Lec 5

2021/Jul/19

Masahito Yamazaki

Today

* Factorization algebra $\rightsquigarrow$ Yangian
  Koszul duality

* Integrable Field Theory $\leftsquigarrow$ 4d CS + surface defect

"uplift" $\downarrow$

* Twistor, ASDYM, hol CS

Lec 5

Lec 6

Yesterday

3d $\mathcal{N}=2$ line

$$Y_\hbar(\mathfrak{g}) : \text{Yangian} \longleftarrow\!\sim\!\sim \text{Feynman diagram computation of 4d } CS_\hbar$$

$$\downarrow \hbar = 0$$

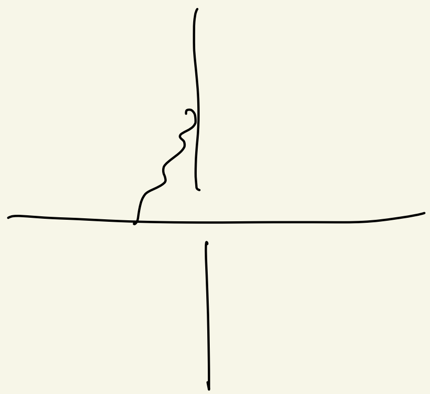
$$U(\mathfrak{g}[[z]])$$

"algebra of loop operators"

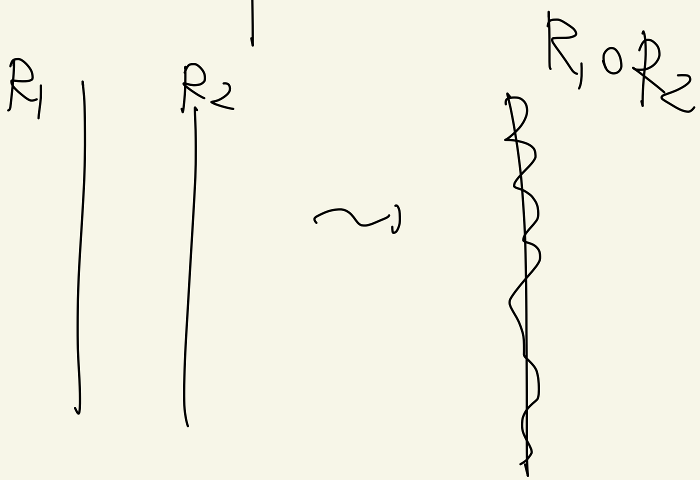
q-cluster alg.

