが望んでいらしたことを達成したとお考えになりますか。

ウィッテン　はい、可解模型にはいろいろな側面があって、一つの方法で全てを理解することはできません。しか



High Energy Physics - Theory

arXiv:1303.2632 (hep-th)

[Submitted on 11 Mar 2013 (v1), last revised 3 Apr 2013 (this version, v2)]

Supersymmetric gauge theory and the Yangian

Kevin Costello

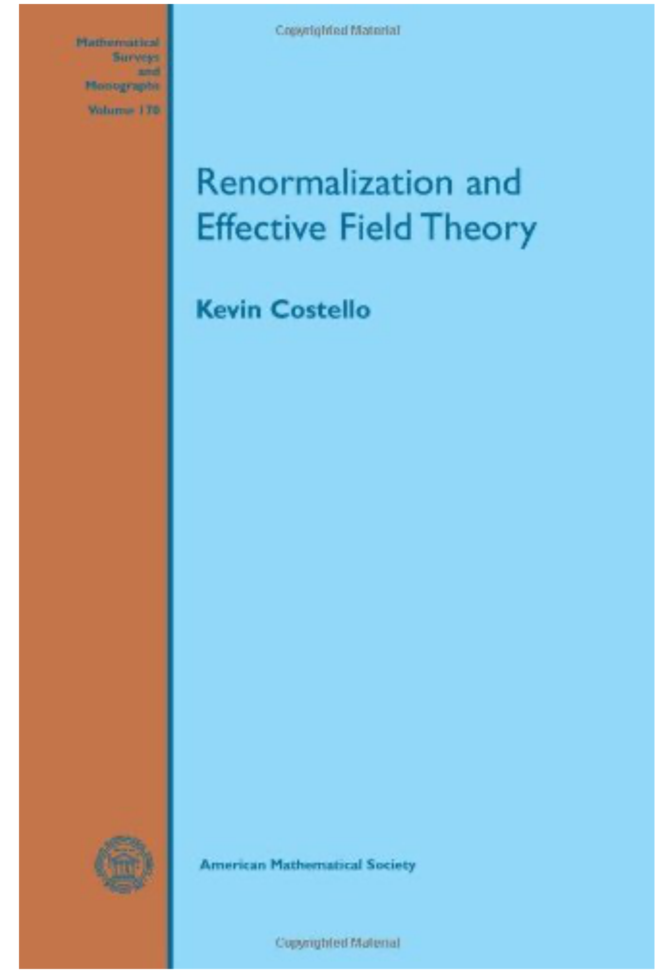
Download PDF

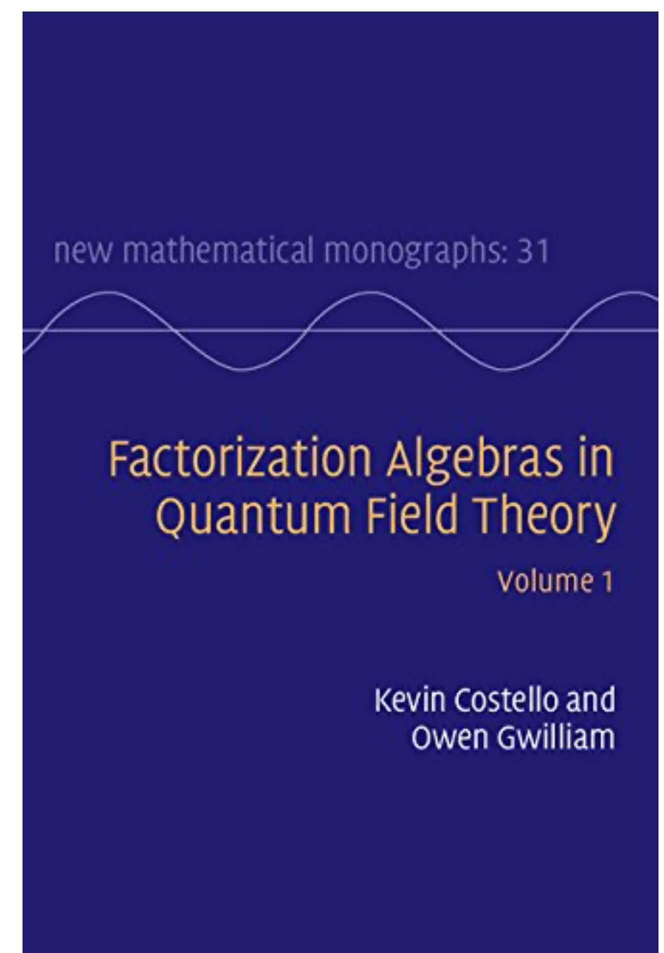
This paper develops a new connection between supersymmetric gauge theories and the Yangian. I show that a twisted, deformed version of the pure N=1 supersymmetric gauge theory is controlled by the Yangian, in the same way that Chern-Simons theory is controlled by the quantum group. This result is used to give an exact calculation, in perturbation theory, of the expectation value of a certain net of n+m Wilson operators in the deformed N=1 gauge theory. This expectation value coincides with the partition function of a spin-chain integrable lattice model on an n-by-m doubly-periodic lattice.

