Lec 4

2021 / Jul / 8

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Integrable Lattice Model ~ conceptual Lec 2
computational Lec 3 + 4

Wilson line
(1d defect)

Lec 4' : factorization algebra

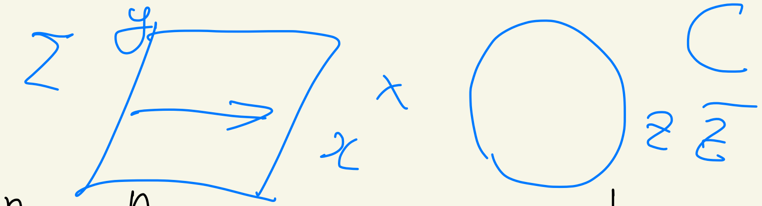
today

Integrable Field Theory ~ Lec 5

Surface defect
(2d defect) ← new!

Twistor, 4D SDYM, hol. CS ~ Lec 6

* Wilson loop



$$W_{\hat R} = \mathrm{Tr}_{\hat R}\, P\exp\left(\int_{\gamma \text{ along } x} \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{n!}\, \partial_z^n A_x(x, y, z, \bar z)\right)\Bigg|_{\substack{z=z_0\\ \bar z=\bar z_0}}$$

Rep $\mathcal{U}(\mathfrak{g}[[z-z_0]])$

Q : $\partial_z g$ when gauge transformation $A \to g^{-1}Ag + g^{-1}dg$?

$$A = A_x dx + A_y dy + A_{\bar z} d\bar z + \cancel{A_z dz}$$

gauge sym,

$$A \to g^{-1}Ag + g^{-1}dg$$

new gauge sym

$$A \to A + x\,dz$$

Gauge Theory $\Rightarrow$ Spectral Parameter in Integrable Model

($\hbar$-expansion)

4d CS $\rightsquigarrow$ quasi-classical R-matrix

6v, 8v, ...

Gauge / YBE $\rightsquigarrow$ non-quasi-classical R-matrix

(review: 1808.04374) (Star-triangle / star-star rel.)

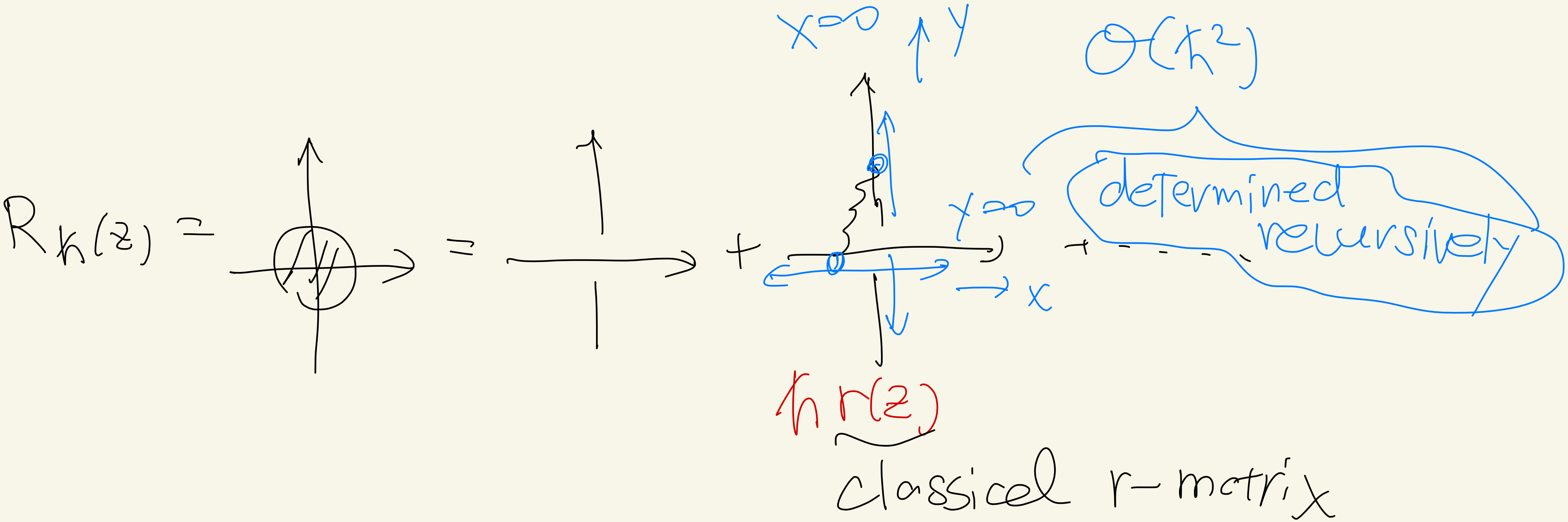
root of unity → chiral Potts Kashiwara-Miwa

