M5-branes / 3d SUSY QFT

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Topological Phases of Matter

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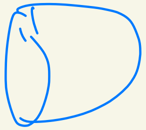
Lec III

"Integrable model"

Recap


$\times$
S^1

3d $\mathcal{N}=2$ SUSY QFT

$S^1 \to \mathcal{M}_{g,p}$
$\downarrow$
Σ_g

F. R. S. T

3d TQFT (MTC)

Blau - Thompson, Hansen(-Takata)

formal series in $\mathfrak{q}, \eta$

$$Z^{\mathcal{M}_{g,p}} \stackrel{?}{=} \sum_\alpha (\mathcal{H}_\alpha)^{g-1} \mathcal{F}_\alpha^{\,p} \quad \stackrel{?}{\longleftrightarrow} \quad Z^{\mathcal{M}_{g,p}} = \sum_\alpha \underbrace{(S_{\alpha 0}^{-2})^{g-1} (T_{\alpha\alpha})^p}$$

α : "vacua" Bethe

$$Z^{S_1 \times S_2} = \cdots \quad \stackrel{?}{\longleftrightarrow} \quad Z^{S^1 \times S^2} = \dim \mathcal{H}(S^2) = 1$$

$$\| $$

$$\mathrm{Tr}_{\mathcal{H}(S^2)} (-1)^{2j_3} \mathfrak{q}^{\frac{R}{2}+j_3} \eta^{A}$$

R-charge

fugacity

flavor charge

$\phi, \partial_\mu \phi, \partial_\mu \partial_\nu \phi, \ldots\ldots$

conformal

$S^1 \times S^2 \simeq \mathbb{R}^3$

scale transformation

H (dilotation)

