Eisenstein Series & Ensemble Averages in Holography

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Based on works with IPMU alumini

Meer Ashwinkumar

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Matthew Dodelson

Jacob M Leedom

Based on

- Ashwinkumar, Dodelson, Kidambi, Leedom + MY 2104.14710
- Ashwinkumar, Leedom + MY 2305.10224
- Ashwinkumar, Kidambi, Leedom + MY to appear

1. Physics  Number Theory

2. Ensemble Averages in Holography

AdS_3

BTZ black hole

initial inspiration:

[Afkhami-Jeddi, Cohen, Hartman, Tajdini ('20)
Maloney - Witten ('20)]

(standard) Narain theory

: even self-dual lattice

generalized Narain theory [Our works]

: general even lattice

Generalized Narain Theories

[Ashwinkumar - Dodelson - Kidambi - Leedom - MY ('21)
Ashwinkumar - Kidambi - Leedom - MY (to appear)]

recall: S^1 Narain theory

moduli = T-duality group

$$\Lambda = \left\{ \left(p^L = \frac{n}{2R} + wR \,,\; p^R = \frac{n}{2R} - \overset{\frown}{wR} \right) \;\middle|\; (n,w) \in \mathbb{Z} \right\}$$

$$Q(\{n,w\}) = p_L^2 - p_R^2 = \underbrace{2nw}_{\text{even quadratic form}} \in 2\mathbb{Z}$$

$$H = p_L^2 + p_R^2 = \frac{n^2}{2R^2} + 2w^2R^2$$

related by T-duality

$R=0$ $\quad$ $R=1/2$ $\quad$ moduli space $\quad$ $R=\infty$

Data

- Q : even quadratic form of signature (p,q) ← lattice Λ
- H : Hamiltonian ← moduli dependent

$$\begin{cases} Q = p_L^2 - p_R^2 \\ H = p_L^2 + p_R^2 \end{cases}$$

$\mathcal{M}_Q$: CFT moduli space

$$\mathcal{M}_Q = \underbrace{O(p,q;\mathbb{Z})}_{\text{(T-duality group)}} \backslash O_Q(p+q) / O(p)\times O(q)$$

Theta function

$$\vartheta_Q(\tau,\bar\tau;m) = \sum_{\ell\in\Lambda} e^{\pi i \tau_1 Q(\ell) - \pi\tau_2 \overbrace{H(\ell)}^{m\text{-dependence}}}$$

$$= \sum_{\ell\in\Lambda} \underbrace{q}_{e^{2\pi i\tau}}^{p_L^2(\ell)/2}\ \underbrace{\bar q}_{e^{2\pi i\bar\tau}}^{p_R^2(\ell)/2}$$

$(\tau = \tau_1 + i\tau_2)$

(※ m : CFT moduli ; τ : spacetime torus moduli)

Theta function

$$\vartheta_{Q,\alpha}(\tau,\bar\tau;m) = \sum_{\ell\in\Lambda} e^{\pi i \tau_1 Q(\ell+\alpha) - \pi\tau_2 H(\ell+\alpha)}$$

$$= \sum_{\ell\in\Lambda} \underbrace{q}_{e^{2\pi i\tau}}{}^{p_L^2(\ell+\alpha)/2}\, \underbrace{\bar q}_{e^{2\pi i\bar\tau}}{}^{p_R^2(\ell+\alpha)/2}$$

a set of basis functions

$$\alpha \in \mathcal{D} := \Lambda^*/\Lambda \quad \text{(discriminant)}$$

modular transformation

$$T: \vartheta_{Q,\alpha}(\tau+1; m) = e^{i\pi Q(\alpha,\alpha)} \vartheta_{Q,\alpha}(\tau, m)$$

$$S: \vartheta_{Q,\alpha}\left(-\frac{1}{\tau}, m\right) = \frac{e^{-i\pi\frac{p-q}{4}}}{\sqrt{|\det Q|}} \tau^{\frac{p}{2}} \bar{\tau}^{\frac{q}{2}} \sum_{\beta\in\mathcal{D}} e^{-2\pi i Q(\alpha,\beta)} \vartheta_{Q,\beta}(\tau; m)$$

T^2 partition function

$$Z_{Q,\alpha}(\tau,\bar{\tau};m) = \frac{\vartheta_{Q,\alpha}(\tau,\bar{\tau};m)}{\eta(\tau)^{p}\,\bar{\eta}(\bar{\tau})^{q}}$$

$$T : Z_{Q,\alpha}(\tau+1; m) = e^{i\pi Q(\alpha,\alpha)} e^{-\frac{2\pi i(p-8)}{24}} Z_{Q,\alpha}(\tau, m)$$

$$S : Z_{Q,\alpha}\left(-\frac{1}{\tau}, m\right)$$

$$= \frac{1}{\sqrt{|\det Q|}} \sum_{\beta\in\mathcal{D}} e^{-2\pi i Q(\alpha,\beta)} Z_{Q,\beta}(\tau; m)$$

" Weil representation " of metaplectic group

Ensemble Average

[Ashwinkumar, Dodelson, Kidambi, Leedom, MY ('21)]

Ensemble Average

Haar measure / Zamolodchikov metric

$$\frac{1}{\mathrm{Vol}(\mathcal{M}_{Q,\alpha})}\underbrace{\int_{\mathcal{M}_{Q,\alpha}} \overbrace{[dm]} \,\vartheta_{Q,\alpha}(\tau,\bar\tau;m)}_{\text{ensemble average over CFT moduli space}} = ??$$