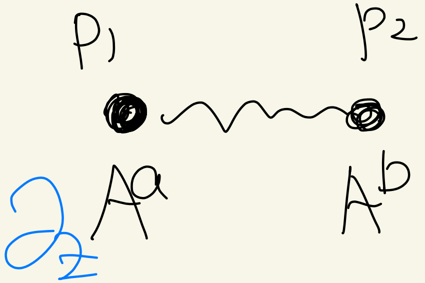
Super-master solution ...

new

~~Yesterday~~

The day before





$$\Rightarrow \partial_z \overset{p_1 - p_2}{P(\overparen{p_1, p_2})}\, \delta^a{}_b \qquad \frac{\sqrt{-1}}{2\pi}\, dz \wedge dP(p_1,p_2) = \delta_{p_1 = p_2}$$

$$P_{(x,y,z,\bar z)} = \frac{1}{2\pi}\, \frac{x\, dy \wedge d\bar z + y\, d\bar z \wedge dx + 2\bar z\, dx \wedge dy}{(x^2+y^2+|z|^2)^2}$$

$$\left(ds^2 = dx^2 + dy^2 + dz\, d\bar z \right)$$

level 1 $\partial_z A^a$

level 0

$$\Rightarrow (t^a_R)_{ij}$$

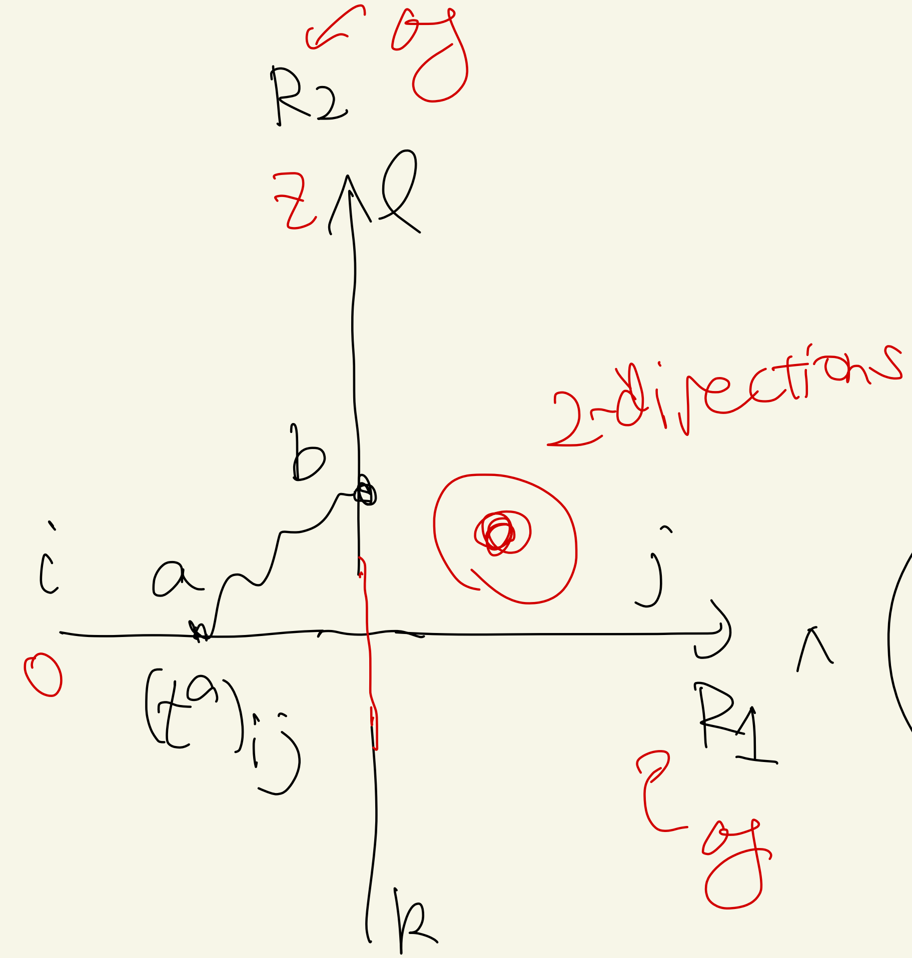
$$g = \{t^a\}_{a \in I} \qquad [t^a, t^b] = i f^{ab}{}_c\, t^c$$