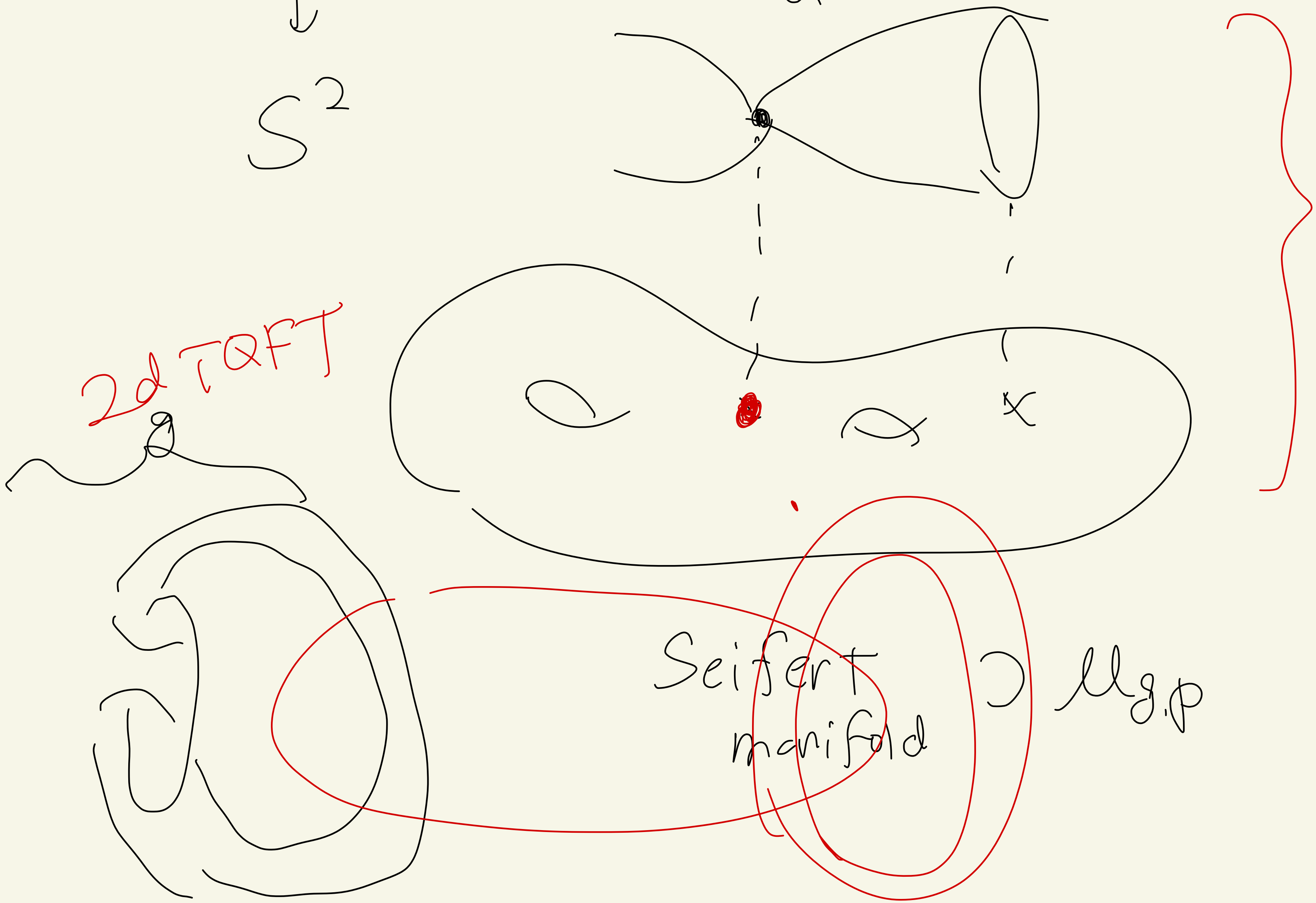
$\downarrow$

(operator dimension)

$$S^1 \to S^3$$
$$\downarrow$$
$$S^2$$

3d $\mathcal{N}=2$

$$Z_{S^3} = \int d\vec{\sigma}\; e^{A\vec{\sigma}^2 + B\vec{\sigma} + C}\, s_b(\vec{\sigma} \cdots)^{\pm 1}$$
$$= \sum_{\alpha} (\mathcal{H}_\alpha)^{-1} \mathcal{F}_\alpha^{1}$$



3d TQFT

3d $\mathcal{N}=4$ theory (+rank = 0)

$(Q_{\underbrace{\alpha}_{1,2}})_{A=1\sim4}$: 8 supercharge "hyperkähler"

R-sym. $\quad SO(4)_R^{\mathcal{N}=4} \simeq SU(2)_L \times SU(2)_R$

$\qquad\qquad\qquad\qquad\qquad\qquad\quad \cup \qquad\qquad \cup$

$\qquad\qquad\qquad\qquad\qquad\qquad U(1)_L \times U(1)_R$

$\qquad\qquad\qquad\qquad\qquad\qquad\quad L \qquad\quad R$

$SO(2)_R^{\mathcal{N}=2} \leadsto V$

$V = L + R$

$A = L - R$

non-R sym, $\quad U(1)_A$

commutes with Q $\quad$ mass parameter

$$Z^{\mathcal{U}}[\underbrace{\nu}, \underbrace{m}; s]$$

$U(1)$ mixing parameter

spin str on $\mathcal{U}$

$$R_\nu = V + \underbrace{\nu} A$$

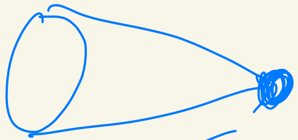
$\nu=1$: $2L$

$\nu=-1$: $2R$

$U(1)$ parametrization

HK manifold

3d $\mathcal{N}=4$ QFT $\rightsquigarrow \mathcal{M}_{VAC} \supset \mathcal{M}_{Higgs} \times \mathcal{M}_{Coulomb}$



CFT

$$SO(4)_R = \underbrace{SU(2)_R}_{\text{B-twist}} \times \underbrace{SU(2)_L}_{\text{A-twist}}$$

$\nu + im$

- $Z^{S^1 \times S^2}(\nu = \pm 1,\ m = 0) = \text{Hilbert series}\left[\mathcal{M}_{Higgs} / \mathcal{M}_{Coulomb}\right]$

ν, m parameter gone

$= 1$

$\mathcal{M}_{Higgs} = \mathcal{M}_{Coulomb} = \text{trivial}$
"rank 0"

TQFT ??

Main claim

unitary

3d $\mathcal{N}=4$ theory

$\mathcal{M}_{\text{Coulomb}} = \mathcal{M}_{\text{Higgs}} = \text{trivial}$

spin str.

