Lec 6

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$\mathfrak{g}[[z]]$ on $\mathbb{R}^2 \times \mathbb{C}$ S^1 integrable QFT

$\mathbb{R}$

4d CS + surface defect $\xrightarrow{S^1}$ 2D WZW (PCM + WZ)

3d CS $\mathfrak{g}$

hol.

6d CS + surface defect on $\mathbb{PT} = \mathbb{CP}^3$

$\mathbb{CP}^1$

[Costello
Bittelston ~ Skinner
Penna]
⋮

4D WZW

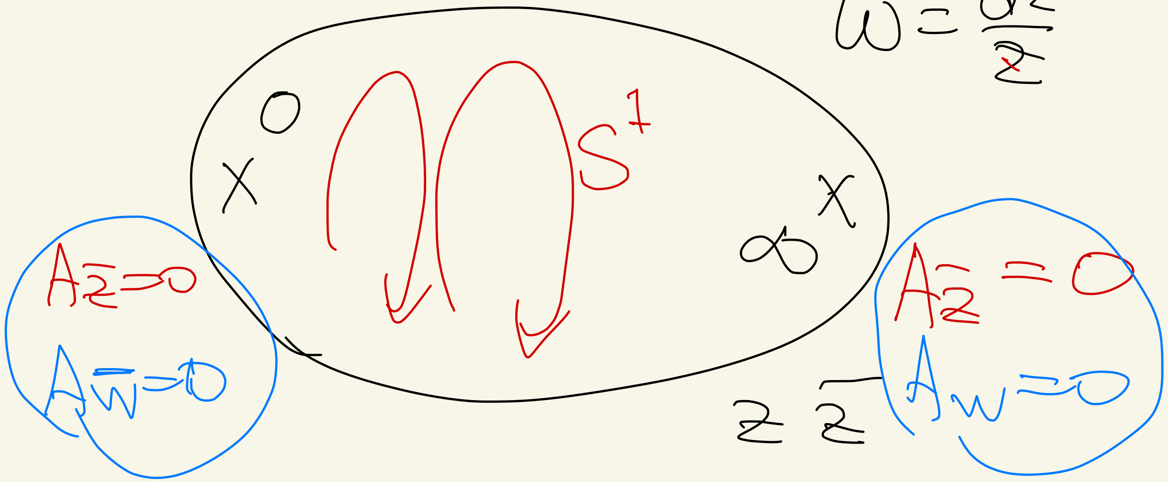
ASD YM ($F^+ = 0$)

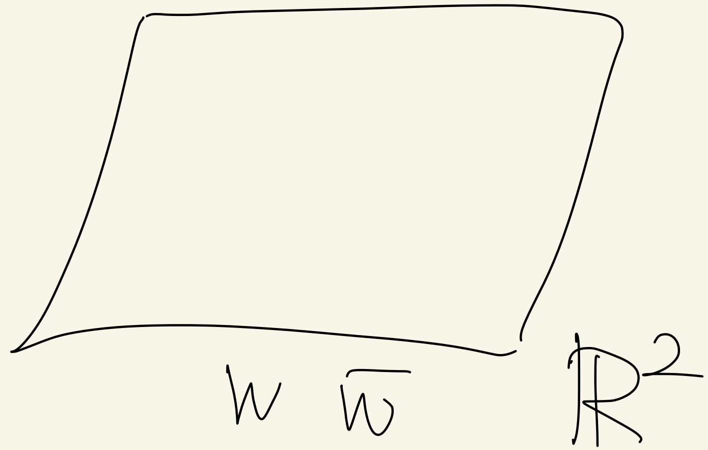
int. eqn.