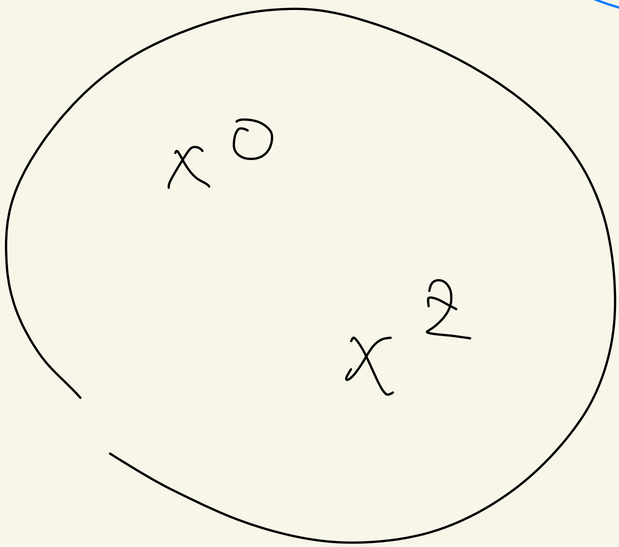
$$\hbar\, r(z) = \hbar \int \underbrace{(t^a_{R_1})_{ij}\, (t^a_{R_2})_{kl}\, \delta^{ab}}\, P(x-y, z, \bar z)\, dx\, dy$$

$$= \cdots \approx \underbrace{\frac{\hbar}{z} \sum_a (t^a_{R_1})_{ij} (t^a_{R_2})_{kl}}_{r(z)}$$