$$Z^{\mathcal{M}_{g,p}}(\nu = \pm 1, m = 0; S)$$

F, R, S, T

spin

non-unitary

3d TQFT $\quad \text{TFT}_{\pm}$

$S_{\alpha\beta}, T_{\alpha\beta}$

$1+\sqrt{2}$

$1-\sqrt{2}$

Galois orbit

unitary TQFT

Examples

$\boxed{F\ R\ S\ T}$, σ

$\sigma(F)$, $\sigma(R)$, $\sigma(S)$, $\sigma(T)$

$\mathcal{T}_{\text{rank } 0}$	$\mathrm{TFT}_{\pm}[\mathcal{T}_{\text{rank } 0}]$	Set of $\{\lvert S^{\pm}_{0\alpha}\rvert\}$ $\mathcal{X}_\alpha$	$\exp(-F)$
$\mathcal{T}_{\min}$	(Lee-Yang)	$\{\sqrt{\frac{5+\sqrt{5}}{10}}, \sqrt{\frac{5-\sqrt{5}}{10}}\}$	$\sqrt{\frac{5-\sqrt{5}}{10}}$
$(U(1)_1+H)$	$\mathrm{Gal}_d(SU(2)_6)/\mathbb{Z}_2^f$ (with $d=\zeta_6^3$)	$\{2\zeta_6^1, 2\zeta_6^3\}$	$2\zeta_6^1$
$SU(2)_k^{\frac{1}{2}\oplus\frac{1}{2}}$ ($\lvert k\rvert>1$)	$\mathrm{Gal}_d(SU(2)_{4\lvert k\rvert-2})/\mathbb{Z}_2^f$ (with $d=\zeta_{4\lvert k\rvert-2}^{2\lvert k\rvert-1}$)	$\{2\zeta_{4\lvert k\rvert-2}^{2n-1}\}_{n=1}^{\lvert k\rvert}$	$2\zeta^1_{4\lvert k\rvert-2}$
$T[SU(2)]_{k_1,k_2}$	See the caption	$\{(\frac{1}{\sqrt{2}}\zeta^n_{\lvert k_1k_2-1\rvert-2})^{\otimes 2}\}_{n=1}^{\lvert k_1k_2-1\rvert-1}$	$\frac{1}{\sqrt{2}}\zeta^1_{\lvert k_1k_2-1\rvert-2}$
$\frac{T[SU(2)]}{SU(2)^{\text{diag}}_{\lvert k\rvert=3}}$	$(\text{Lee-Yang})^{\otimes 2}\otimes U(1)_2$	$\{\frac{1}{\sqrt{10}}^{\otimes 4}, \frac{5+\sqrt{5}}{10\sqrt{2}}^{\otimes 2}, \frac{5-\sqrt{5}}{10\sqrt{2}}^{\otimes 2}\}$	$\frac{5-\sqrt{5}}{10\sqrt{2}}$
$\frac{T[SU(2)]}{SU(2)^{\text{diag}}_{\lvert k\rvert=4}}$	$\frac{\mathrm{Gal}_{\zeta_{10}^7}(SU(2)_{10})\times SU(2)_2}{\mathbb{Z}_2^{\text{diag}}}$	$\{\frac{1}{2}, \frac{1}{2\sqrt{3}}^{\otimes 5}, \frac{3+\sqrt{3}}{12}^{\otimes 2}, \frac{3-\sqrt{3}}{12}^{\otimes 2}\}$	$\frac{3-\sqrt{3}}{12}$
$\frac{T[SU(2)]}{SU(2)^{\text{diag}}_{\lvert k\rvert=5}}$	$\mathrm{Gal}_d((G_2)_3)\otimes U(1)_{-2}$ $\left(d=\sqrt{\frac{5}{84}+\frac{1}{4\sqrt{21}}}\right)$	$\{\frac{1}{\sqrt{6}}^{\otimes 2}, \frac{1}{\sqrt{14}}^{\otimes 6}, \sqrt{\frac{5}{84}\pm\frac{1}{4\sqrt{21}}}^{\otimes 2}\}$	$\sqrt{\frac{5}{84}-\frac{1}{4\sqrt{21}}}$
$\frac{T[SU(2)]}{SU(2)^{\text{diag}}_{\lvert k\rvert\geq 6}}$	?	$\{\frac{1}{\sqrt{2\lvert k\rvert-4}}^{\otimes(\lvert k\rvert-3)}, \frac{1}{\sqrt{2\lvert k\rvert+4}}^{\otimes(\lvert k\rvert+1)}, (\frac{1}{\sqrt{8\lvert k\rvert-16}}+\frac{1}{\sqrt{8\lvert k\rvert+16}})^{\otimes 2}, (\frac{1}{\sqrt{8\lvert k\rvert-16}}-\frac{1}{\sqrt{8\lvert k\rvert+16}})^{\otimes 2}\}$	$\frac{1}{\sqrt{8\lvert k\rvert-16}}-\frac{1}{\sqrt{8\lvert k\rvert+16}}$

$\sqrt{3} \to -\sqrt{3}$

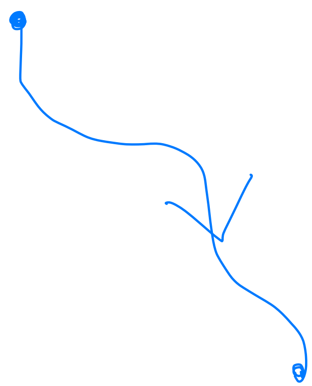
k: Chern-Simons level

Example 1

"minimal" $\mathcal{N}=4$ theory $\mathcal{T}_{min}$ [Gong-MY '18]

$$(\text{3D } \mathcal{N}=2 \text{ gauge theory, } U(1)_{k=-3/2} \text{ coupled to a chiral multiplet } \Phi \text{ of charge } +1)$$
$$\xrightarrow{\text{at IR}} (\text{3D } \mathcal{N}=4 \text{ superconformal field theory } \mathcal{T}_{\min}) \, .$$

