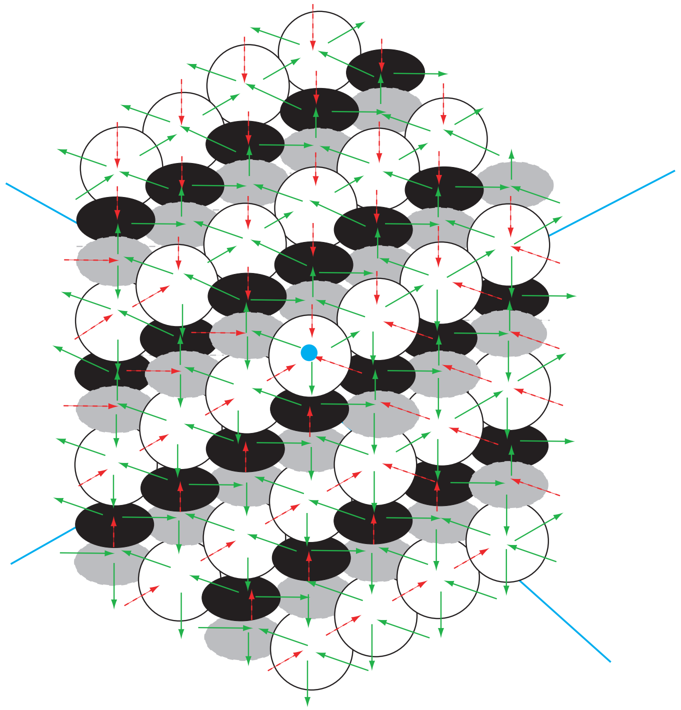

$$\psi^{(a)}(z)\,\psi^{(b)}(w) = \psi^{(b)}(w)\,\psi^{(a)}(z)\ ,$$
$$\psi^{(a)}(z)\,e^{(b)}(w) \simeq \varphi^{b\Rightarrow a}(\Delta)\,e^{(b)}(w)\,\psi^{(a)}(z)\ ,$$
$$e^{(a)}(z)\,e^{(b)}(w) \sim (-1)^{|a||b|}\varphi^{b\Rightarrow a}(\Delta)\,e^{(b)}(w)\,e^{(a)}(z)\ ,$$
$$\psi^{(a)}(z)\,f^{(b)}(w) \simeq \varphi^{b\Rightarrow a}(\Delta)^{-1}\,f^{(b)}(w)\,\psi^{(a)}(z)\ ,$$
$$f^{(a)}(z)\,f^{(b)}(w) \sim (-1)^{|a||b|}\varphi^{b\Rightarrow a}(\Delta)^{-1}\,f^{(b)}(w)\,f^{(a)}(z)\ ,$$
$$\left[e^{(a)}(z), f^{(b)}(w)\right\} \sim -\delta^{a,b}\frac{\psi^{(a)}(z)-\psi^{(b)}(w)}{z-w}\ ,$$

From Counting to Algebras: Tale of Calabi-Yau Geometries

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ICBS, BIMSA

July 18, 2024

Hirosi Ooguri + MY

(0811.2810 [hep-th])

MY's Ph.D. thesis

(1002.1709 [hep-th])

MY's Master thesis

(0803.4474 [hep-th])

Wei Li + MY

(2003.08909 [hep-th])

Dimitry Galakhov + MY

(2008.07006 [hep-th])

Dimitry Galakhov+Wei Li + MY

(2106.01230 [hep-th])

(2108.10286 [hep-th])

(2206.13340 [hep-th])

Jiakang Bao + Rak-Kyeong Seong + MY

(2401.02792 [hep-th])

Jiakang Bao + MY

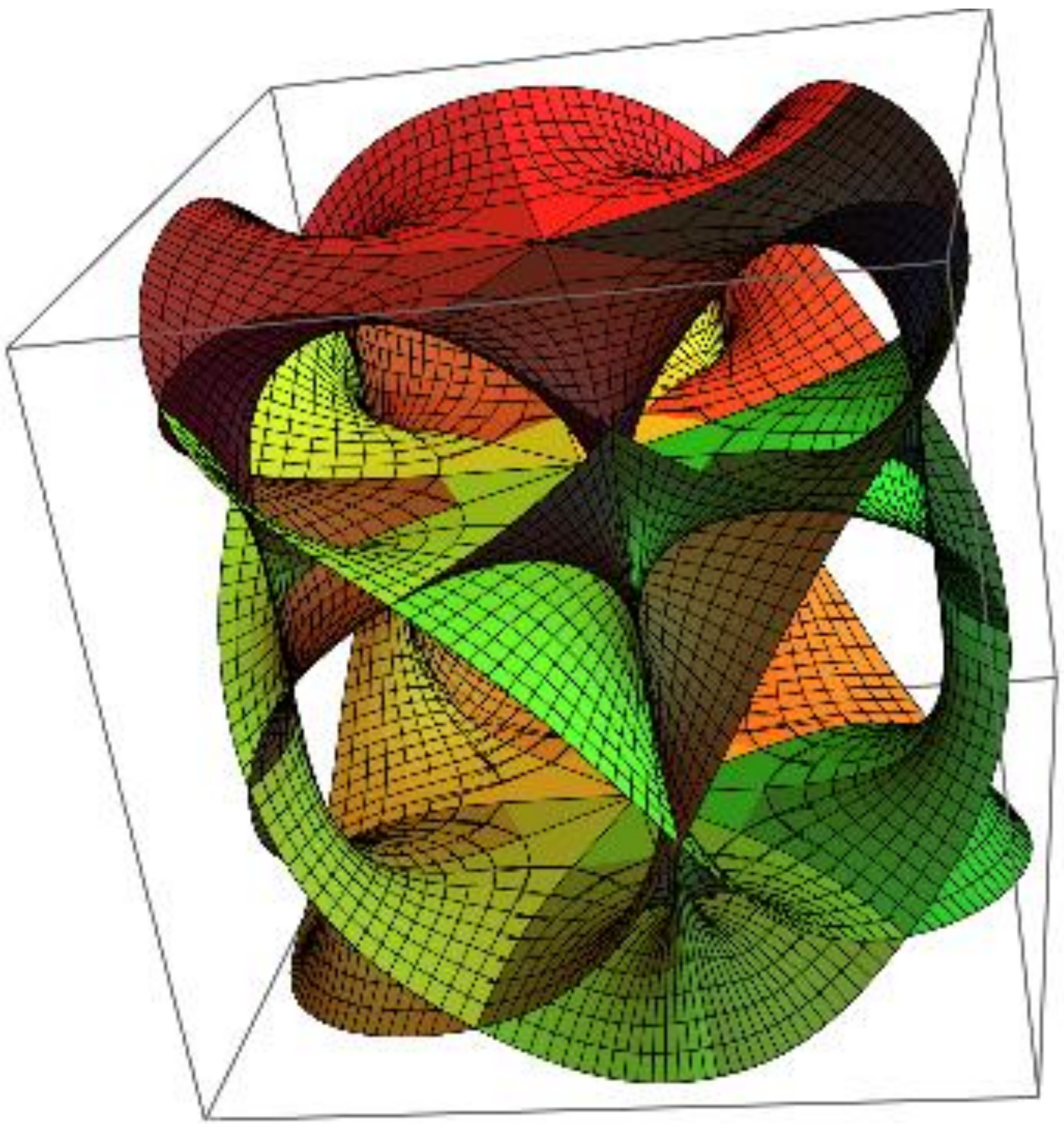
(To Appear)