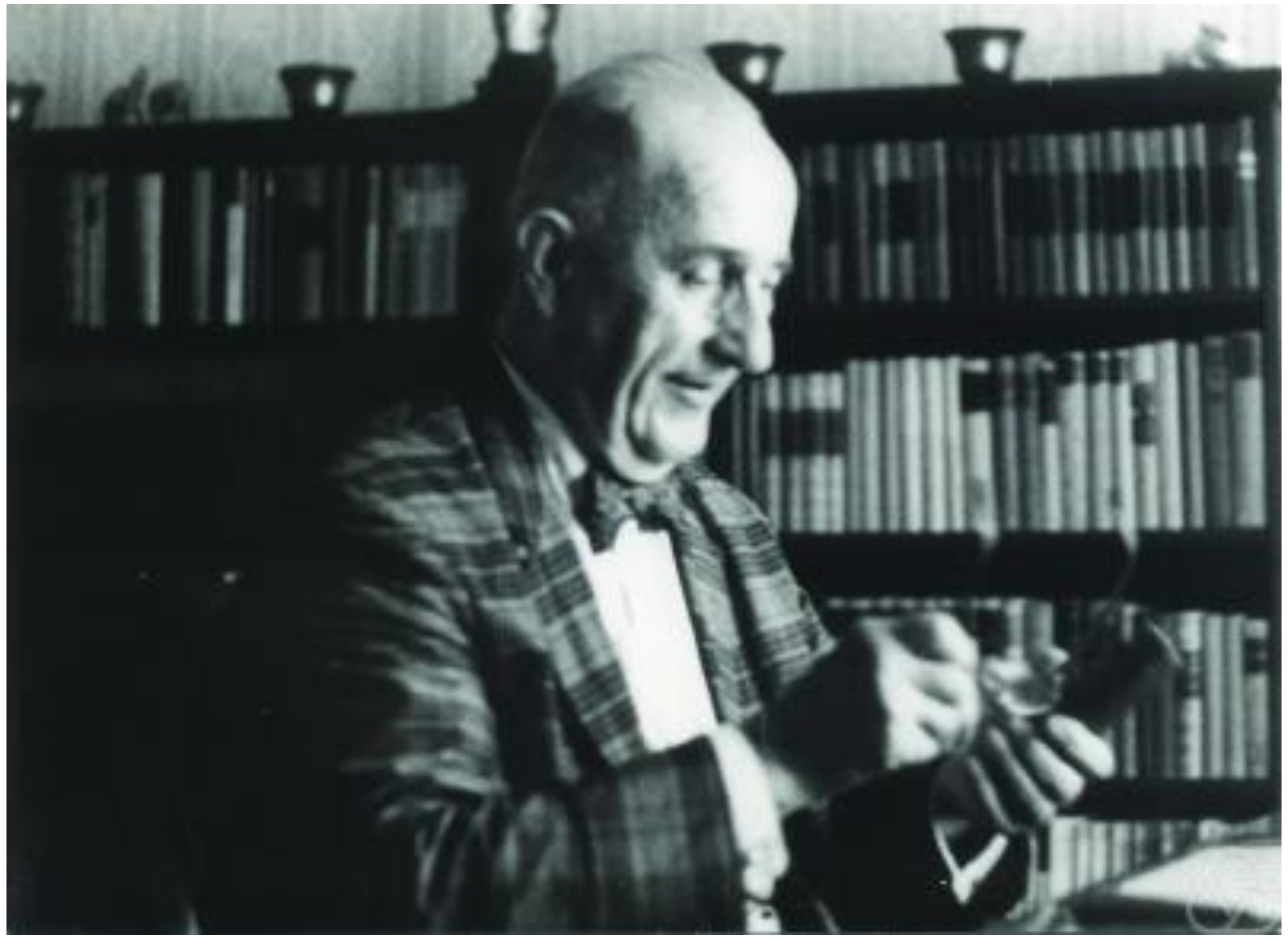
T-duality group should preserve α

Siegel-Weil formula

[Siegel ('51)
Weil ('64)]

$$\underbrace{\frac{1}{\mathrm{Vol}(\mathcal{M}_{Q,\alpha})}\int_{\mathcal{M}_{Q,\alpha}}[dm]\,\vartheta_{Q,\alpha}(\tau,\bar{\tau};m)}_{\text{ensemble average over CFT moduli space}} = \underbrace{E_{Q,\alpha}(\tau,\bar{\tau})}_{\text{non-hol. Eisenstein series}}$$

Carl Ludwig Siegel (1896 – 1981)



"modern version of Gauss"

Taniyama (translation by MY):

"While Weil is a genius in putting together existing theories, he lacks true originalities. and cannot match the creativity of Siegel."