$$C = \sum_a t^a \otimes t^a \quad \text{on } R_1 \otimes R_2$$

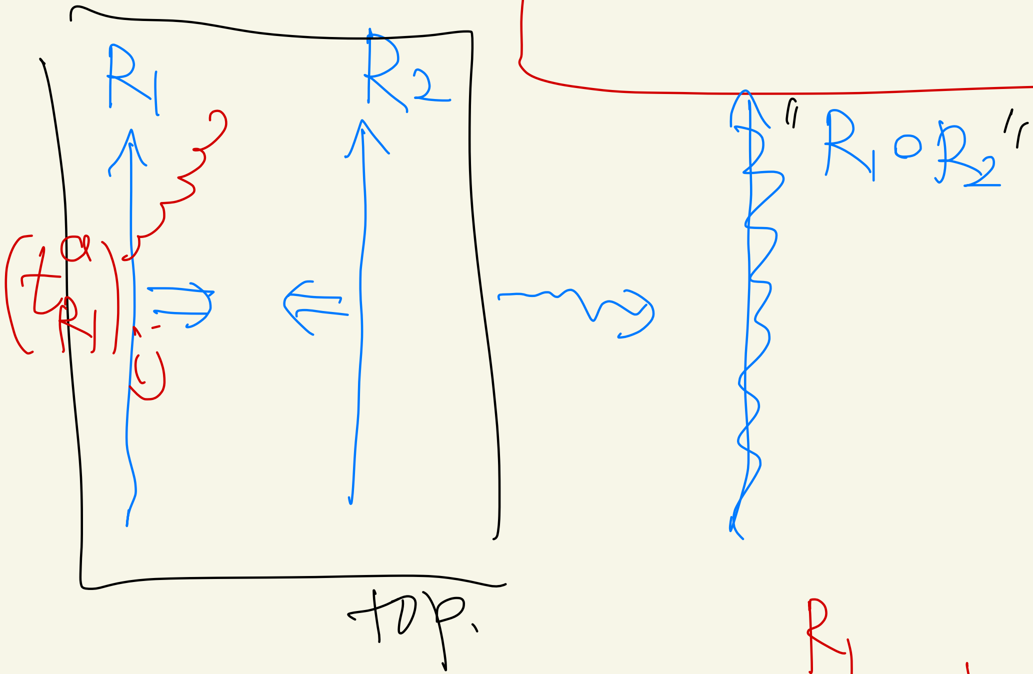
Casimir



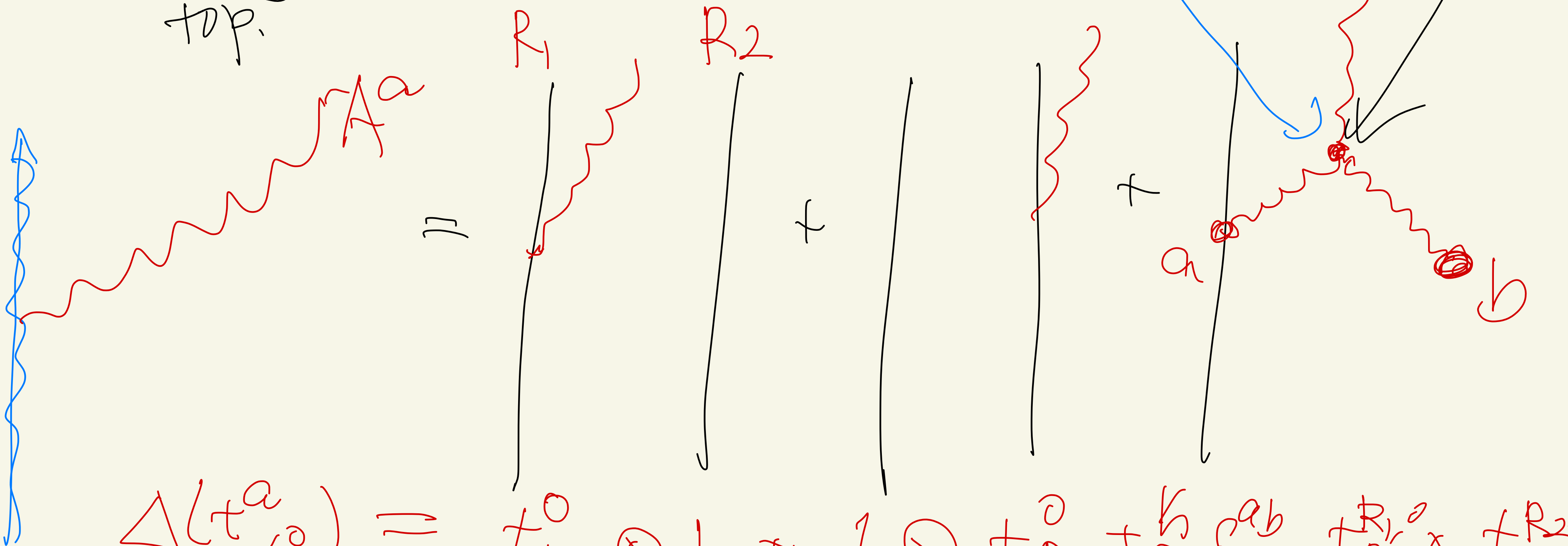


matches with known answer for $r(z)$

Yangian : $Y_\hbar(\mathfrak{g})$ = algebra of Wilson line



$$S_{cs} = \int dz \wedge \left(\mathrm{Tr}\, A \wedge dA + \frac{2}{3} A^3\right)$$



$$\Delta(t^a_{,0}) = t^0_{R_1} \otimes 1 + 1 \otimes t^0_{R_2} + \frac{\hbar}{2} f^{ab}{}_c\, t^{R_1,0}_{R_1} \times t^{R_2,0}_{c}$$

Yangian relation $t_a^n \Leftarrow t_a z^n$ ($n \geq 0, 1 \cdots$; $t_a \in \mathfrak{g} \leftarrow n=0$)

$\mathcal{U}(\mathfrak{g}[[z]])$

$$t_a^0 = t_a, \quad t_a^1 = J(t_a)$$

$$[J(t_a), J([t_b, t_c])] + (\text{cyclic}) = \underbrace{\frac{\hbar^2}{4}\left([t_a, t_d], [[t_b, t_e], [t_c, t_f]]\right)\{t_d, t_e, t_f\}}_{\text{quantum correction!}}$$

$$\{t_d, t_e, t_f\} \Rightarrow \frac{1}{6}\{t_d t_e t_f + (\text{perm})\}$$

$$J([t_b, t_c]) = J(i f_{bc}{}^a t_a^0) = i f_{bc}{}^a t_a^1$$

$Y_{\hbar=0}$

$Y(e_8)$ 248 $\oplus$ 1

$\mathfrak{g}$-repr R $\rightsquigarrow$ $\mathcal{U}(\mathfrak{g}[[z]])$-repr.

? $\cancel{\rightsquigarrow}$ $Y_{\hbar\neq 0}(\mathfrak{g})$ - repr.