3-mfd

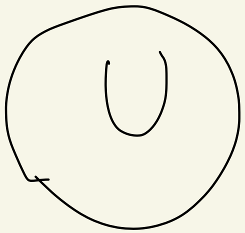
R-matrix

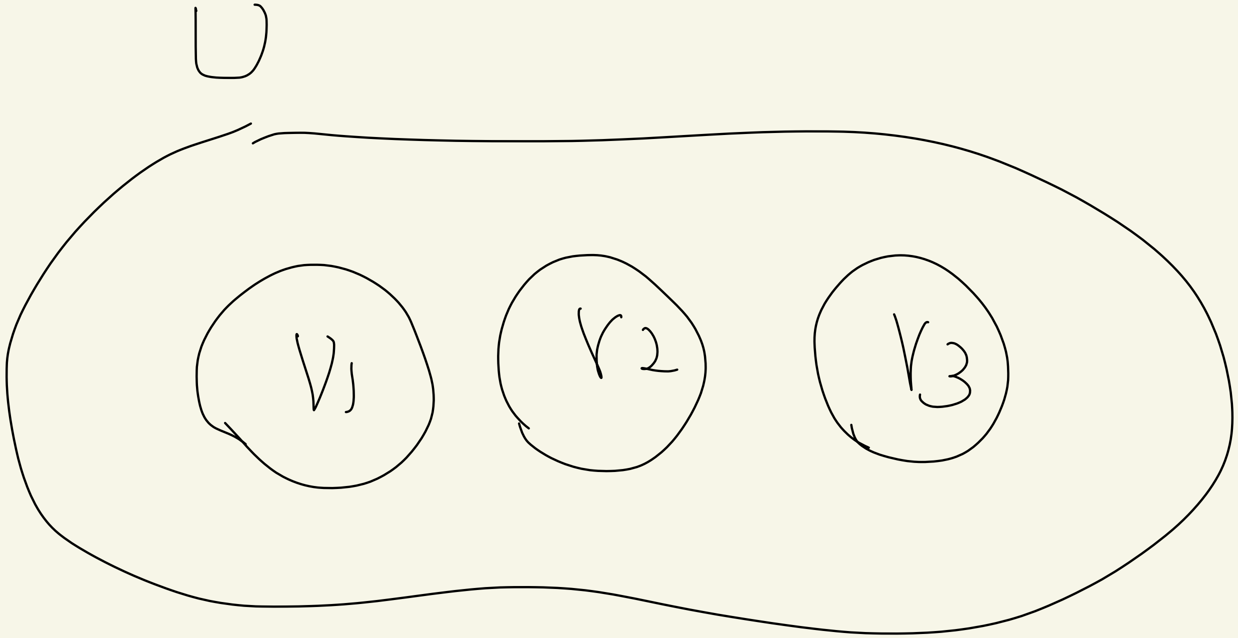


coproduct



(Pr)FA





$\mathcal{F}(U)$

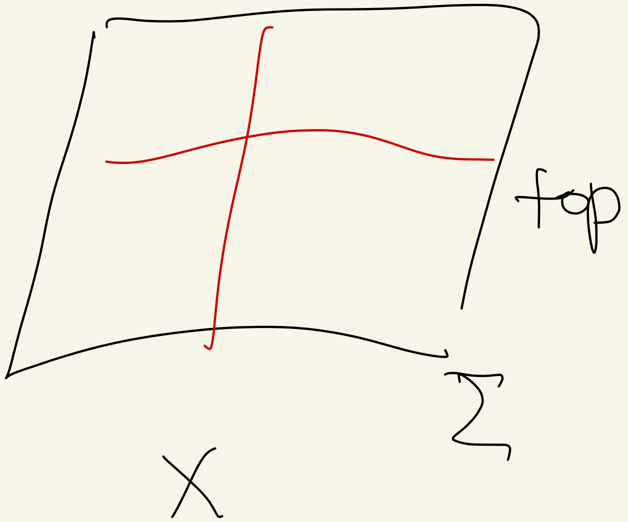
$\Updownarrow$

cochain complex

operators on U

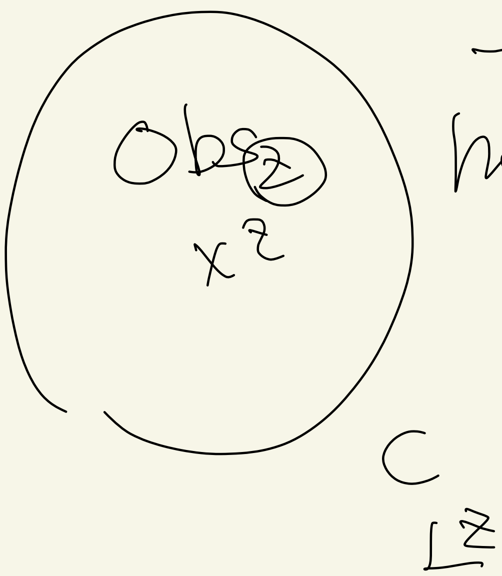
$$\bigotimes_i \mathcal{F}(V_i) \to \mathcal{F}(U)$$

4d CS



$\to$ 2d TQFT : locally const. FA / E_2-alg

hybrid $\updownarrow$ $\hbar$



hol. $\to$ 2d $\underbrace{\text{holomorphic}}$ / $\underbrace{\text{translationally-inv.}}$

$\downarrow$

OPE / VOA

$T_\lambda : z \to z+\lambda$

$$Obs_0 \otimes Obs_0 \to Obs_0 \otimes \mathbb{C}((z))$$

classical, $\hbar = 0$

$$\mathrm{Obs}^{\hbar=0}_{z=0} = \{\text{classical alg of local op. of 4d CS at } z=0\} \quad (\text{over } \mathbb{C})$$

4d bulk

E_2-alg $= C^*(\mathfrak{g}[[z]], d_{CE})$

$\mathfrak{g}$

$Q_{BRST} C^a = f^a{}_{bc} c^b c^c$

$[c, c]$

$S \sim dz \wedge A \wedge dA$

$\cancel{\partial_{\bar z}}$

$A \to g^{-1} A g + g^{-1} dg$

G

$G[[z]]$

Koszul duality

1d defect

$z = z_0$

$U(\mathfrak{g}[[z]])$

$= \{\text{classical alg of loop op located at } z = 0 \in \mathbb{C}\}$

z_0

Hopf alg

$\boxed{\text{Koszul duality : (universal) coupling between bulk \& defect}}$

Phys

A : bulk alg. $\xrightarrow{\text{KD}}$ $A^!$

B : line defect alg.