Siegel-Weil formula

[Siegel ('51)
Weil ('64)]

$$\underbrace{\frac{1}{\mathrm{Vol}(\mathcal{M}_{Q,\alpha})}\int_{\mathcal{M}_{Q,\alpha}}[dm]\,\vartheta_{Q,\alpha}(\tau,\bar\tau;m)}_{\text{ensemble average over CFT moduli space}} = \underbrace{E_{Q,\alpha}(\tau,\bar\tau)}_{\text{non-hol. Eisenstein series}}$$

Poincaré sum

$$\begin{pmatrix} * & * \\ c & d \end{pmatrix} : \quad SL(2,\mathbb{Z}) / \underset{\mathbb{Z}}{\underset{\cong}{\Gamma_\infty}}$$

$$E_{Q,\alpha}(\tau) := \delta_{\alpha \in \Lambda} + \sum_{\substack{(c,d)=1 \\ c>0}} \frac{\gamma_{Q,\alpha}(c,d)}{(c\tau+d)^{\frac{p}{2}} (c\bar{\tau}+d)^{q/2}}$$

$$\gamma_{Q,\alpha}(c,d) = \frac{e^{\frac{\pi i (p-q)}{4}}}{\sqrt{|\det Q|}} \, c^{-\frac{p+q}{2}} \sum_{\ell \in \Lambda / c\Lambda} \exp\left[-\pi i \frac{d}{c} Q(\ell + \alpha) \right]$$

In fact, modular form expected

$$\vartheta_{Q,\alpha}(\tau+1;m) = \underbrace{e^{i\pi Q(\alpha,\alpha)}} \vartheta_{Q,\alpha}(\tau,m)$$

$$\underbrace{\vartheta_{Q,\alpha}\left(-\frac{1}{\tau},m\right) = \frac{e^{-i\pi\frac{p-q}{4}}}{\sqrt{|\det Q|}} \tau^{\frac{p}{2}} \bar{\tau}^{\frac{q}{2}} \sum_{\beta\in\mathcal{D}} e^{-2\pi i Q(\alpha,\beta)}} \vartheta_{Q,\beta}(\tau;m)$$

moduli independent!

$$\langle\vartheta_{Q,\alpha}\rangle(\tau+1) = e^{i\pi Q(\alpha,\alpha)} \langle\vartheta_{Q,\alpha}\rangle(\tau)$$

$$\langle\vartheta_{Q,\alpha}\rangle\left(-\frac{1}{\tau}\right) = \frac{e^{-i\pi\frac{p-q}{4}}}{\sqrt{|\det Q|}} \tau^{\frac{p}{2}} \bar{\tau}^{\frac{q}{2}} \sum_{\beta\in\mathcal{D}} e^{-2\pi i Q(\alpha,\beta)} \langle\vartheta_{Q,\beta}\rangle(\tau)$$

"modular form" for $\Gamma(\underset{\sim}{L})$

$L Q^{-1}$: integral

$$\langle\vartheta_{Q,\alpha}\rangle(\tau+1) = \overbrace{e^{i\pi Q(\alpha,\alpha)}}\langle\vartheta_{Q,\alpha}\rangle(\tau)$$

$$\langle\vartheta_{Q,\alpha}\rangle\left(-\frac{1}{\tau}\right) = \underbrace{\frac{e^{-i\pi\frac{p-q}{4}}}{\sqrt{|\det Q|}}\,\tau^{\frac{p}{2}}\,\bar{\tau}^{\frac{q}{2}}\sum_{\beta\in\mathcal{D}} e^{-2\pi i Q(\alpha,\beta)}\langle\vartheta_{Q,\beta}\rangle(\tau)}$$

$$E_{Q,\alpha}(\tau) := \delta_{\alpha\in\Lambda} + \sum_{\substack{(c,d)=1\\ c>0}} \frac{\overbrace{\gamma_{Q,\alpha}(c,d)}}{(c\tau+d)^{\frac{p}{2}}(c\bar{\tau}+d)^{q/2}}$$

reproduce behavior at cusps

$\tau \sim -\frac{d}{c}$

proof of Siegel-Weil formula

$$f(\tau,\bar\tau) := \langle \vartheta_{Q,\alpha} \rangle (\tau,\bar\tau) - E_{Q,\alpha}(\tau,\bar\tau)$$

(i) $f(\tau,\bar\tau)$: non-hol. modular form wt $(p,8)$ for $\Gamma(L) \subset SL(2,\mathbb{Z})$

(ii) $f(\tau,\bar\tau)$: square-integrable in $\mathbb{H}/\Gamma(L)$

(iii) $$\left(\square_{\frac{p-8}{2}} + \frac{\left(\frac{p+8}{4}-1\right)(p+8)}{4} \right) \left(\tau_2^{\frac{p+8}{2}} f(\tau) \right) = 0$$

$\rightsquigarrow$ $f = 0$ //

Holographic Dual

[Ashwinkumar, Dedelson, Kidambi, Leedom, MY ('21)] boundary

bulk

generalized

Narain theory

$\theta_{Q,\alpha}(\tau,\bar{\tau};m)$

average $\Downarrow$

averaged

Narain theory

$E_{Q,\alpha}(\tau,\bar{\tau})$

boundary

bulk

generalized

Narain theory

$\theta_{Q,\alpha}(\tau,\bar\tau;m)$

average $\Downarrow$

averaged

Narain theory

$E_{Q,\alpha}(\tau,\bar\tau)$ $\Longleftrightarrow$ $S_{CS}=\sum_{I,J}\frac{Q_{IJ}}{8\pi}\int A_I\wedge dA_J$

"Abelian CS"