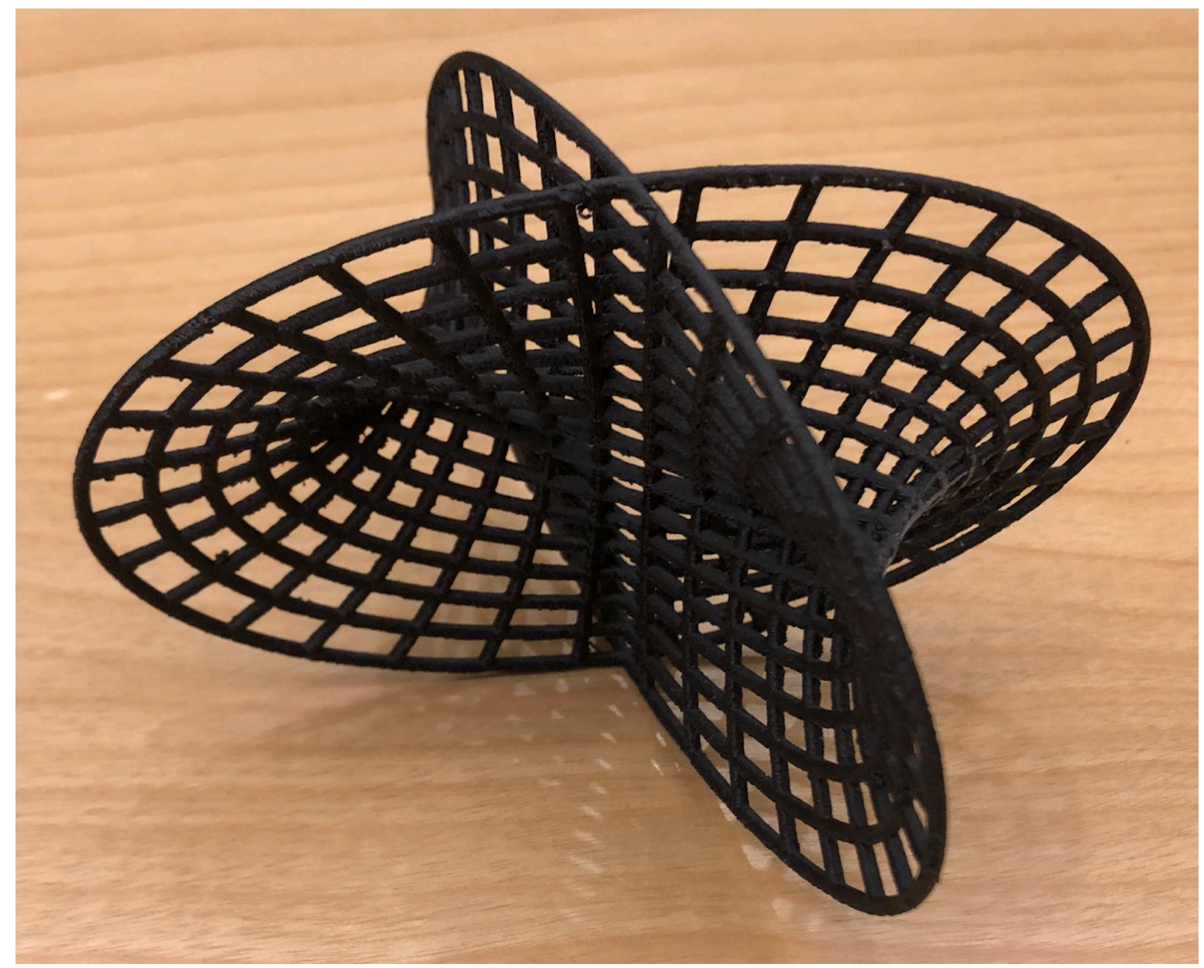
⋯ and many works in the literature

Overview


Eugenio CALABI

Calabi-Yau





Quantum Calabi-Yau?