Kevin Joseph Costello

Citation

The Leonard Eisenbud Prize for Mathematics and Physics is awarded to Kevin Costello for his contributions to the mathematical foundations of quantum field theory and his gauge-theoretic explanation of solutions to the quantum Yang–Baxter equations. These have appeared in the works *Factorization Algebras in Quantum Field Theory I*, published by Cambridge University Press in 2017, *Renormalization and Effective Field Theory*, published by the American Mathematical Society in 2011, and the influential article posted on the arXiv: 1303.2632, "Supersymmetric Gauge Theory and the Yangian."





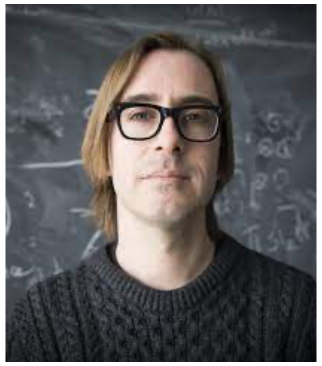
Costello氏の2013年の論文は
まだ「ブラックボックス」
を含んでいた

しかし，既に場の理論はあるのだから，
それを標準的な場の理論の方法で調べれば，
自動的に可積分性が従うはず

arXiv:1709.09993 (hep-th)

[Submitted on 28 Sep 2017 (v1), last revised 4 Feb 2018 (this version, v2)]

Gauge Theory and Integrability, I



Kevin Costello, Edward Witten, Masahito Yamazaki

Several years ago, it was proposed that the usual solutions of the Yang-Baxter equation associated to Lie groups can be deduced in a systematic way from four-dimensional gauge theory. In the present paper, we extend this picture, fill in many details, and present the arguments in a concrete and down-to-earth way. Many interesting effects, including the leading nontrivial contributions to the R-matrix, the operator product expansion of line operators, the framing anomaly, and the quantum deformation that leads from $\mathfrak{g}[[z]]$ to the Yangian, are computed explicitly via Feynman diagrams. We explain how rational, trigonometric, and elliptic solutions of the Yang-Baxter equation arise in this framework, along with a generalization that is known as the dynamical Yang-Baxter equation.