$U(1)^{p+q}$

boundary

generalized Narain theory

$\theta_{Q,\alpha}(\tau,\bar{\tau};m)$

average

$\Downarrow$

averaged Narain theory

$E_{Q,\alpha}(\tau,\bar{\tau})$

bulk

"Abelian Maxwell – CS"

$S_{CS}+S_{kin}$ $\quad \sum_{I,J} m_{IJ}\int F_I\wedge *F_J$

$\Downarrow$ average

"Abelian CS"

$$S_{CS}=\sum_{I,J}\frac{Q_{IJ}}{8\pi}\int A_I\wedge dA_J$$

$U(1)^{p+q}$

Holographic dual after averaging:

$$S_{CS} = \sum_{I,J=1}^{p+q} \frac{1}{4\pi} Q_{IJ} \int A_I \wedge dA_J$$

* defined from Q

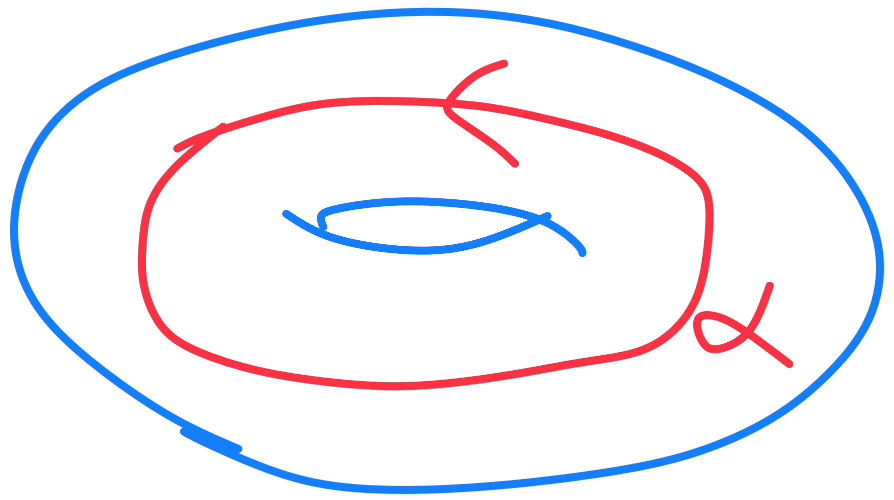
* CS term matches non-modular inv.

* Sum over geom: $SL(2,\mathbb{Z})$ BHs (BH Farey Tail)

"exotic gravity" (solid torus)

thermal AdS_3, BTZ BH, ...

* $|E_{Q,\underbrace{\alpha}_{\text{Wilson line insertion}}}(\tau)\rangle$: wavefunction of Abelian CS on $T^2 = \partial(D^2 \times S^1)$



anyon

$$\alpha \in \underbrace{\Lambda^*}_{\text{charge}} / \underbrace{\Lambda}_{\text{gauge equiv.}}$$

(Supplemented by $\dfrac{1}{\eta^p \bar{\eta}^8}$: Brown-Henneaux Virasoro modes)

$$* \begin{cases} T: Z_{Q,\alpha}(\tau+1) = e^{i\pi Q(\alpha,\alpha)} e^{-\frac{2\pi i(p-q)}{24}} Z_{Q,\alpha}(\tau, m) \\ S: Z_{Q,\alpha}(-\frac{1}{\tau}) = \frac{1}{\sqrt{|\det Q|}} \sum_{\beta\in\mathcal{D}} e^{-2\pi i Q(\alpha,\beta)} Z_{Q,\beta}(\tau; m) \end{cases}$$

matched by modular T/S matrix
for Abelian anyons

$$T_{\alpha\beta} = \underbrace{e^{i\pi Q(\alpha,\alpha)}}_{\text{topological spin}} e^{-\frac{2\pi i(p-q)}{24}} \delta_{\alpha\beta} \; ; \; S_{\alpha\beta} = e^{-2\pi i Q(\alpha,\beta)}$$