"Geon" (Wheeler '55), "Space-time Foam" (Hawking '78),⋯

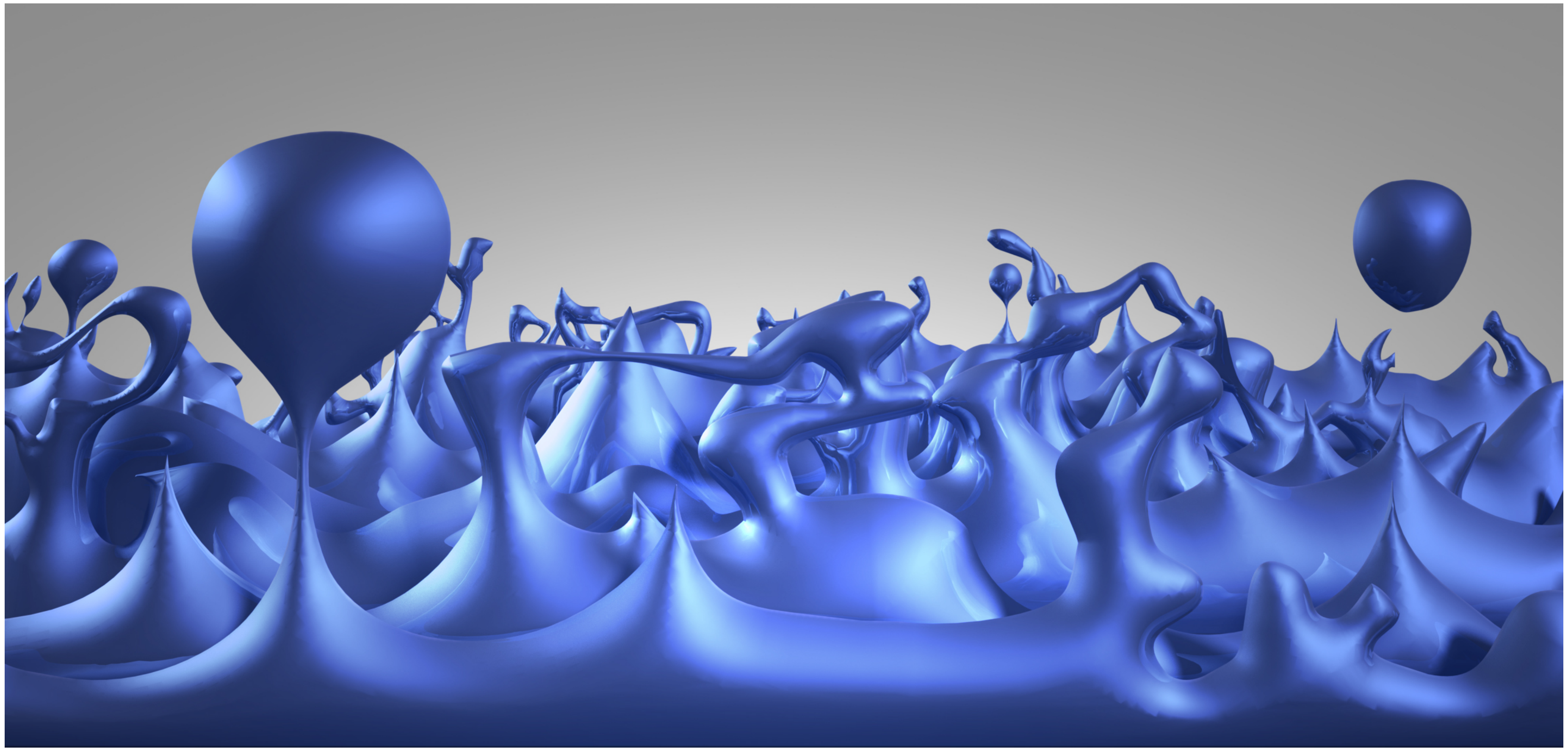


figure from NASA

Quantum Calabi-You?

“Geon” (Wheeler ’55), “Space-time Foam” (Hawking ’78),⋯

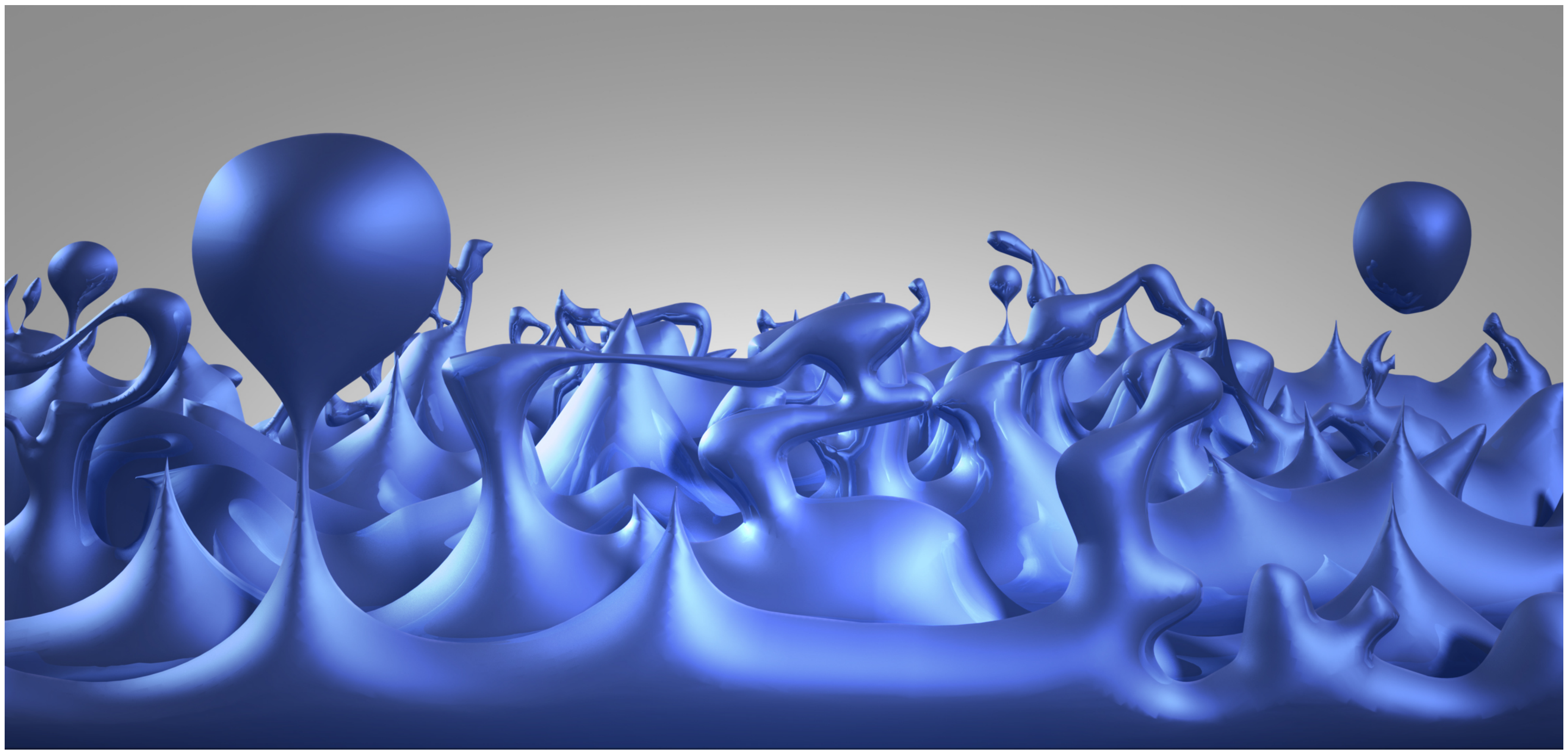
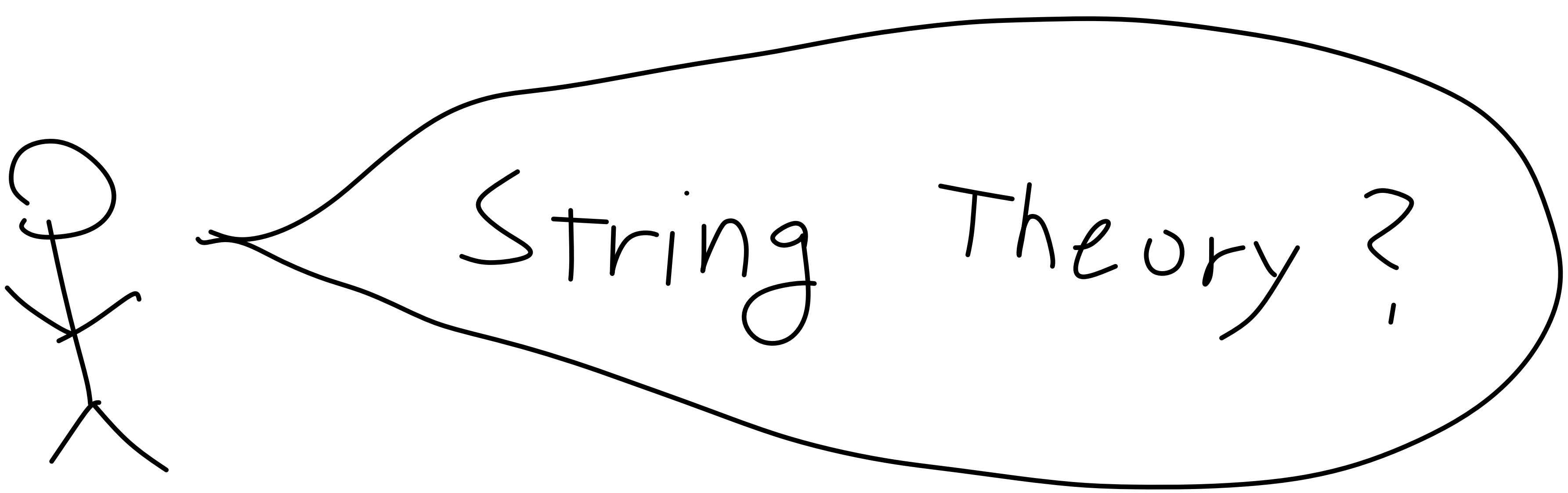
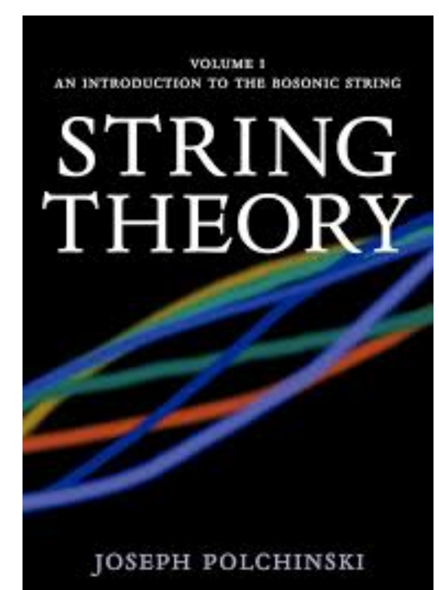