UV 3d $\mathcal{N}=2$ $\quad SO(2)_R \times U(1)_J$



$\mathcal{M}_{\text{Coulomb}} = \mathcal{M}_{\text{Higgs}}$

$=$ trivial

4d $\mathcal{N}=4$

S^1

IR 3d $\mathcal{N}=4$

$SO(4)_R \simeq SU(2)_R \times SU(2)_R$

$Z^{S^1 \times S^2}$

$$\mathcal{I}^{\text{sci}}_{\mathcal{T}_{\min}}(q,\eta,\nu;s=1)=\sum_{\mathfrak{m}\in\mathbb{Z}}\oint_{|a|=1}\frac{da}{2\pi i a}\,q^{\frac{|\mathfrak{m}|}{6}}\left(a(-1)^{\mathfrak{m}}\right)^{-\frac{3\mathfrak{m}}{2}-\frac{|\mathfrak{m}|}{2}}\left(\eta q^{\frac{\nu}{2}}\right)^{-\mathfrak{m}}\text{P.E.}[f_{\text{single}}(q,a;\mathfrak{m})]$$

$$\text{with } f_{\text{single}}(q,a;\mathfrak{m}) := \frac{q^{\frac{1}{6}+\frac{|\mathfrak{m}|}{2}}a}{1-q}-\frac{q^{\frac{5}{6}+\frac{|\mathfrak{m}|}{2}}a^{-1}}{1-q}\,.$$

$\exp\left(\sum_{n=0}^{\infty}\frac{f_{\text{single}}(q^n;a^n,m^n)}{n}\right)$

$\nu=0$

$$\mathcal{I}^{\text{sci}}_{\mathcal{T}_{\min}}(q,u,\nu=0;s=1)=1-q+\left(\eta+\frac{1}{\eta}\right)q^{3/2}-2q^2+\left(\eta+\frac{1}{\eta}\right)q^{5/2}-2q^3+\ldots$$

$\nu=\pm1$

$\eta = e^m \to 1$

$$\mathcal{I}^{\text{sci}}_{\mathcal{T}_{\min}}(q,\eta,\nu,s=1)\big|_{\nu\to\pm1}$$

$$=1+\underbrace{\left(-1+\eta^{\mp1}\right)}_{\downarrow\ 0}q+\underbrace{\left(-2+\eta+\frac{1}{\eta}\right)}_{\downarrow\ 0}q^2+\underbrace{\left(-2+\eta+\frac{1}{\eta}\right)}_{\downarrow\ 0}q^3+\ldots$$

$\to 1$

S^3 partition function

U(1) q-dilog

$$\mathcal{Z}^{S^3_b}_{\mathcal{T}_{\min}}(b,m,\nu) = \int \frac{dZ}{\sqrt{2\pi\hbar}}\, \underbrace{e^{-\frac{Z^2+2Z\left(m+(i\pi+\frac{\hbar}{2})\nu\right)}{2\hbar}}\, \psi_\hbar(Z)}_{\mathcal{I}_\hbar}\ .$$

$\hbar = 2\pi i b^2$

$$\log \mathcal{I}_\hbar(Z,m,\nu) = \log\left(e^{-\frac{Z^2+2Z\left(m+(i\pi+\frac{\hbar}{2})\nu\right)}{2\hbar}}\, \psi_\hbar(Z)\right)$$

$$\xrightarrow{\hbar\to 0} \frac{1}{\hbar}\mathcal{W}_0(Z,m,\nu) + \mathcal{W}_1(Z,m,\nu) + \dots \quad \text{with}$$

$$\mathcal{W}_0 = \mathrm{Li}_2(e^{-Z}) - \frac{Z^2}{2} - Z(m+i\pi\nu)\ , \quad \mathcal{W}_1 = -\frac{1}{2}\log(1-e^{-Z}) - \frac{Z\nu}{2}\ .$$

$$\text{Bethe-vacua of } \mathcal{T}_{\min}\ :\ \left\{ z\ :\ \frac{(z-1)e^{-m-i\pi\nu}}{z^2} = 1 \right\}.$$