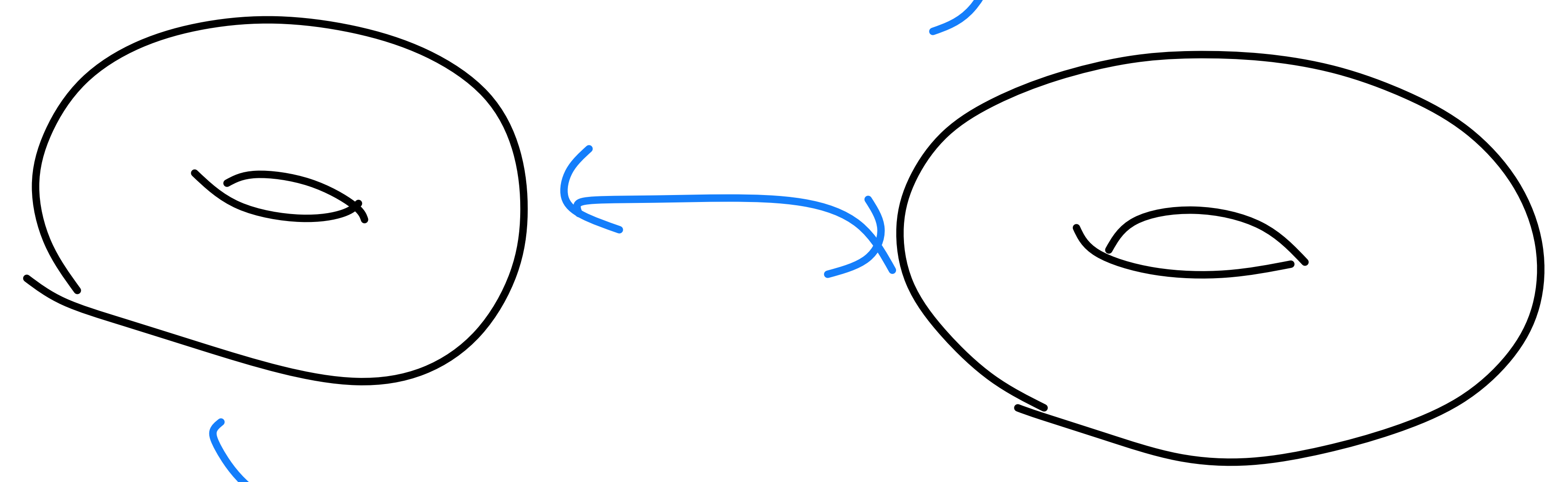
$M_{(c,d)}$

sum over

$SL(2,\mathbb{Z})$ BH

lens space partition function

$$\langle \psi_0 | \begin{pmatrix} * & * \\ c & d \end{pmatrix} | \psi_\alpha \rangle$$



$M_{(c,d)}$

$$E_{Q,\alpha}(\tau) := \delta_{\alpha\in\Lambda} + \sum_{\substack{(c,d)=1 \\ c>0}} \frac{\gamma_{Q,\alpha}(c,d)}{(c\tau+d)^{\frac{p}{2}}\,(c\bar{\tau}+d)^{\frac{q}{2}}}$$

$$\gamma_{Q,\alpha}(c,d) = \frac{e^{\frac{\pi i(p-q)}{4}}}{\sqrt{|\det Q|}}\, c^{-\frac{p+q}{2}} \sum_{\ell\in\Lambda/c\Lambda} \exp\left[-\pi i \frac{d}{c} Q(\ell+\alpha)\right]$$

Emergent Global Symmetries

[Ashwinkumar, Leedom, MY ('23)]

QG : "No exact global symmetries"

[Swampland conjecture: e.g. Harlow-Ooguri ('18)
for recent discussion]

QG: "No exact global symmetries"

[Swampland conjecture: e.g. Harlow-Ooguri ('18) for recent discussion]

However, Abelian CS theories have zero-form / one-form symmetries

($\because$ not an immediate contradiction since bulk is not Einstein gravity)

Symmetries of anyons

- $\alpha \in \mathcal{D} = \Lambda^*/\Lambda$; anyons w/ spin $\theta(\alpha) = e^{\pi i Q(\alpha)}$

- symmetry : $\sigma \in \mathrm{Aut}(\mathcal{D})$

  $\alpha \longmapsto \sigma \cdot \alpha$ s.t. $\theta(\alpha) = \theta(\sigma \cdot \alpha)$

$$\alpha \xrightarrow{\sigma} \sigma(\alpha)$$

$\sigma \in \mathrm{Aut}(\mathcal{D})$ then

After average

- $E_{Q,\, \sigma \cdot \alpha}(\tau, \bar{\tau}) = E_{Q,\, \alpha}(\tau, \bar{\tau})$ ☺

$\sigma \in \mathrm{Aut}(\mathcal{D})$ then

After average

- $E_{Q,\sigma\cdot\alpha}(\tau,\bar{\tau}) = E_{Q,\alpha}(\tau,\bar{\tau})$

Before average

$$\vartheta_{Q,\sigma\cdot\alpha}(\tau,\bar{\tau};m) \neq \vartheta_{Q,\alpha}(\tau,\bar{\tau};m)$$??

NOT a symmetry of a given CFT

$\sigma \in \mathrm{Aut}(\mathcal{D})$ then

After average

• $E_{Q,\sigma\cdot\alpha}(\tau,\bar{\tau}) = E_{Q,\alpha}(\tau,\bar{\tau})$

Before average

$$\vartheta_{Q,\sigma\cdot\alpha}(\tau,\bar{\tau};m) \neq \vartheta_{Q,\alpha}(\tau,\bar{\tau};m)$$

??

$$\vartheta_{Q,\sigma\cdot\alpha}(\tau,\bar{\tau};\underbrace{\sigma\cdot m}) = \vartheta_{Q,\alpha}(\tau,\bar{\tau};m)$$