$$(A \,\&\, B : \text{coupling}) \longleftrightarrow \mathrm{Hom}(A^!, B) = MC(A \otimes B)$$

Maurer-Carton

Koszul duality

V: vect sp.

math

$$A = T(V) = \bigoplus_{n \geq 0} V^{\otimes n} \Big/ \underbrace{R}_{} \subset \underbrace{V \otimes V}_{}$$

$$\updownarrow$$

$$A^! = T(V^*) \Big/ \underbrace{R^\perp}_{} \subset \underbrace{V^* \otimes V^*}_{}$$

$$\boxed{\mathcal{U}(\mathfrak{g}[z])} \xrightarrow{\text{Abelian}} \mathrm{Sym}(\mathfrak{g}[z])$$

$$\mathrm{Sym}(V) \overset{\text{KD}}{\longleftrightarrow} \Lambda^*(V^*)$$

$$\mathrm{Sym}(\mathfrak{g}[z]) \overset{\text{KD}}{\longleftrightarrow} \Lambda^*(\mathfrak{g}^*[z])$$

$$\Lambda^*(\mathfrak{g}^*[z]) \xleftarrow{(\text{Abelian } d_{CE} = 0)} \boxed{C^*(\mathfrak{g}[z], d_{CE})}$$

Quantum mechanically

$\mathrm{Obs}^{\hbar\neq 0}_{z=0} \leftarrow\!\!\!\!/\ \mathbb{C}[[\hbar]]$

$\boxed{\text{bulk}}$

- : E_2 - algebra (w/ augmentation) top

- : $T_\lambda : \mathrm{Obs}^{\hbar}_{z=0} \to \mathrm{Obs}^{\hbar}_{z=\lambda}$ translation transl. hol.

- : OPE $\mathrm{Obs}_0 \otimes \mathrm{Obs}_0 \to \mathrm{Obs}_0 \otimes \mathbb{C}((z))$

KD

$z \to z+\lambda$

$U(\mathfrak{g}[[z]])$ $U(\mathfrak{g}[[z]])$

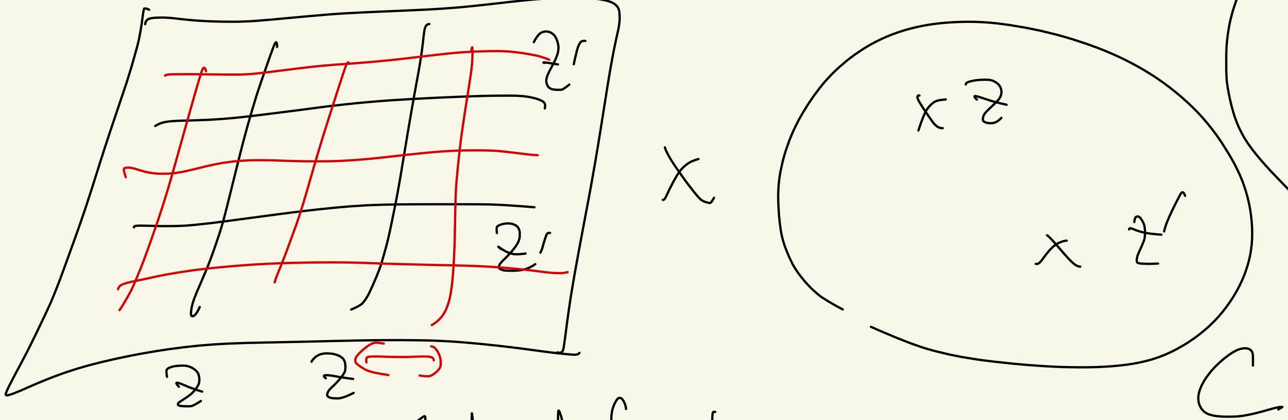
$\boxed{\text{defect}}$ "$Y_\hbar(\mathfrak{g})$" : Hopf alg $Y_\hbar(\mathfrak{g}) \otimes_{\mathbb{C}[[\hbar]]} \mathbb{C}$

: $T_\lambda \quad Y_\hbar(\mathfrak{g}) \to Y_\hbar(\mathfrak{g})$

"Yangian"