$\exp\left(\frac{\partial \mathcal{W}_0}{\partial Z}\right) = 1$

$m=0,\ \nu=\pm 1$

$$z_{\alpha=0} \to \frac{1}{2}\left(\sqrt{5}-1\right)\ , \quad z_{\alpha=1} \to \frac{1}{2}\left(-\sqrt{5}-1\right)$$

$$\log Z^{S^3_b} \underset{Z=Z_\alpha}{\sim} \frac{1}{\hbar} S_0^\alpha + S_1^\alpha + \hbar S_2^\alpha + \hbar^2 S_3^\alpha + \cdots$$

$$\begin{aligned}
&\mathcal{S}_0^{\alpha=0} \to \frac{7\pi^2}{30}, \quad \mathcal{S}_1^{\alpha=0} \to -\frac{1}{2}\log\left(\frac{5-\sqrt{5}}{2}\right), \quad \mathcal{S}_2^{\alpha=0} \to -\frac{7}{120}\,, \\
&\mathcal{S}_0^{\alpha=1} \to -\frac{17\pi^2}{30}, \quad \mathcal{S}_1^{\alpha=1} \to -\frac{1}{2}\log\left(\frac{5+\sqrt{5}}{2}\right)\,, \quad \mathcal{S}_2^{\alpha=1} \to -\frac{7}{120}\,, \\
&\mathcal{S}_{n\geq 3}^{\alpha} \to 0\,.
\end{aligned}$$

$T_{\alpha\alpha}$

$$\begin{aligned}
&\left\{\mathcal{F}_\alpha(m=0,\nu\to\pm 1, s=-1)\right\}_{\alpha=0,1} \longrightarrow \left\{\exp\left(-\frac{7i\pi}{60}\right), \exp\left(\frac{17i\pi}{60}\right)\right\}\,, \\
&\left\{\mathcal{H}_\alpha(m=0,\nu\to\pm 1, s=-1)\right\}_{\alpha=0,1} \longrightarrow \left\{\frac{5-\sqrt{5}}{2}, \frac{5+\sqrt{5}}{2}\right\}\,.
\end{aligned}$$

$S_{0\alpha}^{-2}$

$$\underbrace{\langle \mathcal{O}_\beta \rangle}_{\text{Wilson/'t Hooft line}} = \sum_\alpha (\mathcal{H}_\alpha)^{g-1} (\mathcal{F}_\alpha)^p \times \underbrace{W_\beta(z_\alpha)}_{\frac{S_{\beta\alpha}}{S_{0\alpha}}}$$

$$\mathcal{O}_{(p,q)} = z^p(1 - z^{-1})^q \ .$$

electric magnetic charge

$$\mathcal{O}_\beta^{A\text{-cycle}} |\alpha\rangle = W_\beta(\alpha) |\alpha\rangle$$

$$\mathcal{O}_{\alpha=0} = (\text{identity operator}) \ , \quad \mathcal{O}_{\alpha=1} = \mathcal{O}_{(p,q)=(1,0)} \ .$$

$$W_{\beta=0,1}(0) = 1 \ , \quad W_{\beta=0}(1) = z_0 = \frac{1}{2}(\sqrt{5} - 1) \ , \quad W_{\beta=1}(1) = z_1 = \frac{1}{2}(-\sqrt{5} - 1) \ .$$