* Perturbation theory of 4d CS around classical solution

$(A=0 \text{ for } C=\mathbb{C})$

isolated

non-isolated

$\Rightarrow$ R-matrix $R_\hbar(z)$

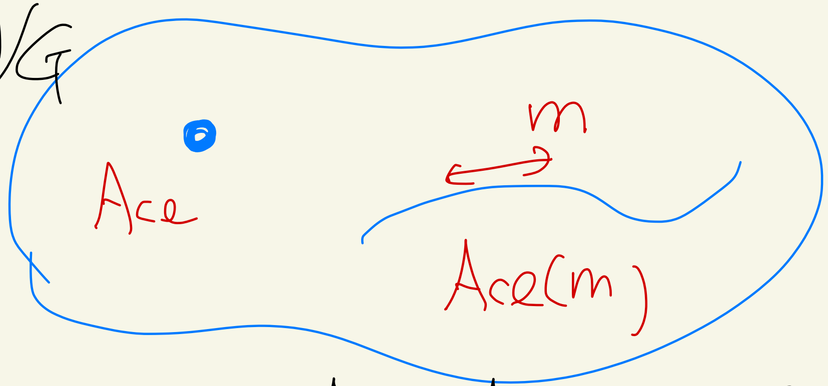
w/ moduli

$R_\hbar(z; m)$

$F_{xy} = F_{x\bar{z}} = F_{y\bar{z}} = 0$

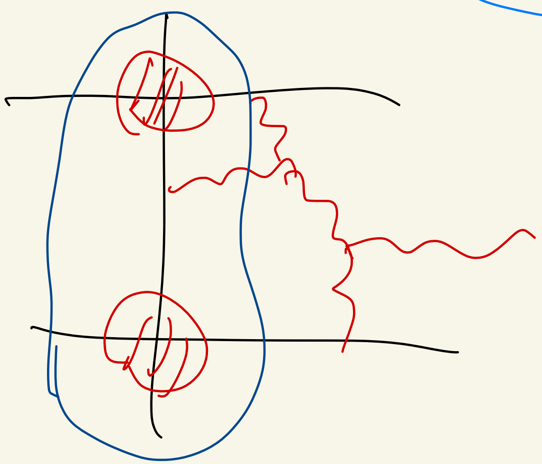
$\mathcal{M}_{flat}^{T^2} = (G\times G)/G$

$= T/W$



$A = A_{cl}(m) + \epsilon\delta A + \cdots$

[e.g. dynamical YBE]



$\underbrace{\text{Dynamical YBE}}_{(\text{I } \S 10)} \leftarrow$ ell. $C := E$, $\omega = dz$

$G = SL_N \supset T$

$G \to T$

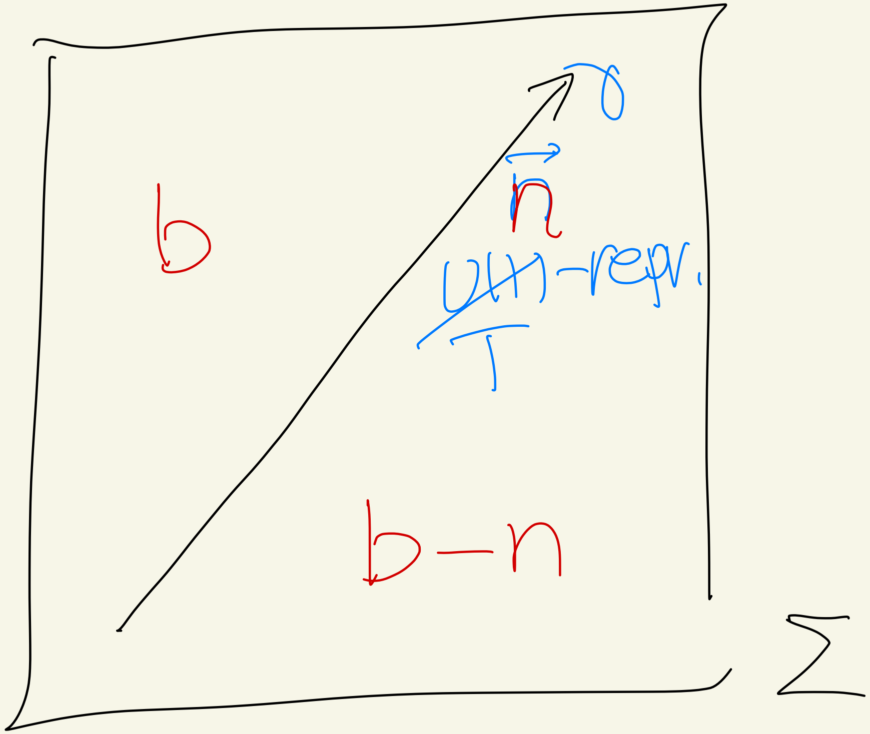
$$\underbrace{A_{\bar z} = \mathrm{diag}(\underbrace{b_1, \dots b_N}_{b})}_{C\text{-direction}} \quad \sum b_i = 0$$

Σ-dependent

$$A = \underbrace{A_x dx + A_y dy}_{A^{2d}} + \underbrace{A_{\bar z}}_{b} d\bar z$$

$$\text{“}S_{4d\,CS}\text{”} \longrightarrow \text{“}S_{2d}\text{”} = \int_\Sigma \mathrm{Tr}(bF)$$