: $R_{V,W} : T_\lambda W \otimes V \simeq V \otimes T_\lambda W$

Integrable Field Theories

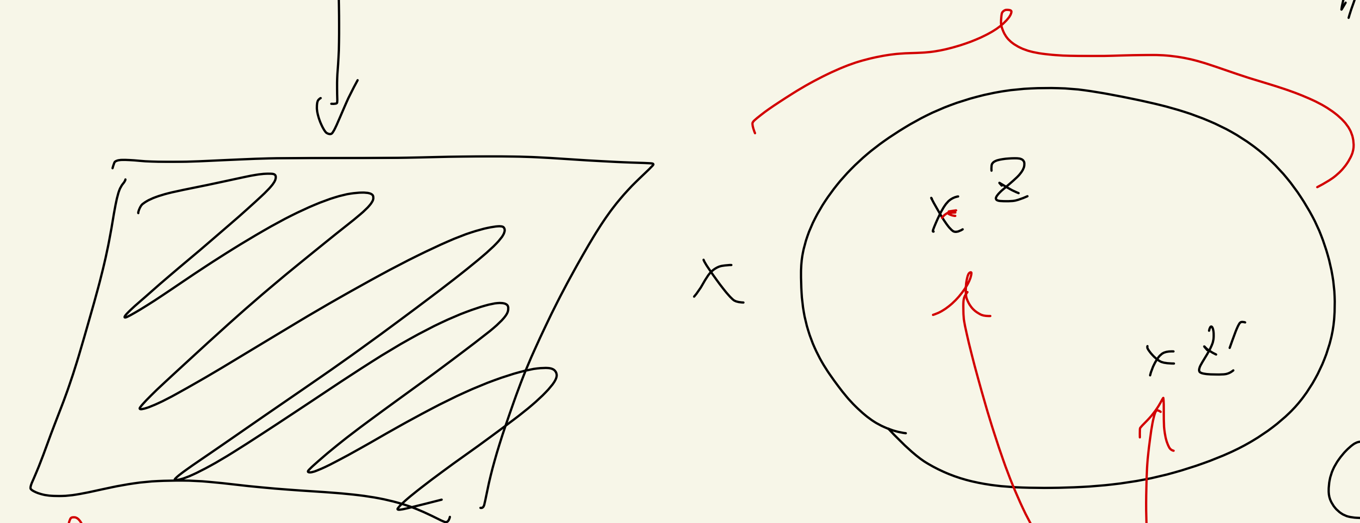


1d defect

$\boxed{\text{disorder} \leftarrow A \text{ b.c.}}$

order $\leftarrow$ coupling to defect theory

"integrate out C"



R^2

2d defect $\subset$ 4d

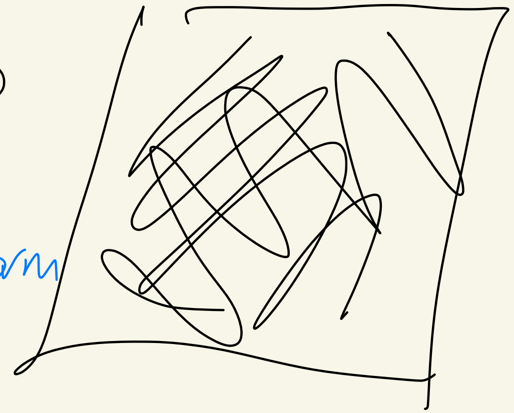
"surface defect"

b.c. for gauge field A

$\rightsquigarrow$

ω rel. 1-form

C'



effective 2d integrable FT

R^2

Why integrable?

4D CS

$\downarrow$

e.o.m. $F_{w\bar{w}} = F_{\bar{z}w} = F_{\bar{z}\bar{w}} = 0$

$\downarrow$ ($A_{\bar{z}} = 0$ gauge)

$*$ $\partial_{\bar{z}} A_w = \partial_{\bar{z}} A_{\bar{w}} = 0$ $\leftarrow$ $A_w, A_{\bar{w}}$: z-hol.

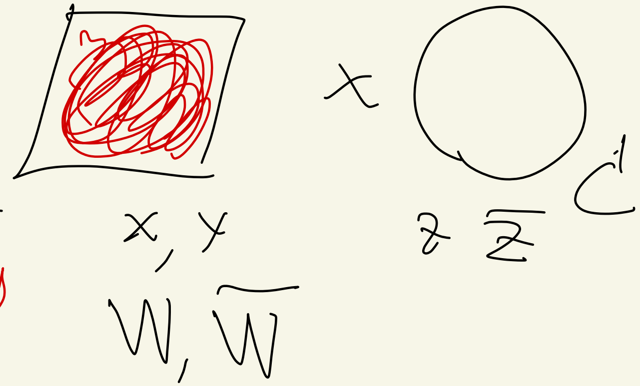
@ $\partial_w A_{\bar{w}} - \partial_{\bar{w}} A_w + [A_w, A_{\bar{w}}] = F_{w\bar{w}} = 0$ $\leftarrow$ $w\bar{w}$-direction flat