$$S_{\beta\alpha} = W_\beta(\alpha) S_{\alpha 0}$$

Recovers S/T of Lee-Yang theory!
(2, 5) minimal model

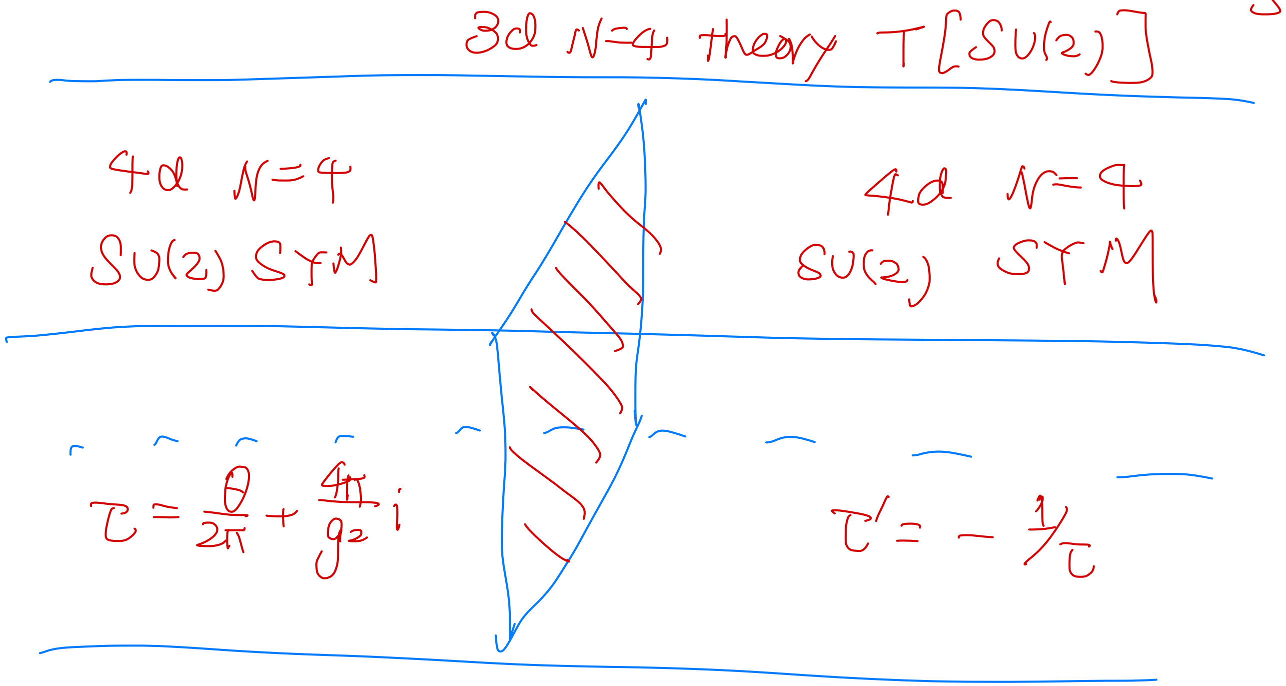
$$S=\begin{pmatrix}\sqrt{\frac{1}{10}\left(\sqrt{5}+5\right)} & -\sqrt{\frac{1}{10}\left(5-\sqrt{5}\right)}\\ -\sqrt{\frac{1}{10}\left(5-\sqrt{5}\right)} & -\sqrt{\frac{1}{10}\left(\sqrt{5}+5\right)}\end{pmatrix}, \quad T=\begin{pmatrix}1 & 0\\ 0 & \exp(-\frac{2\pi i}{5})\end{pmatrix}$$

More checks, e.g.

$$\left|\mathcal{Z}^{S^3_b}_{\mathcal{T}_{\rm min}}(b, m=0, \nu=\pm 1)\right| = \left|\mathcal{Z}^{S^3}_{\rm Lee-Yang}\right|.$$

Example 2

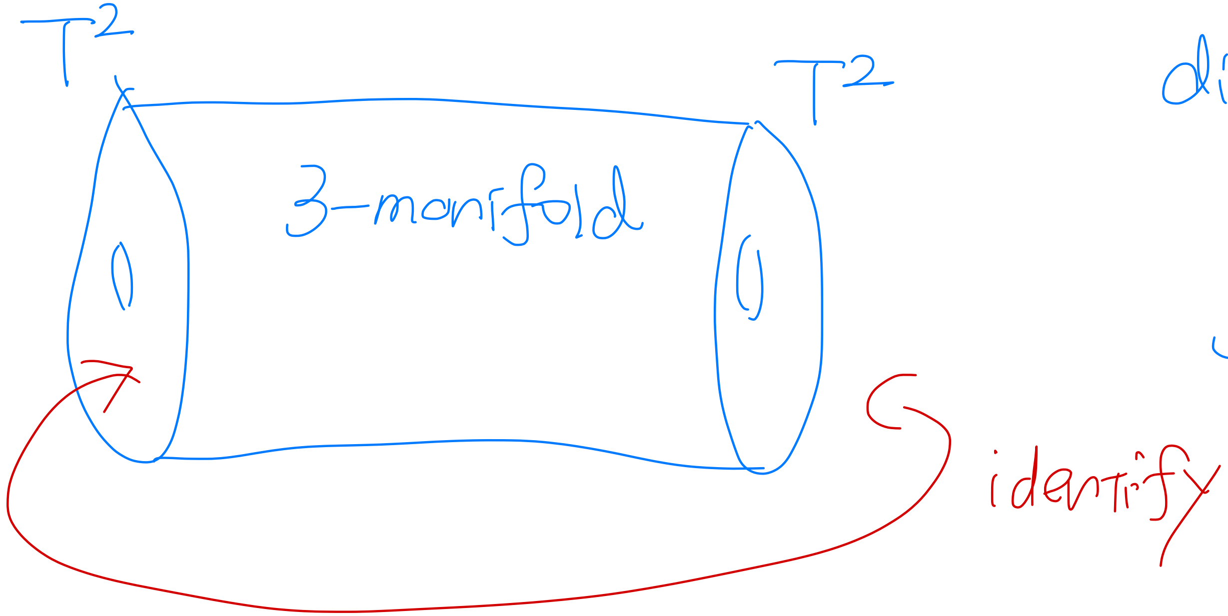
S-fold SCFT : $\mathcal{S}_{k:|k|\geq 3} := \dfrac{T[SU(2)]}{SU(2)_k^{\mathrm{diag}}}$



$SU(2)_C \times SU(2)_H$ flavor sym.