$\omega = \frac{dz}{z}$ $\quad C = \mathbb{CP}^1$

$\mathbb{C}^\times$

X





① solve e.o.m. $(A_w, A_{\bar w}, A_{\bar z})$

② Plug A's into $S_{4d\,CS}$ ⇝ 2d action

$A_{\bar z}$ along C : $\boxed{\sigma : \mathbb{R}^2 \to G_{\mathbb{C}}} = (G_{\mathbb{C}} \times G_{\mathbb{C}})/G_{\mathbb{C}}$

$$\left(\begin{array}{l} \hat\sigma : \mathbb{R}^2 \times \mathbb{CP}^1 \to G_{\mathbb{C}} \quad\quad \sigma^{-1}\hat\sigma|_{\{\infty\}} = 1 \\ \text{s.t.} \ \bullet \ U(1) \text{ inv,} \\ \quad\quad \bullet \ \hat\sigma|_{\{0\}} = 1 \quad\quad \hat\sigma|_{\{\infty\}} = \sigma \end{array} \right)$$

$$\begin{cases} A_{\bar z} = \boxed{\hat\sigma^{-1}\partial_{\bar z}\hat\sigma} = (\sigma^{-1}\hat\sigma)\partial_{\bar z}(\sigma^{-1}\hat\sigma) \\ A_{\bar w} = \boxed{\hat\sigma^{-1}\partial_{\bar w}\hat\sigma} \to 0 \quad @\ z=\infty \\ A_w \neq \hat\sigma^{-1}\partial_w\hat\sigma \not\to 0 \quad @\ z=\infty \\ \quad \| \\ \quad (\sigma^{-1}\hat\sigma)^{-1}\partial_w(\sigma^{-1}\hat\sigma) \to 0 \quad @\ z=\infty \end{cases}$$

$$F_{\bar z \bar w} = 0$$

$$\| $$

$$\boxed{\hat\sigma^{-1}\partial_w\hat\sigma} - \boxed{\hat\sigma^{-1}(\partial_w\sigma)\sigma^{-1}\hat\sigma}$$

$$d\theta \qquad CS(A) \qquad \hat\sigma^{-1}\partial\hat\sigma$$

plug in

$$S_{4d\,CS} = \frac{1}{2\pi\hbar}\int_{\mathbb{R}^2\times} \frac{dz}{z}\wedge \mathrm{Tr}\left(A\wedge dA + \frac{2}{3}A^3\right)$$

$0 \quad \infty \quad \cong S^1$

$\rightsquigarrow S_{2d} \supset \cup$ (**)

$$\underbrace{\int_{\mathbb{R}^2\times[0,\infty]} \mathrm{Tr}\left(\hat{\sigma}^{-1} d\hat{\sigma}^{-1}\right)^3}_{\text{WZ-term}}$$

$\mathbb{R}_{\geq 0}$

WZ

(**)

$$\int_{\mathbb{R}^2} \mathrm{Tr}(\sigma^{-1}\partial_w \sigma)(\sigma^{-1}\partial_{\bar{w}}\sigma)$$

kinetic term

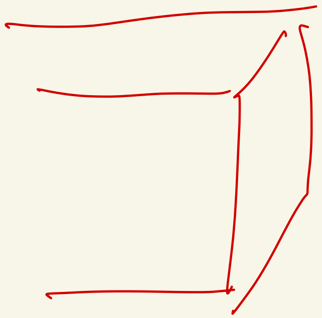
$$\int_{\mathbb{R}^2\times\mathbb{C}^1} \frac{dz}{z}\wedge d_w(\cdots)$$

integrate by part

$$\partial_{\bar{z}}\left(\frac{dz}{z}\right) = 2\pi i\left(\delta_{z=0} - \delta_{z=\infty}\right)$$