figure from NASA





Quantum vs. Stringy

$$\ell_p \ll \ell_s$$

Planck length | String length

	Quantum	vs.	Stringy
g_s : small	l_p	$\ll$	l_s
g_s : large	l_p	$\gg$	l_s

string coupling constant

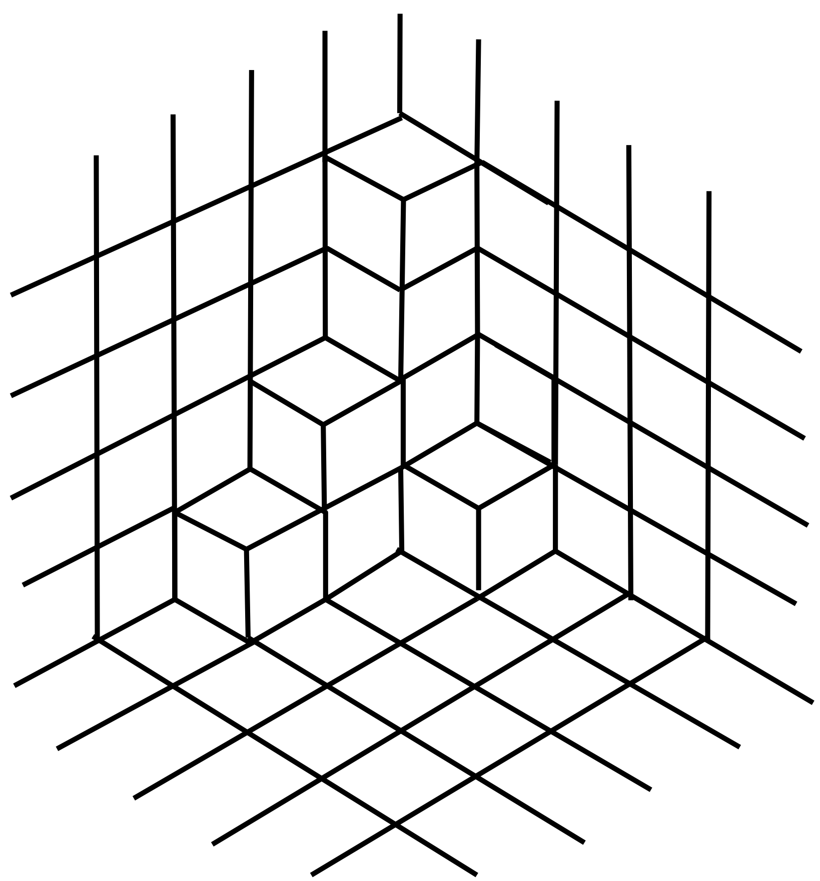
	Quantum vs. Stringy
g_s : small	$Z = \exp\left(\sum_{g \geq 0} g_s^{2g-2} F_g\right)$ "stringy"
g_s : large	$Z = \sum_{n \geq 0} \Omega_n \left(e^{-g_s}\right)^n$ "quantum"

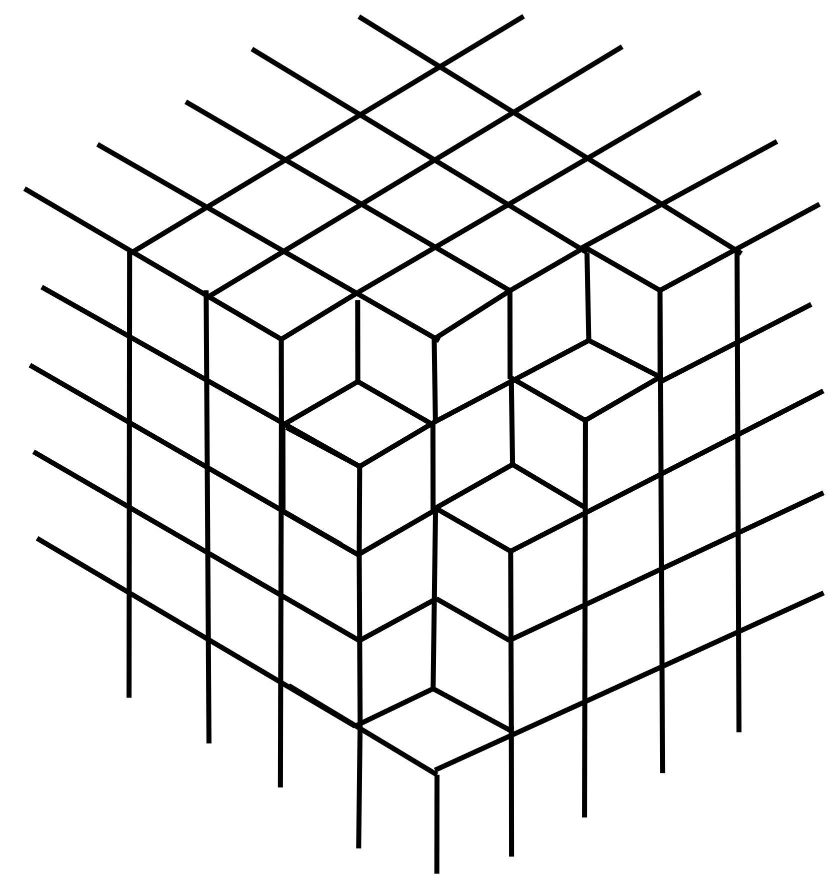
"GW"

MNOP

"GV
DT
PT"