Comments:	141 pp
Subjects:	**High Energy Physics - Theory (hep-th)**; Statistical Mechanics (cond-mat.stat-mech); Quantum Algebra (math.QA)
Journal reference:	ICCM Not. 6, 46-119 (2018)

High Energy Physics – Theory

[Submitted on 5 Feb 2018]

Gauge Theory and Integrability, II

Kevin Costello, Edward Witten, Masahito Yamazaki

Starting with a four-dimensional gauge theory approach to rational, elliptic, and trigonometric solutions of the Yang-Baxter equation, we determine the corresponding quantum group deformations to all orders in $\hbar$ by deducing their RTT presentations. The arguments we give are a mix of familiar ones with reasoning that is more transparent from the four-dimensional gauge theory point of view. The arguments apply most directly for $\mathfrak{gl}_N$ and can be extended to all simple Lie algebras other than $\mathfrak{e}_8$ by taking into account the self-duality of some representations, the framing anomaly for Wilson operators, and the existence of quantum vertices at which several Wilson operators can end.

Comments: 51 pp

Subjects: **High Energy Physics - Theory (hep-th)**; Statistical Mechanics (cond-mat.stat-mech); Quantum Algebra (math.QA)

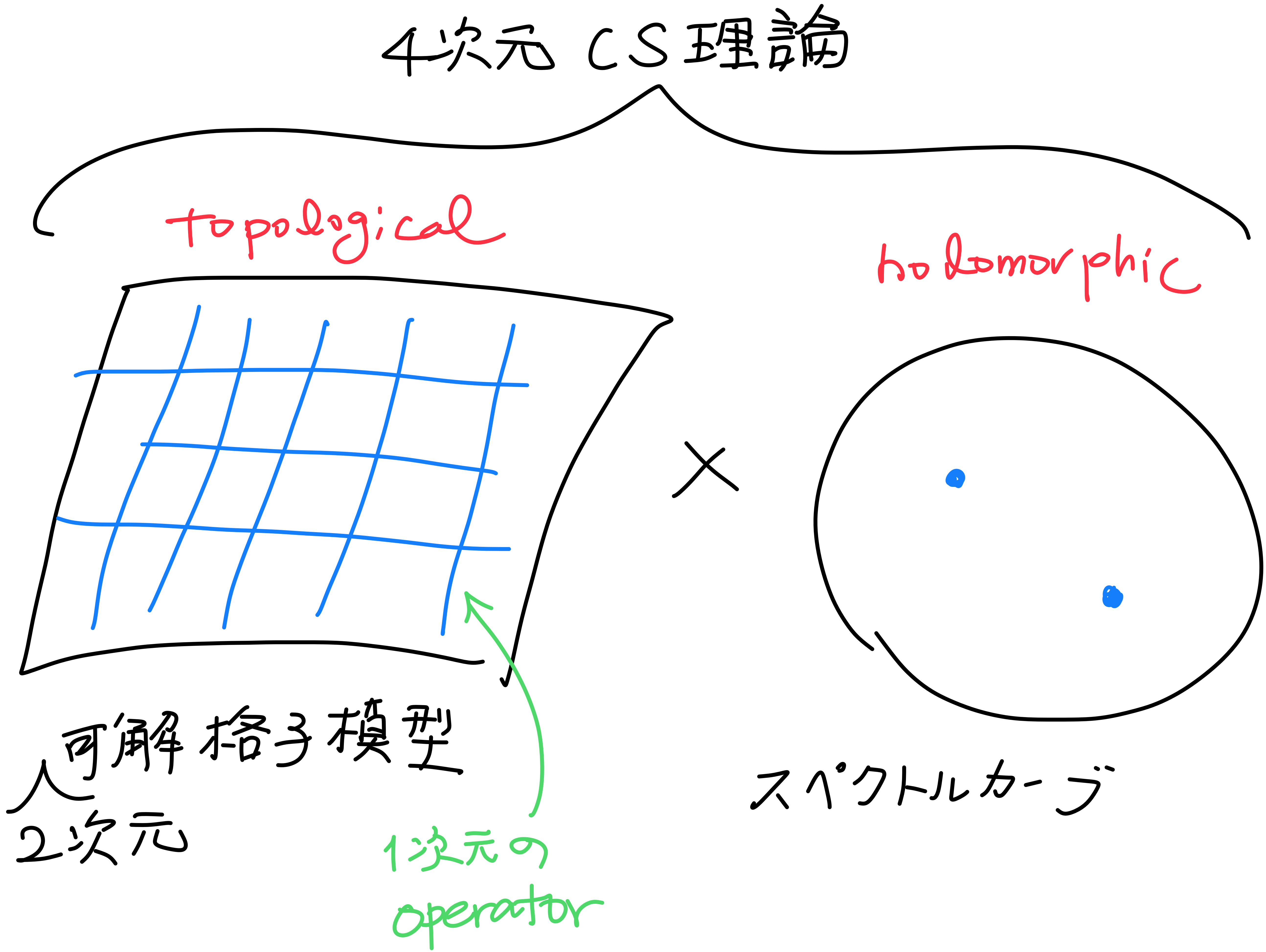
Journal reference: ICCM Not. 6, 120-146 (2018)