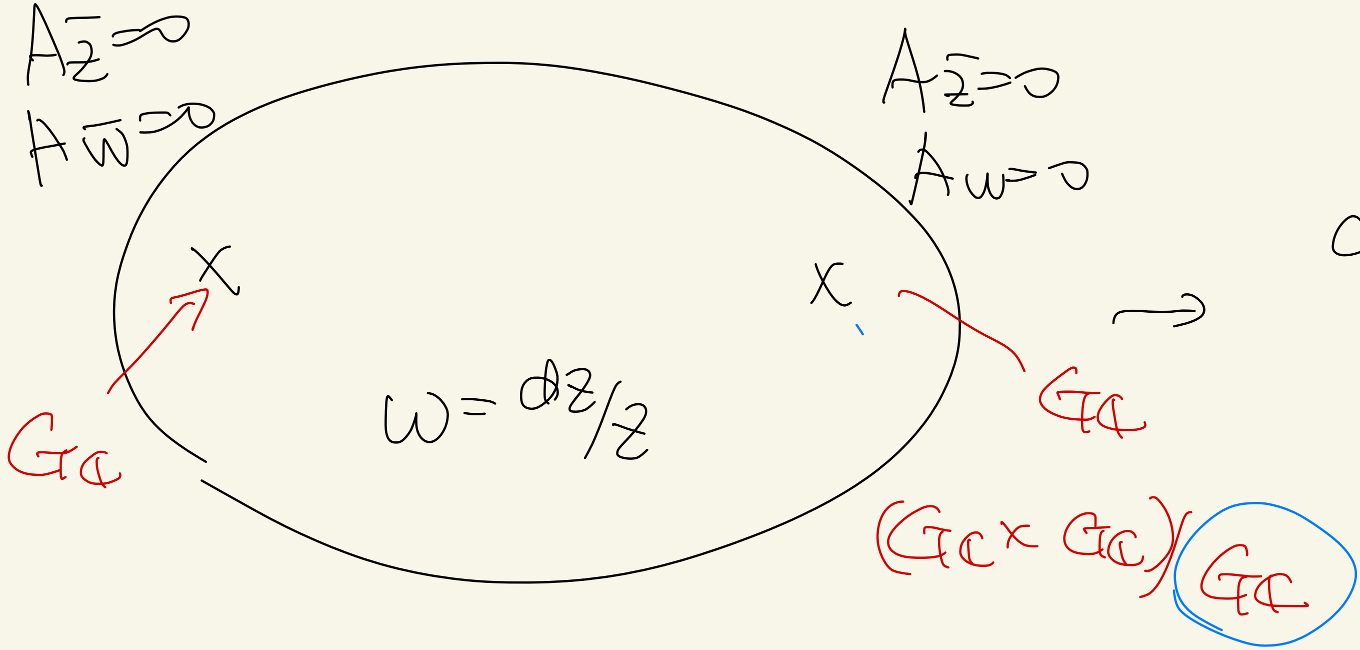
PCM

$\mathbb{R}^2$



$\mathbb{R}^2\times\mathbb{R}_{\geq 0}$

WZNW = PCM + WZ

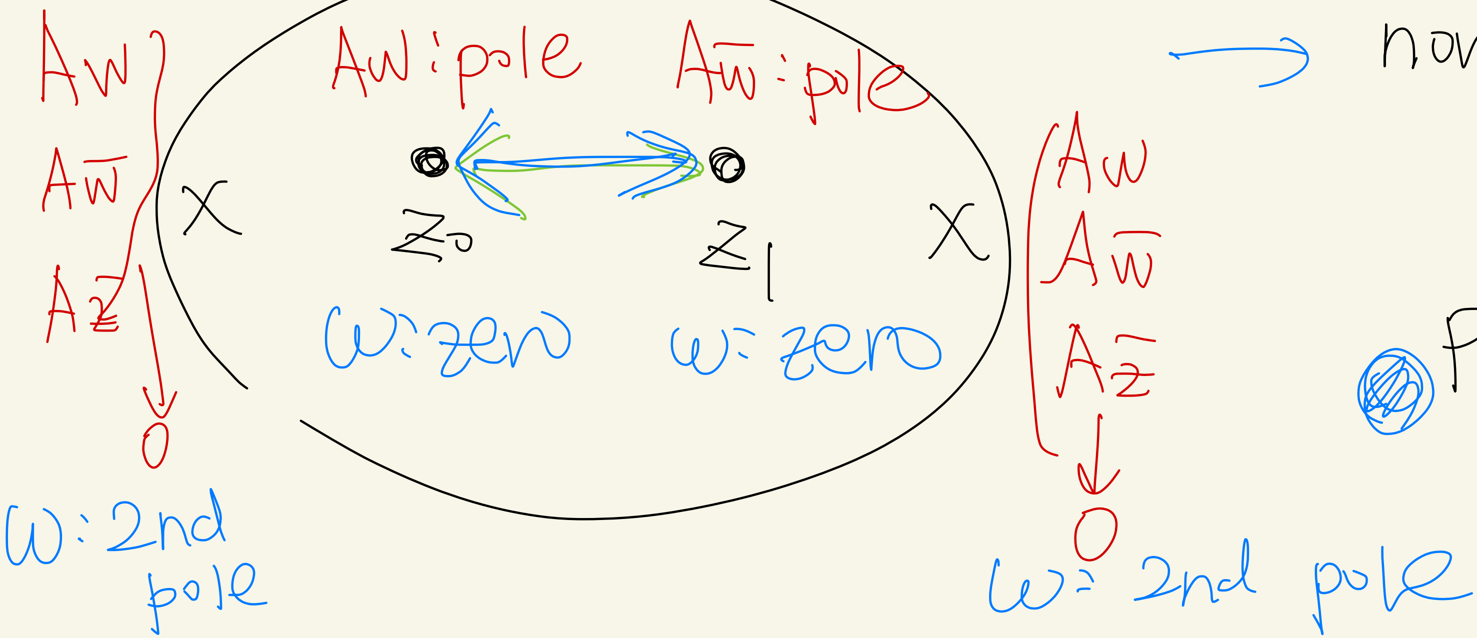


$\longrightarrow$ conformal WZW $G_{\mathbb{C}}$

$\|$

PCM + WZ

$$\omega = \frac{(z - z_0)(z - z_1)}{z^2} dz$$



$\longrightarrow$ non-conformal WZW

$\|$

PCM + WZ

open A-model $\xleftrightarrow{\text{T-duality}}$ AAB/3 $\xleftrightarrow{T}$ ABB/3 $\xleftrightarrow{T}$ open B-model

3d CS ⟷ 4d CS ⟷ 5d CS ⟷ 6d CS hol.

$\boxed{\hbar_2 : NC}$

$$\int_{\mathbb{R}^3} CS_3 \qquad \int_{\mathbb{R}^2\times \mathbb{C}_z} \omega\wedge CS_3 \qquad \frac{1}{\boxed{\hbar_1}}\int \underbrace{\omega}_{\text{hol. 2-form}} \wedge CS_3 \qquad \int \underbrace{\Omega}_{\text{hol. 3-form}}\wedge CS_3$$

hol.

$\mathbb{R}\times\mathbb{C}_{z_1,z_2}$ $[\dim_{\mathbb{C}} C = 2]$ $[\dim_{\mathbb{C}} = 3]$