"Quantum Toric Calabi-Yau"

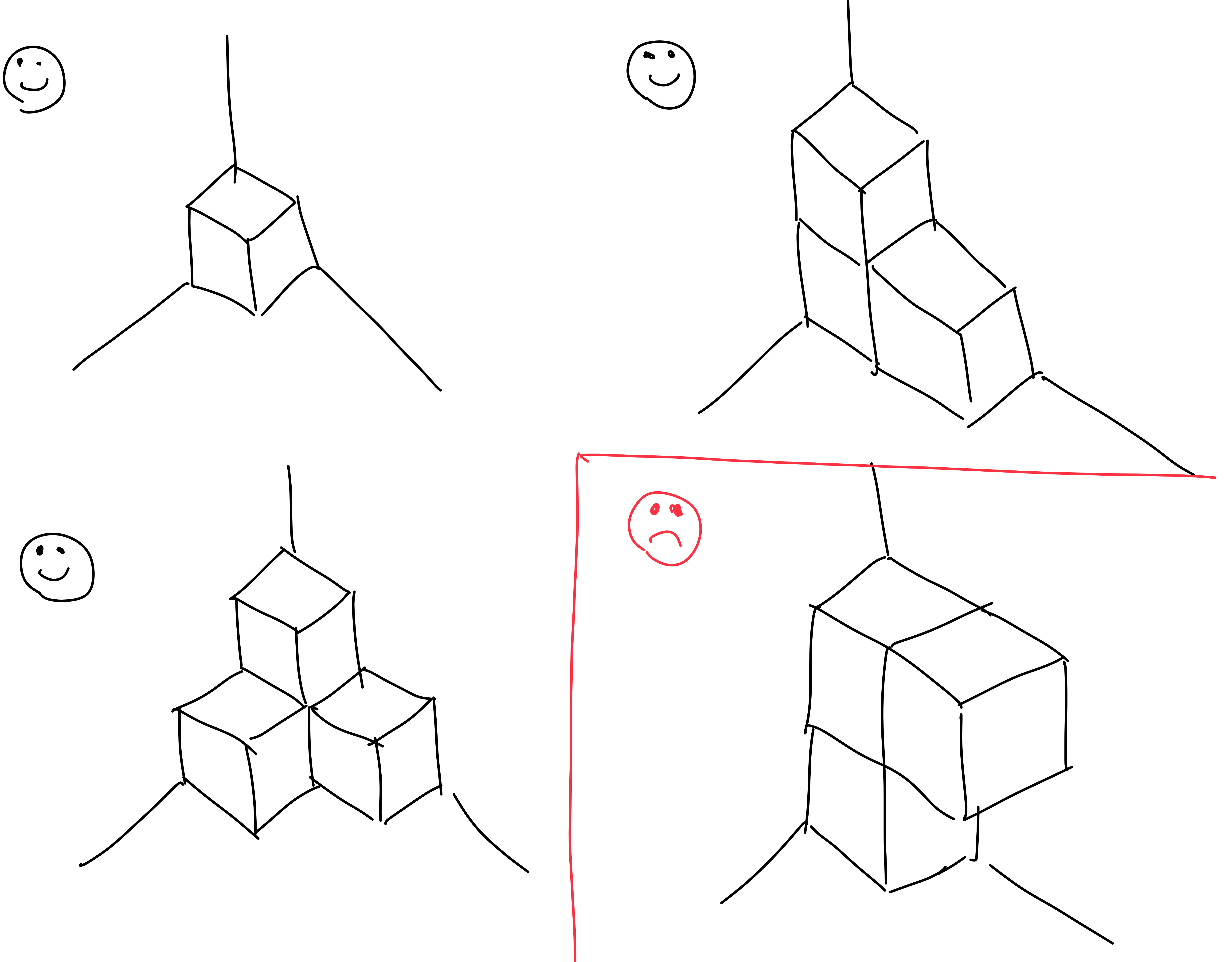


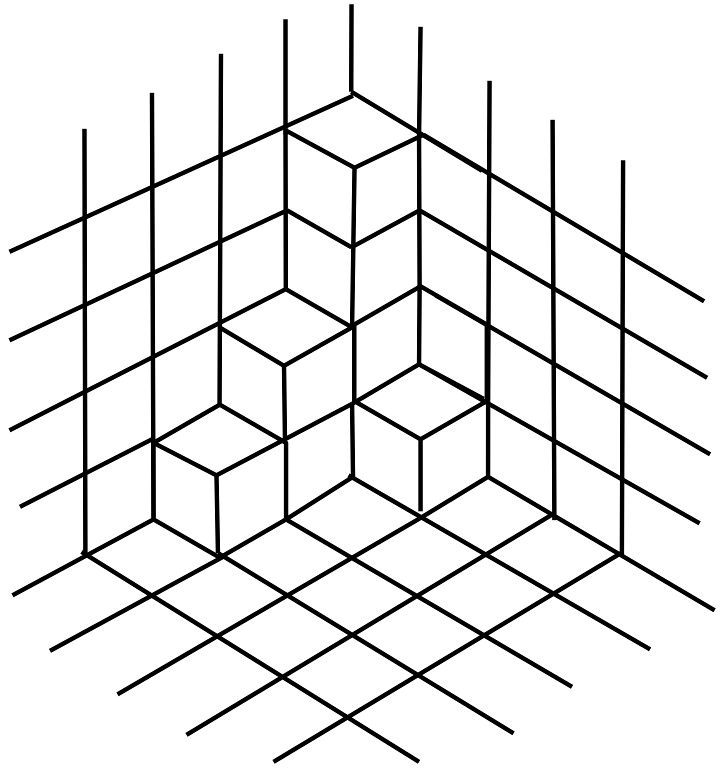
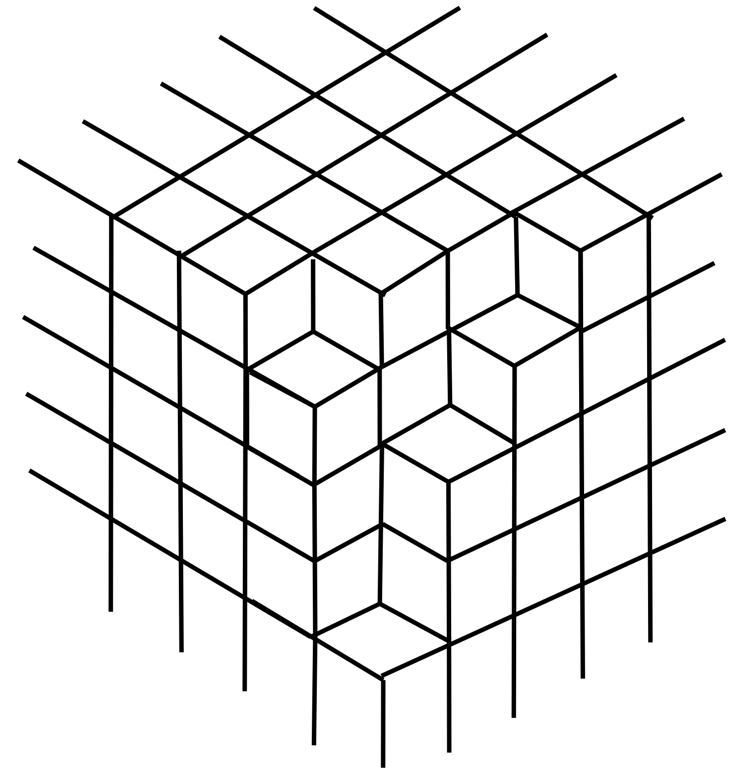