if T-duality origin relation between different theories

In general,

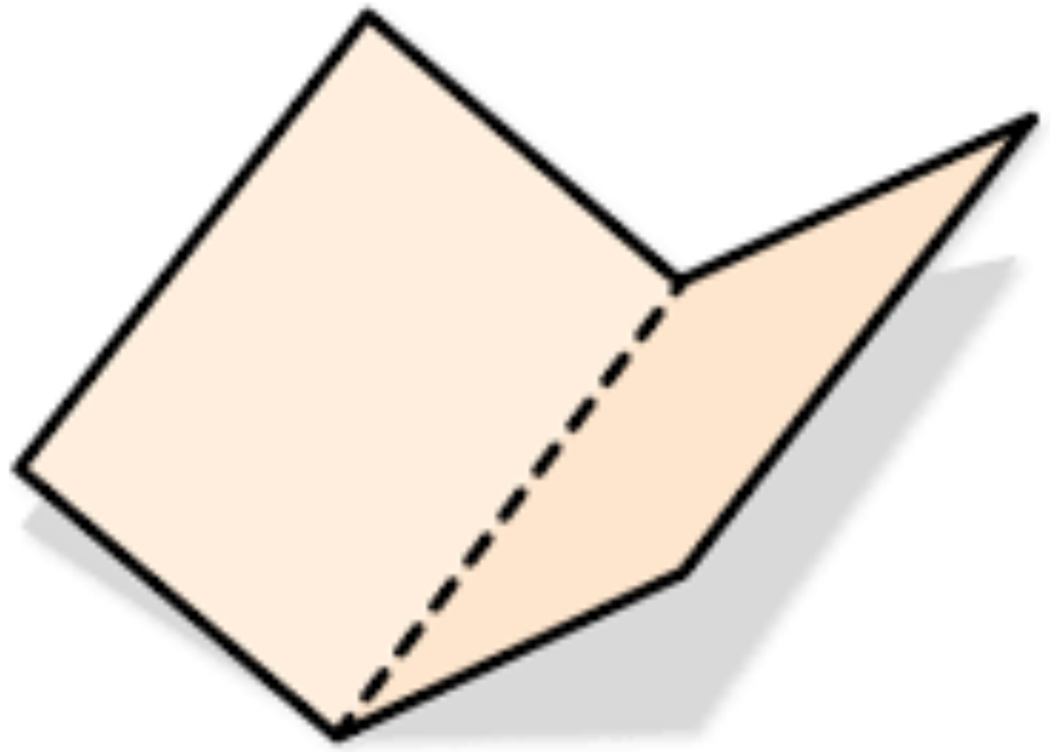
"ensemble sym."

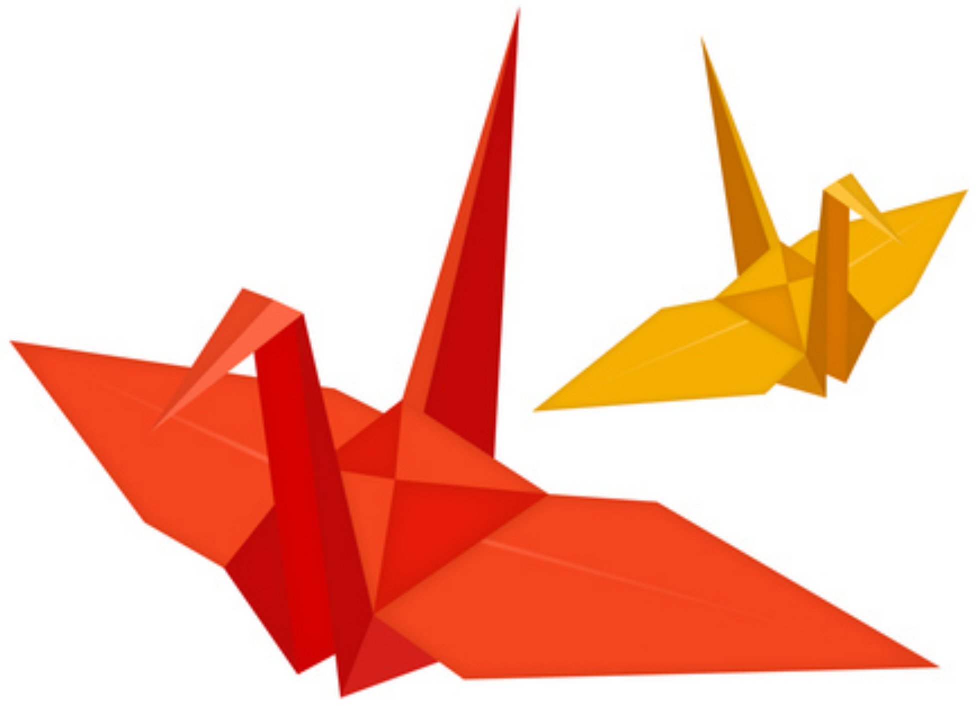
average over $\mathcal{M} \circlearrowleft \sigma \in G$ s.t. $[d(\sigma(m))] = [dm]$

of $\mathcal{O}(m, \alpha)$ s.t. $\mathcal{O}(\sigma \cdot m, \sigma \cdot \alpha) = \mathcal{O}(m, \alpha)$

$$\rightsquigarrow \langle \mathcal{O} \rangle (\alpha) = \int [dm]\, \mathcal{O}(m, \alpha) = \langle \mathcal{O} \rangle (\sigma \cdot \alpha)$$

"duality origami"