$\downarrow$

$*$ $\mathcal{L}(z) = A_w(z)\,dw + A_{\bar{w}}(z)\,d\bar{w}$: 1-form on $\mathbb{R}^2_{w\bar{w}} = \Sigma$

@ $d\mathcal{L}(z) + \mathcal{L}(z) \wedge \mathcal{L}(z) = 0$: Lax connection
"classical integrable"

$\mathbb{R}^2 = \Sigma$



lattice $\rightarrow$ top $\rightarrow$ hol

FT. $\rightarrow$ hol.

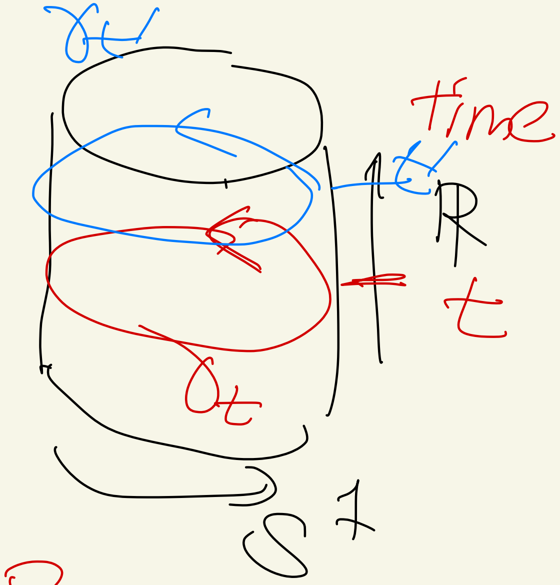
$$\partial_t \left(\mathrm{Tr}_R \, P\exp \int_{\gamma_t} \mathcal{L}(z) \right) = 0$$

Wh

z-expand

$$\exp\left(\sum_{n\geq 0}^{\infty} Q_n / z^n \right)$$

$\mathbb{R}^2 \to$



$$\partial_t Q_n = 0$$

$Y_n(g)$ $\quad t_a^{(n)}$ ← level

Yangian

non-local charge of

∞ - many charges
conserved

integrable lattice model (top. along $\Sigma = \mathbb{R}^2$)

ω: hol. 1-form $\leftarrow$ no zero.

$\Downarrow$

C: $\mathbb{C}$, $\mathbb{C}^\times$, E

A_w
$A_{\bar{w}}$

integrable field theory (hol. along Σ)

ω: hol. 1-form $\leftarrow$ zeros allowed

$\Downarrow$

C: higher genus ok.

$$S \supset \boxed{\int \underbrace{\omega}_{\infty \quad 0} \wedge \operatorname{Tr}(\underbrace{A}_{0 \quad \infty} \wedge dA)}$$

$\exists \phi \supset \infty$

general data

translation surface

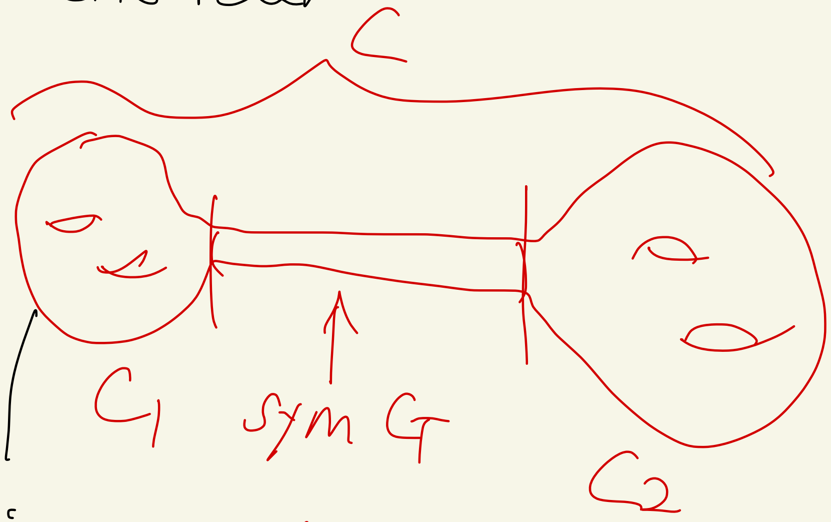
$g>1$

- spectral curve C (higher genus OK)
- hol. one-form ω on C (pole, zero ...)
  2nd 1st
  1st
- b.c. for A
  consistent ω