鍵：3次元のtopological invarianceを諦める

4次元理論で,

２次元部分：topological

残りの２次元：holomorphic


4次元 CS理論
topological
holomorphic
×
可解格子模型
2次元
1次元の
operator
スペクトルカーブ

Holomorphic Quantum Field Theories (Postponed)

25-29 May 2020
Kavli IPMU, Kashiwa, Japan
Asia/Tokyo timezone

- Overview
- Registration
- Participant List
- Timetable
- Accommodation
- Access
- Visa
- Link
- Measures against coronavirus outbreak
- Kavli IPMU Code of Conduct

Due to the current worldwide situation with the coronavirus, this conference is postponed to sometime in the future.

Dates: May 25-29, 2020

Venue: Lecture Hall, Kavli IPMU, Kashiwa, Japan

Overview:

Holomorphic quantum field theories (HQFT) are c
such as correlation functions, being holomorphic
dimensional analogs of chiral conformal field the
topological eld theories (where the correlators are
in the topology of smooth manifolds, HQFTs are c
instance, anomalies in such theories are related t
manifolds. An early example of a HQFT was the K

Invited speakers:
Mykola Dedushenko (Simons Center, Stony Brook)
Richard Eager (KIAS)
Chris Elliott (UMass Amherst)
Boris Feigin (HSE Moscow and RIMS)
John Francis (Northwestern)
Dennis Gaitsgory (Harvard)
Benjamin Hennion (Paris-Orsay)
Olaf Hohm (Berlin)
Anton Khoroshkin (HSE)
Si Li (Tsinhua)
Natalie Paquette (Caltech)
Nick Rozenblyum (Chicago)
Ingmar Saberi (Heldelberg)
Pavel Safronov (Zurich)
Matt Szczesny (Boston)
Yuji Tachikawa (IPMU)
Brian Williams (Northeastern)
Philsang Yoo (Yale)
Barton Zwiebach* (MIT)
(*: to be confirmed)

High Energy Physics - Theory

arXiv:1908.02289 (hep-th)

[Submitted on 6 Aug 2019]