$\mathbb{C}^3$

Crystal Melting as Quantum Foam

Okounkov-Reshetikhin-Vafa ('03), also Iqbal, Nekrasov,···



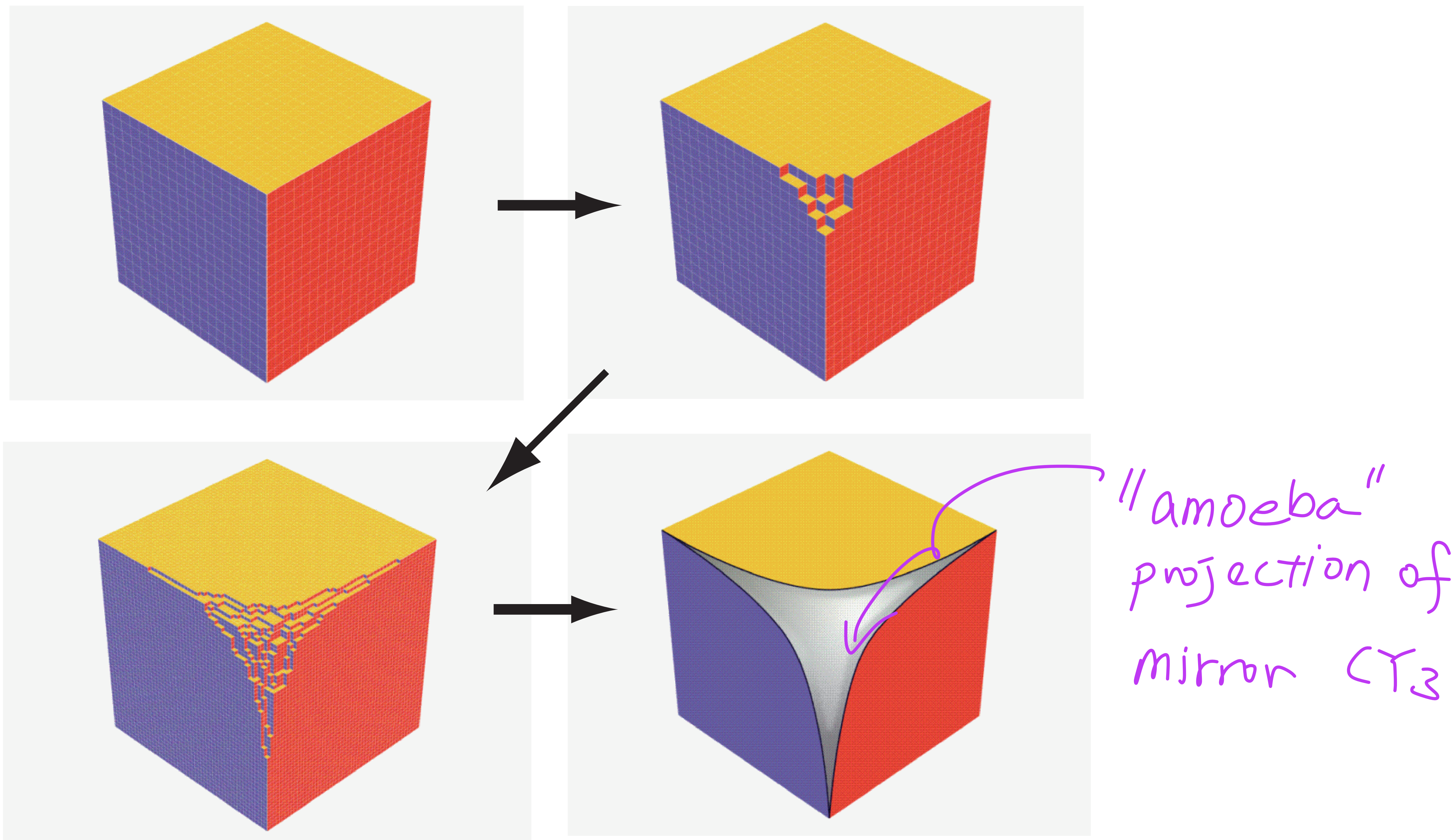
 

$$Z_{top} = Z_{crystal} = \sum_{\Lambda:\ \text{plane partition}} q^{|\Lambda|} \qquad (q = e^{-g_s})$$

$$= \prod_{k=1}^{\infty} (1-q^k)^{-k} \qquad \text{(MacMahon function)}$$

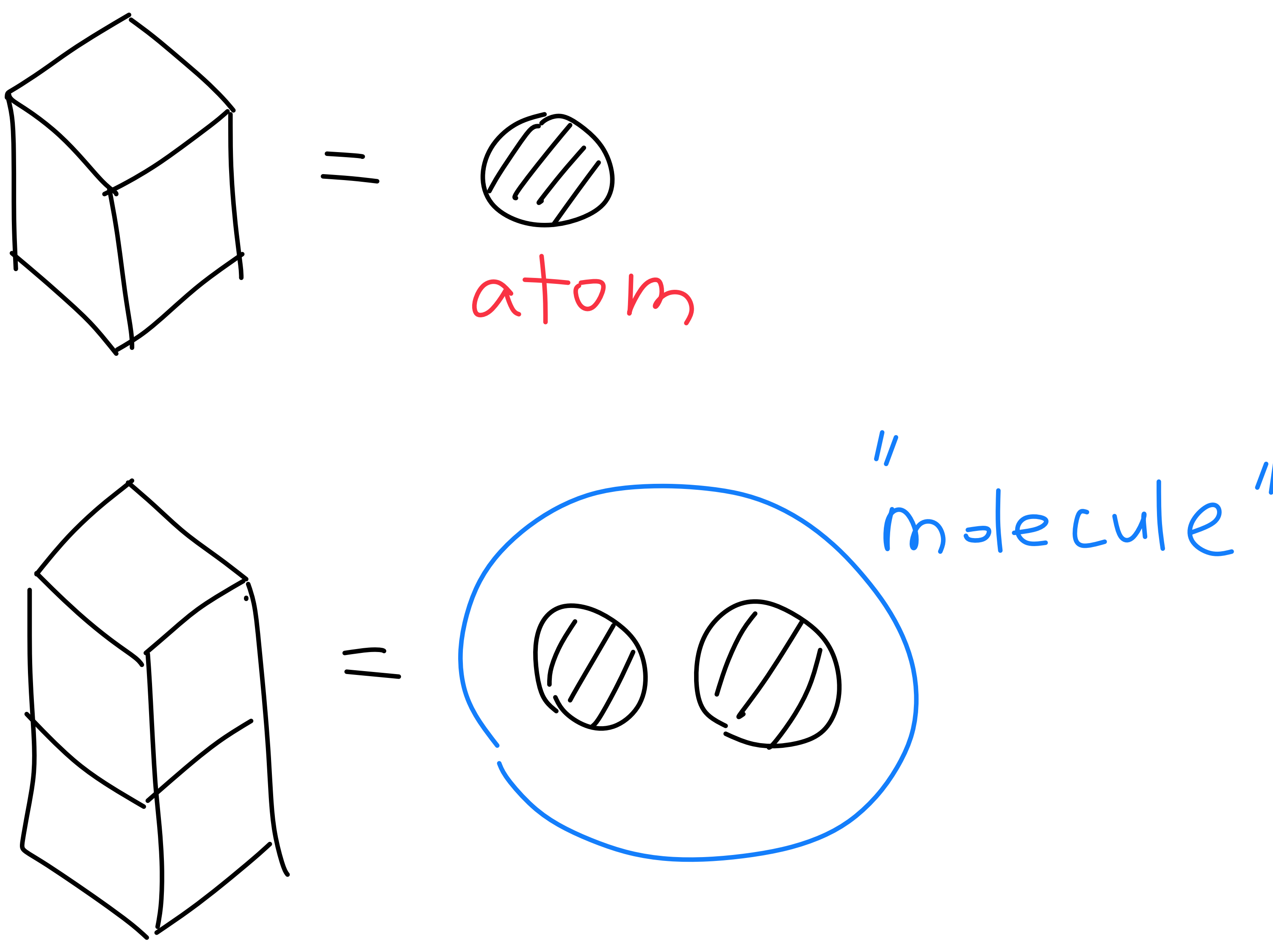
$$= 1 + q + 3q^2 + 6q^3 + 13q^4 + \cdots$$