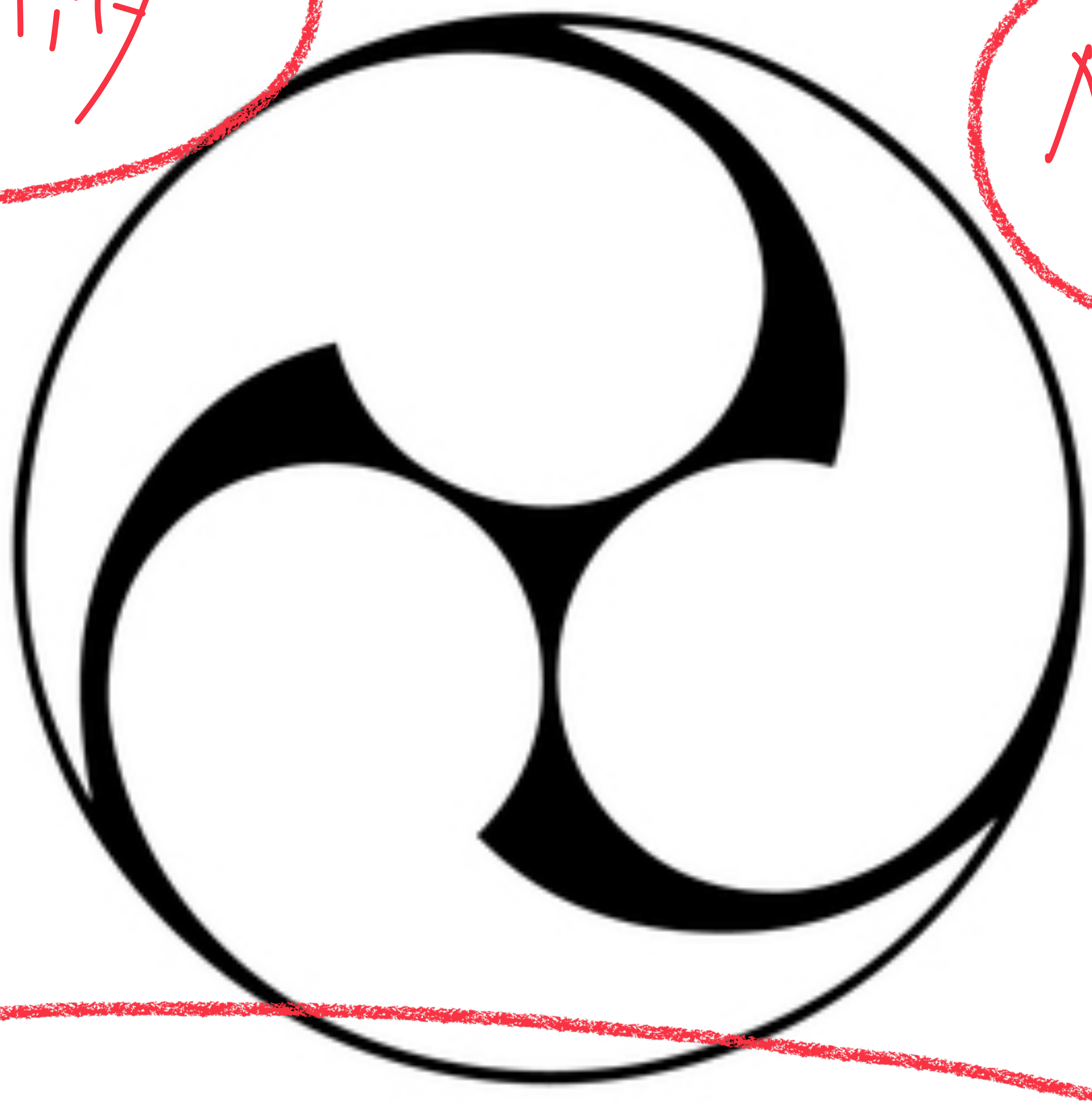


ensemble symmetry $\xrightarrow{\text{ensemble average}}$ global symmetry

(e.g. T-duality)

T-duality

No global sym,



distance conjecture

$\vartheta_{Q,\alpha}(\tau)$ $\xrightarrow{\infty\text{-distance}}$ asymptotic at cusps as $\tau \to -d/c$

ensemble average

lens inv.

$\gamma_{Q,\alpha}(c,d)$

$$\langle \vartheta_{Q,\alpha} \rangle(\tau) = \delta_{\alpha\in\Lambda} + \sum_{\substack{(c,d)=1 \\ c>0}} \frac{\gamma_{Q,\alpha}(c,d)}{(c\tau+d)^{\frac{p}{2}}\,(c\bar{\tau}+d)^{q/2}}$$

building block

Deviations from ensemble average?

Spectral Decomposition [Benjamin, Collier, Fritzpatrick, Maloney, Perlmutter ('21)]

$$f(\tau,\bar\tau) := \tau_2^{\frac{p+q}{2}}\left(\vartheta_{Q,\alpha}(\tau,\bar\tau;m) - E_{Q,\alpha}(\tau,\bar\tau)\right)$$

$$= \sum_{i=1}^{\infty}\langle f, u_i\rangle u_i(\tau) + \sum_{j=1}^{N}\langle f, v_j\rangle v_j(\tau)$$

$$+ \sum_{\mathbb{R}} \frac{1}{4\pi}\int_{-\infty}^{\infty}\left\langle f, E_{a_k}\left(\tau, \tfrac{1}{2}+it\right)\right\rangle E_{a_k}\left(\tau; \tfrac{1}{2}+it\right)dt$$

subleading corrections break emergent global sym, Our setup: proposed for

$AdS_3 \times K_7$ compactifications in actual string theory!

[Gukov-Martinec-Moore-Strominger '04]

Ideal setup for precision

Swampland / Ensemble discussion?

Gauging Global Symmetries

[Ashwinkumar, Kidambi, Leedom, MY (to appear)]

cf. [Dong, Hartman, Jiang ('21)]

* Gauge global sym. in the bulk?