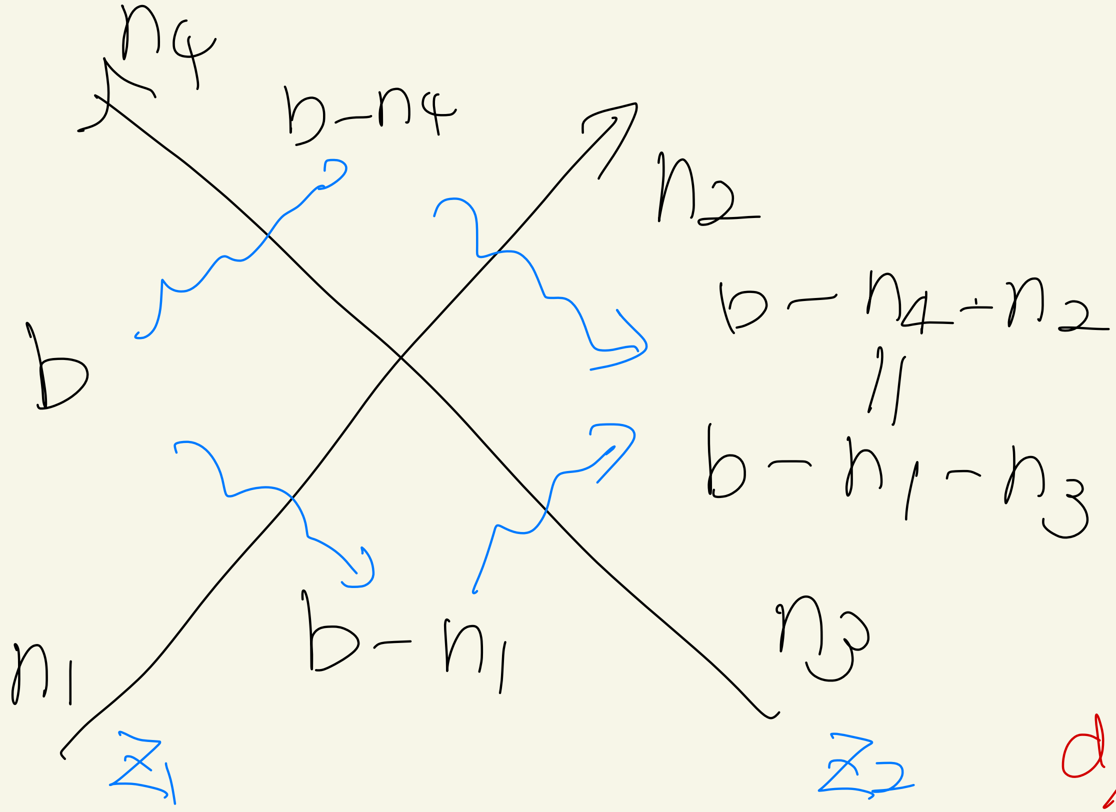
$G \qquad T$



$$S = \int_\Sigma \mathrm{Tr}(bF) + \underbrace{n\int_\gamma A}_{\int_\Sigma A \wedge \delta_\gamma}$$

$$\rightsquigarrow \quad \underset{\sim\sim}{db} + \underset{\sim\sim}{n\delta_\gamma} = 0$$