$\boxed{2}_C - (1) - \boxed{2}_H$

$SU(2)_k^{\mathrm{diag}}$: diagonal gauging of $SU(2)_C \times SU(2)_H$



More complicated formulas, e.g.

$$\mathcal{Z}_{S_k}^{S_b^3}(m,\nu) = \frac{1}{2}\int \frac{dXdZ}{2\pi\hbar}\mathcal{I}_\hbar(X,Z;W)\big|_{W=m+\nu(i\pi+\frac{\hbar}{2})}\ , \ \text{where}$$

$$\mathcal{I}_\hbar(X,Z;W) = 4\sinh(X)\sinh\left(\frac{2\pi i X}{\hbar}\right)\exp\left(\frac{2kX^2+2(X-Z)^2+W^2-(i\pi+\frac{\hbar}{2})W}{2\hbar}\right)$$

$$\times\left(\prod_{\epsilon_1,\epsilon_2=\pm1}\psi_\hbar\left(\epsilon_1 Z+\epsilon_2 X+\frac{W+i\pi+\hbar/2}{2}\right)\right)\psi_\hbar(-W+i\pi+\frac{\hbar}{2})\ .$$

$$\{\mathcal{H}_\alpha(m=0,\nu=\pm1)\}_{\alpha=0}^{2k+1} = \{(a_0^{-2})^{\otimes 2}, (a_1^{-2})^{\otimes(k-3)}, (a_2^{-2})^{\otimes(k+1)}, (a_3^{-2})^{\otimes 2}\}\,,$$

$$\{\mathcal{F}_\alpha(m=0,\nu=\pm1)\}_{\alpha=0}^{2k+1} = \{e^{2\pi i\delta}\exp(2\pi i h_\alpha)\}_{\alpha=0}^{2k+1} \quad \text{with}$$

$$\{h_\alpha\}_{\alpha=0}^{2k+1} = \left\{0, \frac{k+2}{4}, \frac{A^2}{4(k-2)}\Big|_{A=1,\ldots,k-3}, \frac{B^2}{4(k+2)}\Big|_{B=1,\ldots,k+1}, \frac{k+2}{4}, 0\right\}.$$

$$(a_0,a_1,a_2,a_3) = \left(\frac{1}{\sqrt{8(k-2)}}+\frac{1}{\sqrt{8(k+2)}}, \frac{1}{\sqrt{2(k-2)}}, \frac{1}{\sqrt{2(k+2)}}, \frac{1}{\sqrt{8(k-2)}}-\frac{1}{\sqrt{8(k+2)}}\right)$$

[$k=3,4,5$ Gang-Kim-Lee-Shim-MY '22

$k\geq 6$ Gang-Kim '23]

$$\{|S_{0\alpha}(\text{of TFT}[\mathcal{S}_k])|\}_{\alpha=0}^{2k+1} = \{a_0^{\otimes 2}, a_1^{\otimes(k-3)}, a_2^{\otimes(k+1)}, a_3^{\otimes 2}\}$$

$$(T_{\alpha\beta}\text{ of TFT}[\mathcal{S}_k]) = \delta_{\alpha,\beta}\exp(2\pi i h_\alpha)$$

[Gang-Kim '23]

$S^4 = 1 \quad (ST)^3 = 1$

(S of TFT[$\mathcal{S}_k$])

$$
= \left(\begin{array}{cc|ccccc|ccccc|cc}
a_0 & a_0 & a_1 & a_1 & a_1 & \cdots & a_1 & -a_2 & -a_2 & -a_2 & \cdots & -a_2 & a_3 & a_3 \\
a_0 & (-1)^k a_0 & -a_1 & a_1 & -a_1 & \cdots & (-1)^{k-3}a_1 & a_2 & -a_2 & a_2 & \cdots & (-1)^{k+2}a_2 & (-1)^k a_3 & a_3 \\ \hline
a_1 & -a_1 & & & & & & & & & & & -a_1 & a_1 \\
a_1 & a_1 & & & & & & & & & & & a_1 & a_1 \\
a_1 & -a_1 & & & 2a_1 \cos \frac{ij\pi}{k-2}\Big|_{1\le i,j\le k-3} & & & & & 0 & & & -a_1 & a_1 \\
\vdots & \vdots & & & & & & & & & & & \vdots & \vdots \\
a_1 & (-1)^{k-3}a_1 & & & & & & & & & & & (-1)^{k-3}a_1 & a_1 \\ \hline
-a_2 & a_2 & & & & & & & & & & & -a_2 & a_2 \\
-a_2 & -a_2 & & & & & & & & & & & a_2 & a_2 \\
-a_2 & a_2 & & & 0 & & & & & 2a_2 \cos \frac{ij\pi}{k+2}\Big|_{1\le i,j\le k+1} & & & -a_2 & a_2 \\
\vdots & \vdots & & & & & & & & & & & \vdots & \vdots \\
-a_2 & (-1)^{k+2}a_2 & & & & & & & & & & & (-1)^{k+1}a_2 & a_2 \\ \hline
a_3 & (-1)^k a_3 & -a_1 & a_1 & -a_1 & \cdots & (-1)^{k-3}a_1 & -a_2 & a_2 & -a_2 & \cdots & (-1)^{k+1}a_2 & (-1)^k a_0 & a_0 \\
a_3 & a_3 & a_1 & a_1 & a_1 & \cdots & a_1 & a_2 & a_2 & a_2 & \cdots & a_2 & a_0 & a_0
\end{array}\right),
$$