$$\mathcal{U}(\mathfrak{g}[[z]]) \qquad \mathcal{U}(\mathfrak{g}[[z_1, z_2]]) \qquad \mathcal{U}(\mathfrak{g}[[z_1,z_2,z_3]])$$

$$Y_\hbar(\mathfrak{g}) \qquad Y_{\hbar_1,\hbar_2}(\mathfrak{g}) \qquad \hbar_1+\hbar_2+\hbar_3 = 0$$

Yangian affine Yangian

6d CS = hCS

$$S_{\substack{6d\\ hCS}} = \int \underbrace{\Omega}_{} \wedge \mathrm{Tr}\left(\mathcal{A} \wedge d\mathcal{A} + \frac{2}{3}\mathcal{A}^3\right)$$

[$\mathcal{A}$: hol. connection]

$A_{\bar{z}_{1,2,3}}$ $\quad \cancel{A_{z_{1,2,3}}}$

$\mathbb{CP}^3 = \mathbb{PT}$ twistor

! not CY_3 $\quad$ ($\mathbb{CP}^{3|4}$: CY_3)

3-form on $\mathbb{CP}^3$

$$\Omega = \frac{D^3 Z \left(= \underbrace{\varepsilon_{\alpha\beta\gamma\delta}} Z^\alpha dZ^\beta dZ^\gamma dZ^\delta / 4!\right)}{(A\cdot Z)^2 (B\cdot Z)^2}$$

$A, B = (\mathbb{CP}^3)^*$

$\lambda [Z^0 : Z^1 \cdots : Z^3]$

homogeneous coord.