surface defect

[- order surface defect (coupling to 2d FT)]   disorder

$\hookrightarrow$ 2d integrable field theory

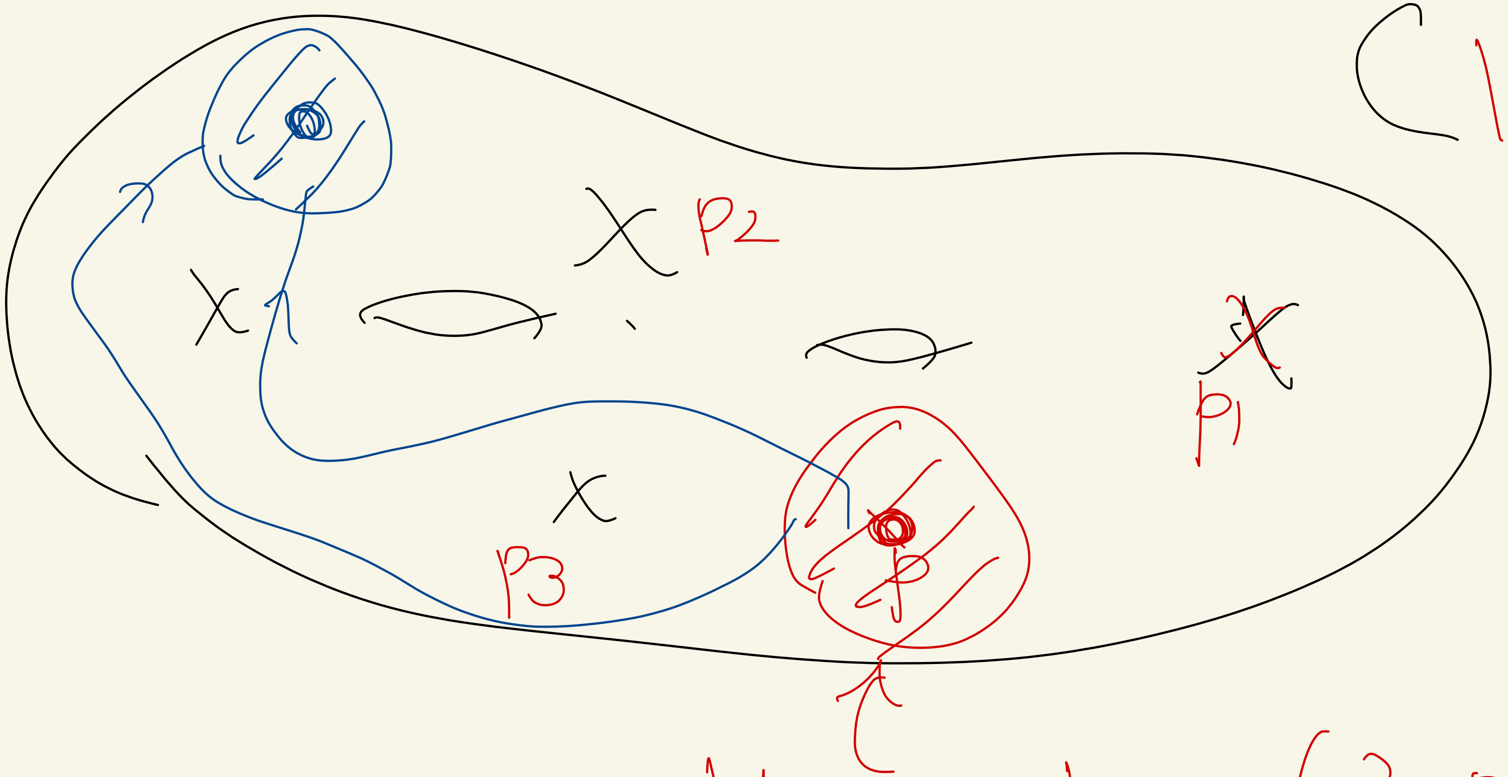


$$T_C \simeq (T_{C_1} \times T_{C_2}) /\!/ G$$

$\mathbb{C}$

$\parallel$

$\mathbb{C}\setminus\{p_i\}$



$$W_\gamma(z)\big|_{z=p} \quad (z-p)^n \underbrace{Q_n}_{\text{conserved charge}}$$