$\rightsquigarrow$ b jumps across γ



$$R^{n_4 n_2}_{n_1 n_3}(z_1 - z_2)$$

b

new parameter

dynamical YBE

Factorization algebra $\longrightarrow$ Yangian

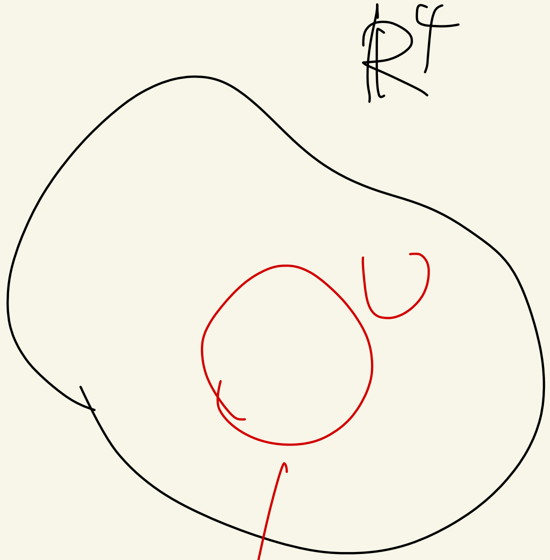
QFT in math $\cup$ VOA

TQFT

Lagrangian $\longrightarrow$ equation of motion

$\longrightarrow$ "derived" solution

cochain complex

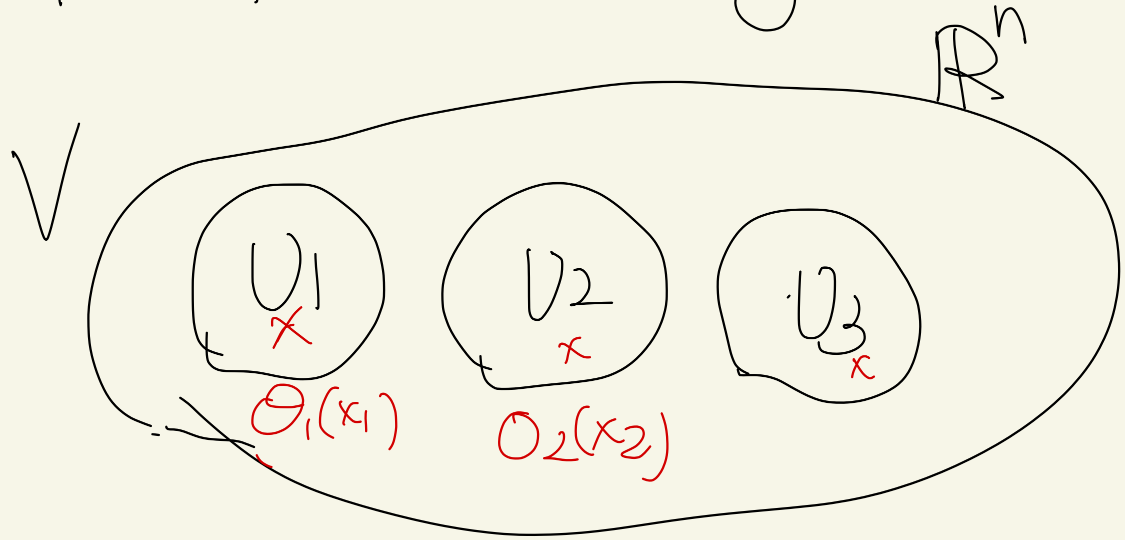


$\mathcal{F}(U)$: "operator on U"

$$\mathcal{F}_{(U)} = \left\{ C^0_{(U)} \to C^1_{(U)} \to \cdots \right\}$$

U : open set in $\mathbb{R}^4$

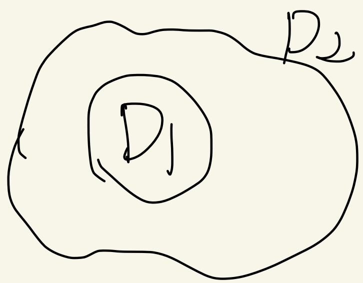
Pre-factorization algebra



$F(U_1)\times F(U_2)\times F(U_3) \to F(V)$

[Costello-Gwilliam]

TQFT: locally constant



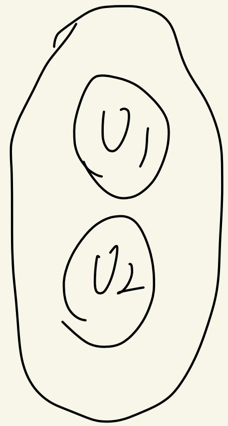
$F(D_1) \to F(D_2)$

quasi-iso.

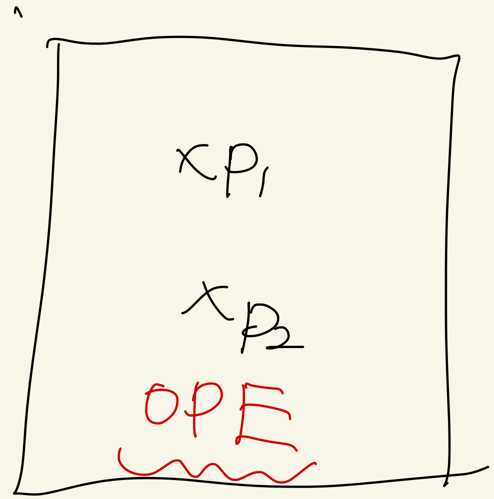
locally const. FA

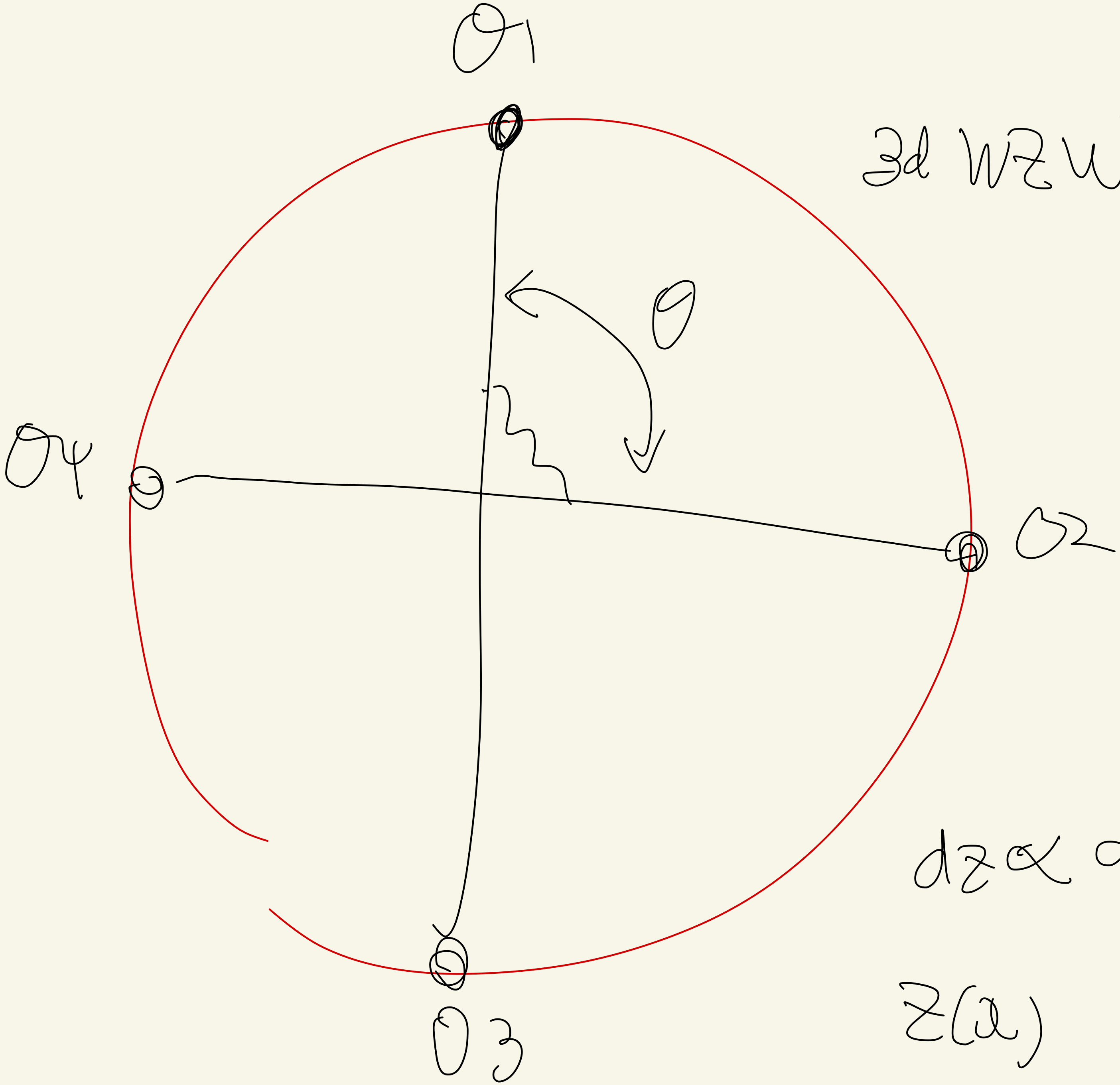
$= E_n$-alg.