(T of TFT[$\mathcal{S}_k$])

$$
= \mathrm{diag}\left[\exp\left(2\pi i \left\{0, \frac{k+2}{4}, \frac{A^2}{4(k-2)}\Big|_{A=1,\dots,k-3}, \frac{B^2}{4(k+2)}\Big|_{B=1,\dots,k+1}, \frac{k+2}{4}, 0\right\}\right)\right].
$$

Haagerup-Izumi modular data for $k = 4m^2 + 4m + 3$, $m \in \mathbb{Z}$!

& decoupled $U(1)_{\pm 2}$

For even k,

$$[\mathbf{1}'] \times [\mathbf{1}'] = [\mathbf{1}] , \quad [\mathbf{1}'] \times [I_i] = [I_{k-2-i}] , \quad [\mathbf{1}'] \times [J_i] = [J_{k+2-i}] ,$$
$$[\mathbf{1}'] \times [V] = [V'] , \quad [\mathbf{1}'] \times [V'] = [V] ,$$
$$[V] \times [I_i] = [I_i] + \sum_{j:i+j=\text{even}} ([I_j] + [J_j]) + \begin{cases} [V] + [V'], & i = \text{even} \\ 0 , & i = \text{odd} \end{cases} ,$$
$$[V] \times [J_i] = -[J_i] + \sum_{j:i+j=\text{even}} ([I_j] + [J_j]) + \begin{cases} [V] + [V'], & i = \text{even} \\ 0 , & i = \text{odd} \end{cases} ,$$
$$[V'] \times [I_i] = [I_{k-2-i}] + \sum_{j:i+j=\text{even}} ([I_j] + [J_j]) + \begin{cases} [V] + [V'], & i = \text{even} \\ 0 , & i = \text{odd} \end{cases} ,$$
$$[V'] \times [J_i] = -[J_{k+2-i}] + \sum_{j:i+j=\text{even}} ([I_j] + [J_j]) + \begin{cases} [V] + [V'], & i = \text{even} \\ 0 , & i = \text{odd} \end{cases} ,$$
$$[V] \times [V] = [V'] \times [V'] = [\mathbf{1}] + [V] + [V'] + \sum_{i:\text{even}} ([I_i] + [J_i]) ,$$
$$[V] \times [V'] = [\mathbf{1}'] + [V] + [V'] + \sum_{i:\text{even}} ([I_i] + [J_i]) ,$$
$$[I_i] \times [I_j] = \delta_{i,j}([\mathbf{1}] + [V]) + \delta_{i+j,k-2}([\mathbf{1}'] + [V']) + \sum_{l\,:\,i+j+l=\text{even}} ([I_l] + [J_l])$$
$$+ \sum_{l:|j-l|=i \text{ or } j+l=\pm i (\text{mod } 2k-4)} [I_l] + \begin{cases} [V] + [V'], & i+j = \text{even} \\ 0, & i+j = \text{odd} \end{cases} ,$$
$$[J_i] \times [J_j] = \delta_{i,j}([\mathbf{1}] - [V]) + \delta_{i+j,k+2}([\mathbf{1}'] - [V']) + \sum_{l\,:\,i+j+l=\text{even}} ([I_l] + [J_l])$$
$$- \sum_{l:|j-l|=i \text{ or } j+l=\pm i (\text{mod } 2k+4)} [J_l] + \begin{cases} [V] + [V'], & i+j = \text{even} \\ 0, & i+j = \text{odd} \end{cases} ,$$
$$[I_i] \times [J_j] = \sum_{l\,:\,i+j+l=\text{even}} ([I_l] + [J_l]) + \begin{cases} [V] + [V'], & i+j = \text{even} \\ 0, & i+j = \text{odd} \end{cases}$$

$$N_{\alpha\beta}^{\gamma} = \sum_{\sigma} \frac{S_{\alpha\sigma} S_{\beta\sigma} S_{\gamma\sigma}^{*}}{S_{0\sigma}}$$

Verlinde formula