Emergence of classical geometry



Okounkov-Reshetikhin-Vafa ('03), ⋯ Ooguri-MY ('09),⋯

Quantum Geometry ~ Molecules consisting of (Calabi-Yau) atoms



The story generalizes to an arbitrary toric CY3

[Ooguri-MY ’08’09]

See also [Szendroi; Bryant, Young; Mozgovoy, Reineke; Nagao, Nakajima; Ooguri, MY; Jafferis, Chuang, Moore; Sulkowski; Aganagic, Vafa; $\cdots$]

Crystal Melting for General Toric CY_3

[Ooguri-MY '08]

$$Z_{BPS} = \sum \underbrace{\Omega(\underbrace{n_0}_{D0}, \underbrace{\vec{n}_2}_{D2}, \underbrace{\vec{n}_4}_{D4})}\, q^{n_0}\, \vec{Q}_2^{\vec{n}_2}\, \vec{Q}_4^{\vec{n}_4}$$

BPS degeneracy of
D0/D2/D4
on 0/2/4-cycles

"generalized
Donaldson-Thomas inv."

Crystal Melting for General Toric CY_3

[Ooguri-MY '08]

$$Z_{BPS} = \sum \Omega(n_0, \vec{n}_2, \vec{n}_4)\, q^{n_0}\, \vec{Q}_2^{\vec{n}_2}\, \vec{Q}_4^{\vec{n}_4}$$

$$= Z_{QM}$$ ← Witten index of QM of Dp on p-cycles

$$= Z_{crystal}$$

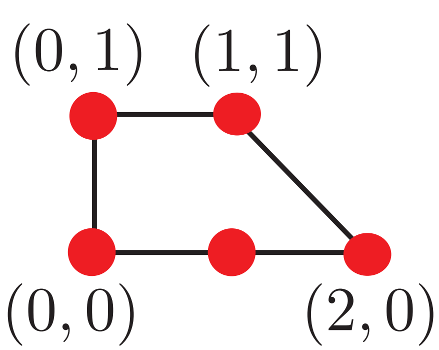
↑ localization

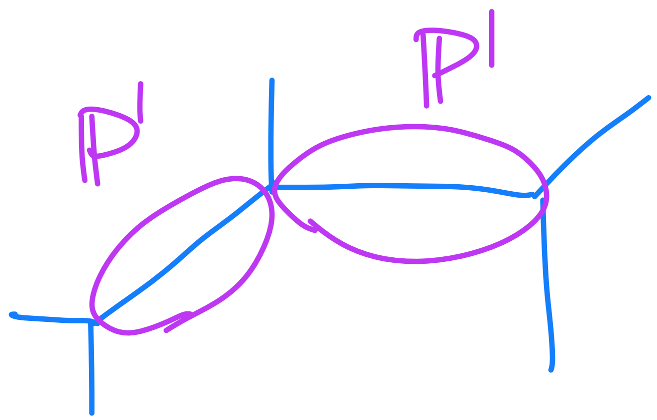
[Ooguri-MY '08]

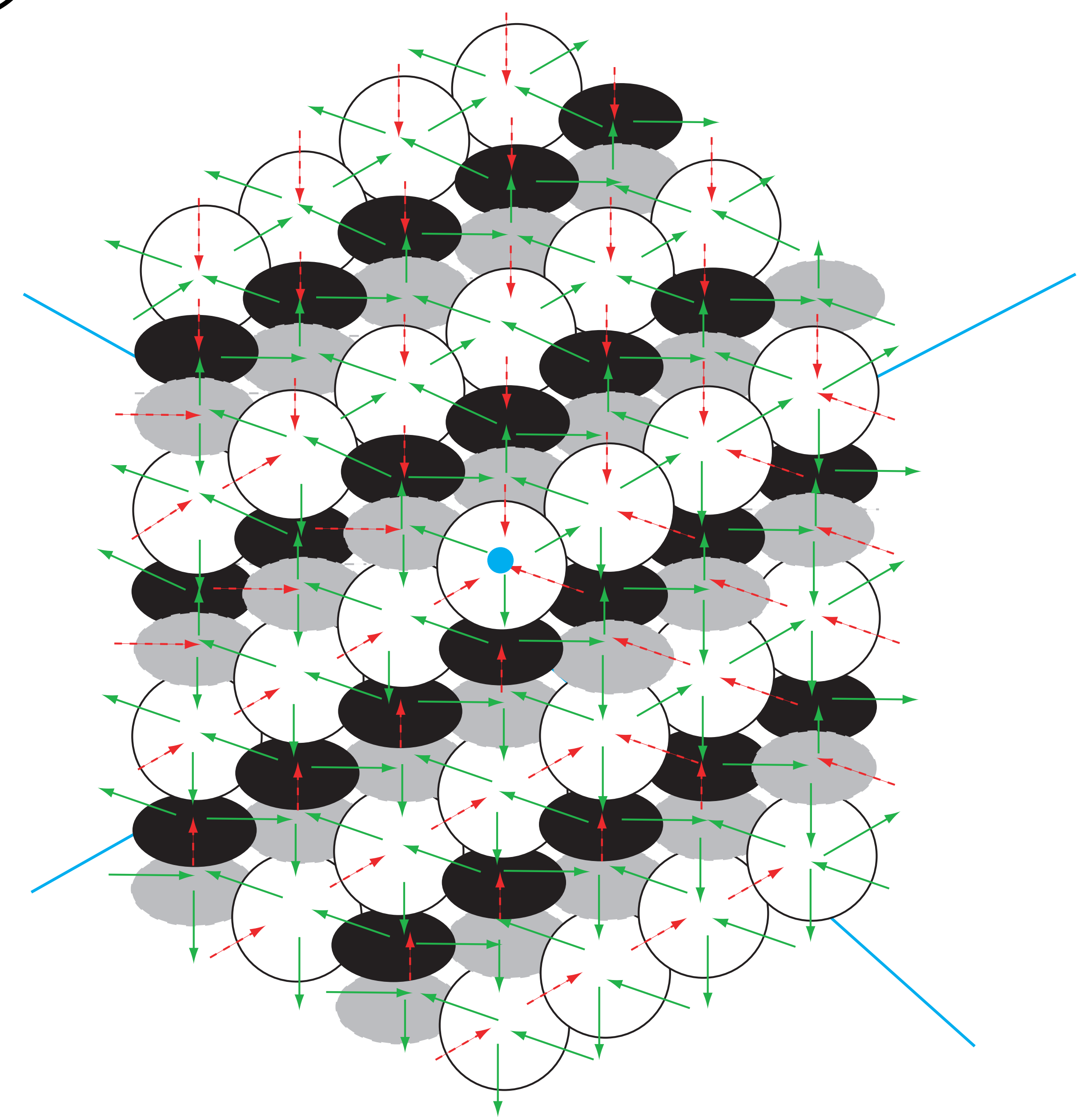
$$Z_{BPS}(\underbrace{\overset{D0}{q}, \overset{D2}{Q_1}, \overset{D2}{Q_2}}) = Z_{crystal}(\underbrace{q_W, q_B, q_G})$$