VOA, translationally-inv, holomorphic PrFA



$\Longrightarrow$

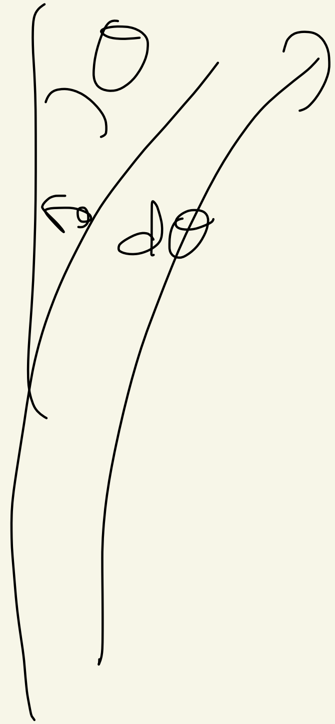




3d WZW

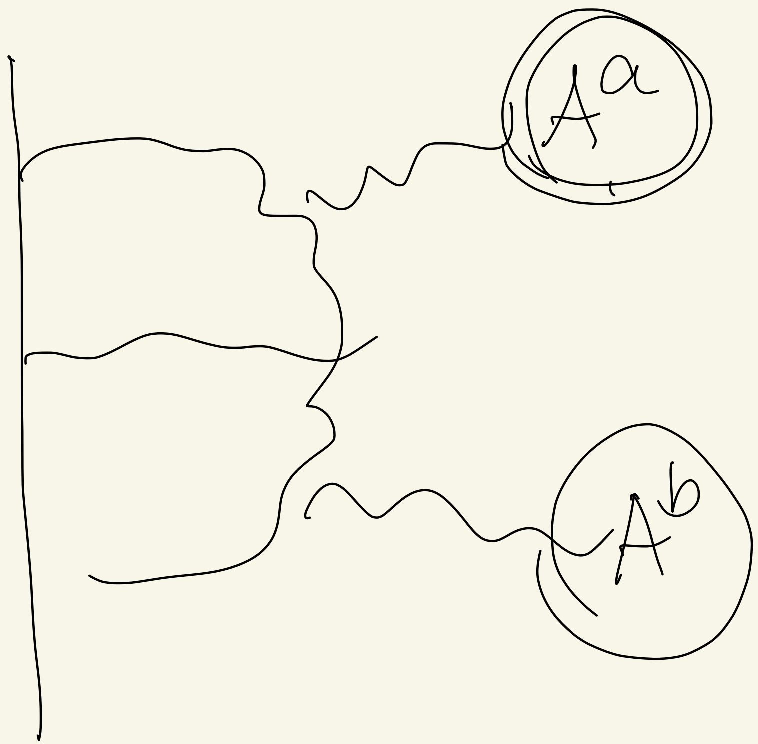
$\theta(x)$

x



$dz \propto d\theta,$

$z(x)$



A^a
A^b