Gauge Theory And Integrability, III

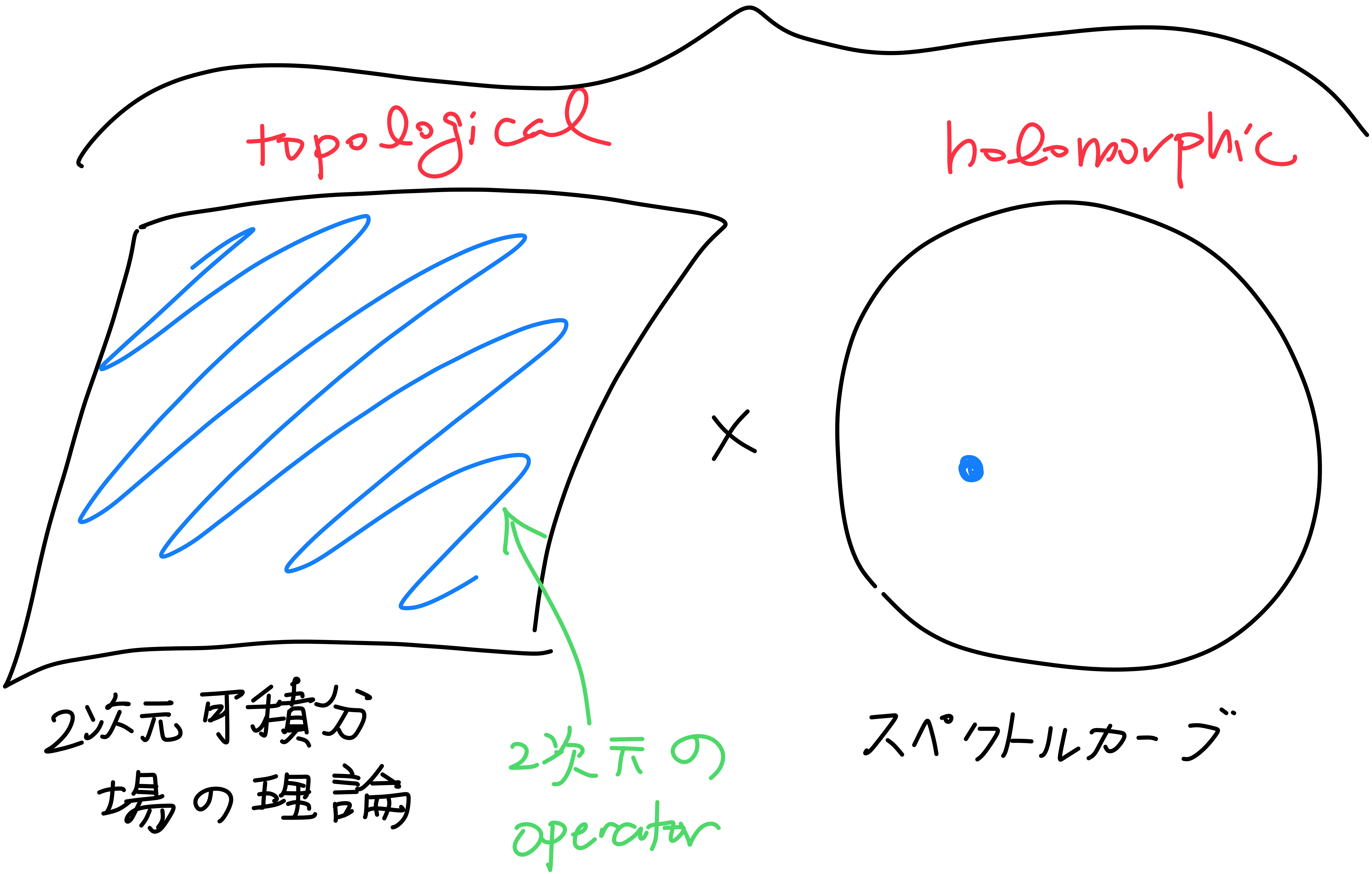
Kevin Costello, Masahito Yamazaki

Download PDF

We study two-dimensional integrable field theories from the viewpoint of the four-dimensional Chern-Simons-type gauge theory introduced recently. The integrable field theories are realized as effective theories for the four-dimensional theory coupled with two-dimensional surface defects, and we can systematically compute their Lagrangians and the Lax operators satisfying the zero-curvature condition. Our construction includes many known integrable field theories, such as Gross-Neveu models, principal chiral models with Wess-Zumino terms and symmetric-space coset sigma models. Moreover we obtain various generalization these models in a number of different directions, such as trigonometric/elliptic deformations, multi-defect generalizations and models associated with higher-genus spectral curves, many of which seem to be new.

Comments: 108 pages, 19 figures

4次元CS理論


topological
holomorphic
×
2次元可積分
場の理論
2次元の
operator
スペクトルカーブ

次の講義からは，板書でこれらの
内容をゆっくり（?）講義します

進度についてのフィードバックは
歓迎します