$$\mathbb{CP}^3 = \frac{SU(4)}{S[U(3)\times U(1)]}$$

$$\Omega = \frac{D^3 z}{(A z)^2 (B z)^2} \quad : \quad \text{pole at} \quad \begin{array}{l} A \cdot z = 0 \\ B \cdot z = 0 \end{array}$$

$\mathbb{CP}^1$

$$\mathbb{CP}^3 \setminus \mathbb{CP}^1$$

$$\cong$$

$$\underbrace{\mathcal{O}(-1) \oplus \mathcal{O}(-1)}_{\mathbb{R}^4} \to \mathbb{CP}^1$$

$$\times \, \mathbb{CP}^1$$

$$\frac{D^3 z}{(A_1 z)(A_2 z) \cdots (A_4 z)}$$

6d hCS

4d CS

$$\mathbb{R}^2 \times \mathbb{CP}^1 \setminus \{0, \infty\} \qquad \omega = \frac{dz}{z} \qquad \begin{array}{l} z = 0 \\ z = \infty \end{array}$$

$$\cong$$

(cylinder)

$$
\begin{tikzcd}[column sep=small]
 & \boxed{\begin{array}{c} \text{6d hCS} \\ \text{on } \underbrace{\mathbb{R}^4}_{\mathbb{R}^2\times\tilde{\mathbb{R}}^2} \times \mathbb{CP}^1 \end{array}} \arrow[dl, "\tilde{\mathbb{R}}^2\ \ T^2"'] \arrow[dr, "\mathbb{CP}^1"] & \mathbb{CP}^3/\mathbb{CP}^1 \arrow[l] \\
\boxed{\begin{array}{c} \text{4d CS} \\ \text{on } \mathbb{R}^2\times\mathbb{CP}^1 \end{array}} \arrow[dr, "\mathbb{CP}^1"'] & & \boxed{\begin{array}{c} \text{4d WZW} \\ \text{on } \mathbb{R}^2\times\tilde{\mathbb{R}}^2 \end{array}} \arrow[dl, "\tilde{\mathbb{R}}_2\ \ T^2"] \\
 & \boxed{\begin{array}{c} \text{2d WZW} \\ \text{on } \mathbb{R}^2 \end{array}} &
\end{tikzcd}
$$

$\int \Omega_\omega \wedge CS_3$

KdV

NLS

$\uparrow$

ASDYM

$\int \omega \wedge CS_3$

2d WZW

$$J = \underset{\mathbb{R}^2}{\sigma^{-1} d\sigma}, \qquad \hat{J} = \underset{\mathbb{R}^2 \times \mathbb{R}_{>0}}{\hat{\sigma}^{-1} d\hat{\sigma}}$$

$$S_{\substack{2d\\ WZW}} = @\int_{\mathbb{R}^2} \mathrm{Tr}\, J \wedge * J + @\int_{\mathbb{R}^2 \times \mathbb{R}_{\geq 0}} \mathrm{Tr}\, \hat{J}^3$$

4d WZW

hol. 2-form.

$$S_{\substack{4d\\ WZW}} = \int_{\mathbb{R}^4} \omega \wedge \mathrm{Tr}\, J \wedge * J \times \int_{\mathbb{R}^4 \times \mathbb{R}_{\geq 0}} \omega \wedge \mathrm{Tr}\, \hat{J}^3$$

4d WZW $\longrightarrow$ e.o.m. Yang's eqn J

$\updownarrow$

4D ASD YM eqn $F^{+} = 0$

$[D_w, D_z]$ $[D_{\bar w}, D_{\bar z}]$ $w, \bar w, z, \bar z$

$\underbrace{F_{wz} = F_{\bar w \bar z} = 0}$ $F_{w\bar w} = F_{z\bar z}$

$D_w h = D_z h = 0$

$D_{\bar w} k = D_{\bar z} k = 0$

$J = h^{-1} k$

$\omega = dw\,d\bar w - dz\,d\bar z$

$\omega \wedge \partial(J^{-1}\bar\partial J) = 0$