- $G \in \mathrm{Aut}(\mathcal{D})$ anyon sym.
- $\mathcal{D} \rightarrow$ " $\underbrace{(\mathcal{D}/\sim)}_{\text{mod by } G} \oplus \underbrace{\mathcal{D}_G}_{G\text{-anyons}}$ "

* Gauge global sym. in the bulk?

- $G \in \mathrm{Aut}(\mathcal{D})$ anyon sym.
- $\mathcal{D} \to$ " $\underbrace{(\mathcal{D}/\sim)}_{\text{mod by } G} \oplus \underbrace{\mathcal{D}_G}_{G\text{-anyons}}$ "

* orbifolding generalized Narain theories

- $G \in \mathrm{Aut}(\mathcal{Q})$ lattice symmetry
- $\underbrace{(\text{untwisted sector})}_{\text{projected by } G} + \underbrace{(\text{twisted sectors})}_{\text{rep. of } G}$

We can orbifold ensemble average stories

Ashwinkumar – Kidambi – Leedom – M.Y.

(to appear)

New generalized $\left\{\begin{array}{l}\text{Eisenstein series} \\ \text{Siegel-Weil formula}\end{array}\right.$

(cf. Zemels ('21) for theta functions)

Summary

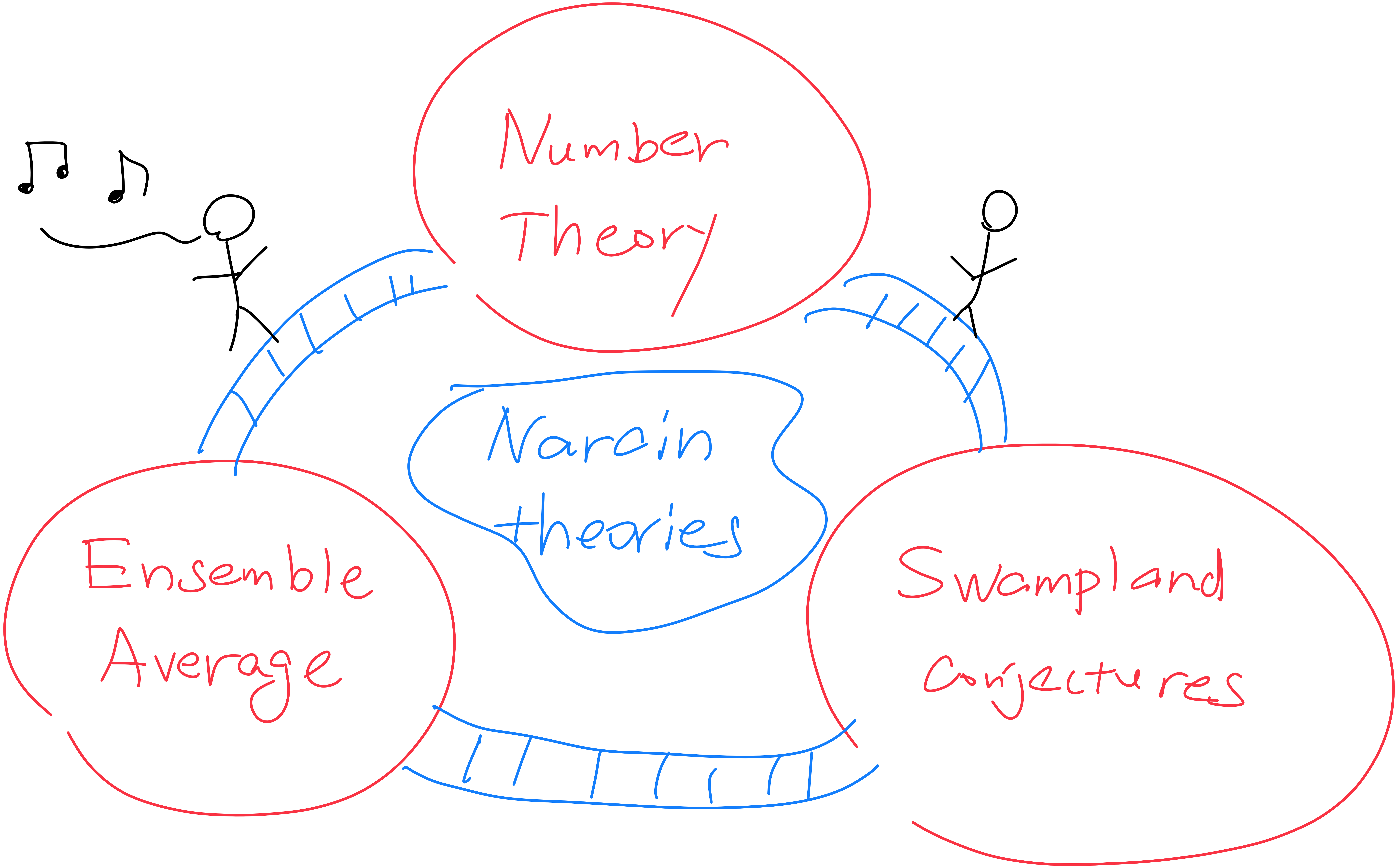
"Precision Ensemble Average"

Generalized Narain theory

Abelian CS in bulk

NT Siegel-Weil formula

non-hol. Eisenstein series, ...


Number Theory
Narain theories
Ensemble Average
Swampland Conjectures