parameters related

e.g. $q = q_W q_B q_G$

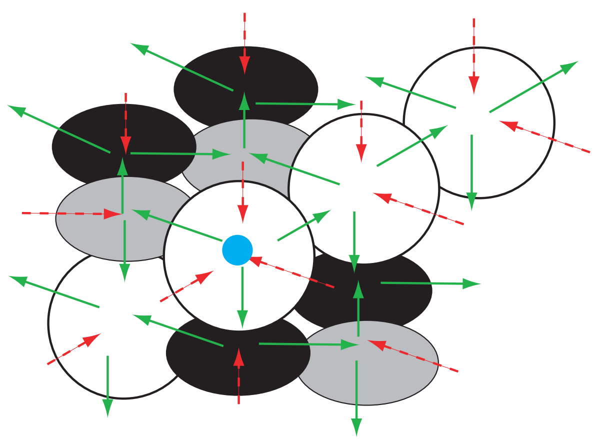






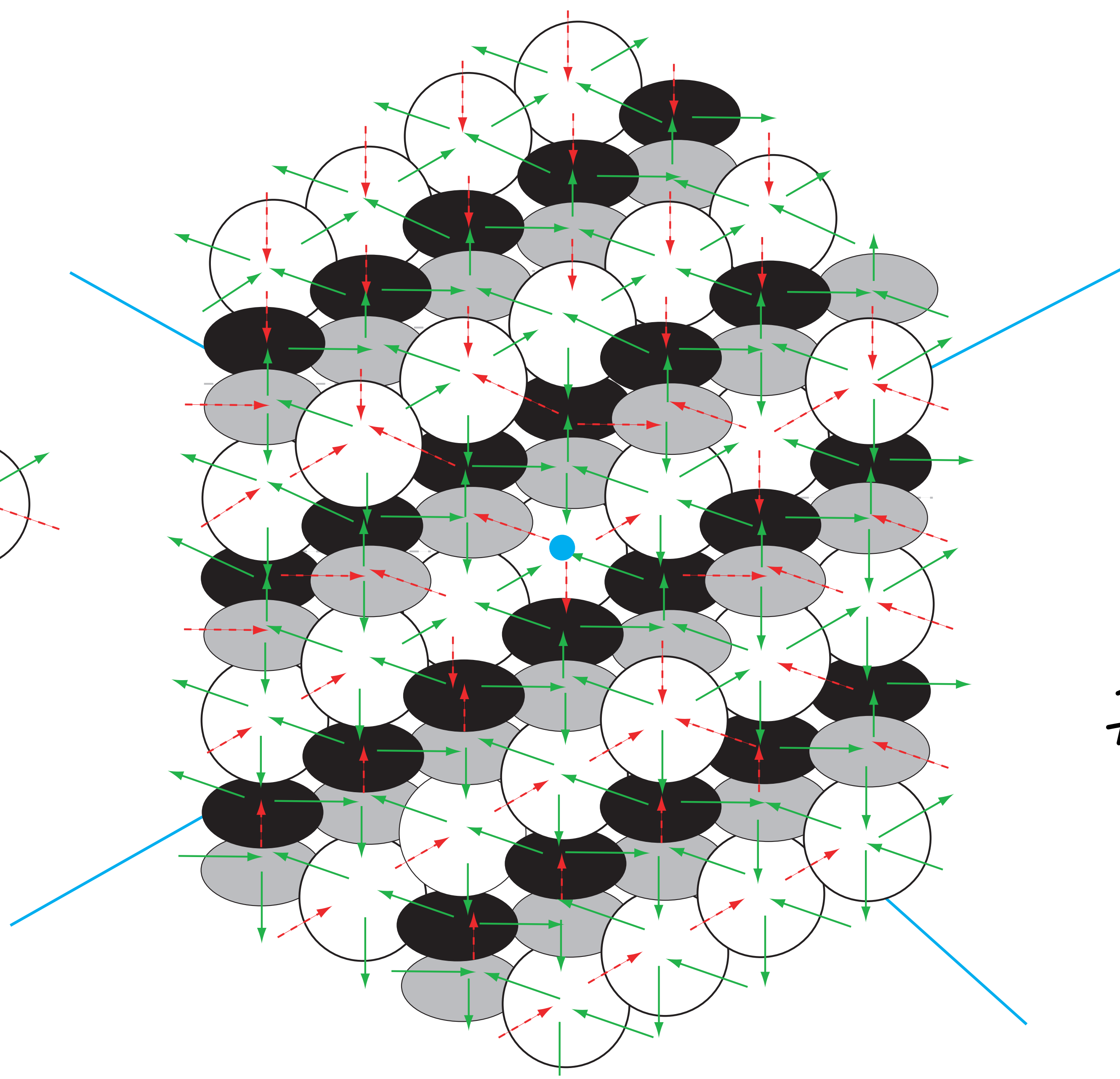
3 atoms: white / black / gray

$$Z_{BPS} = Z_{crystal}$$



K

"atoms from top"



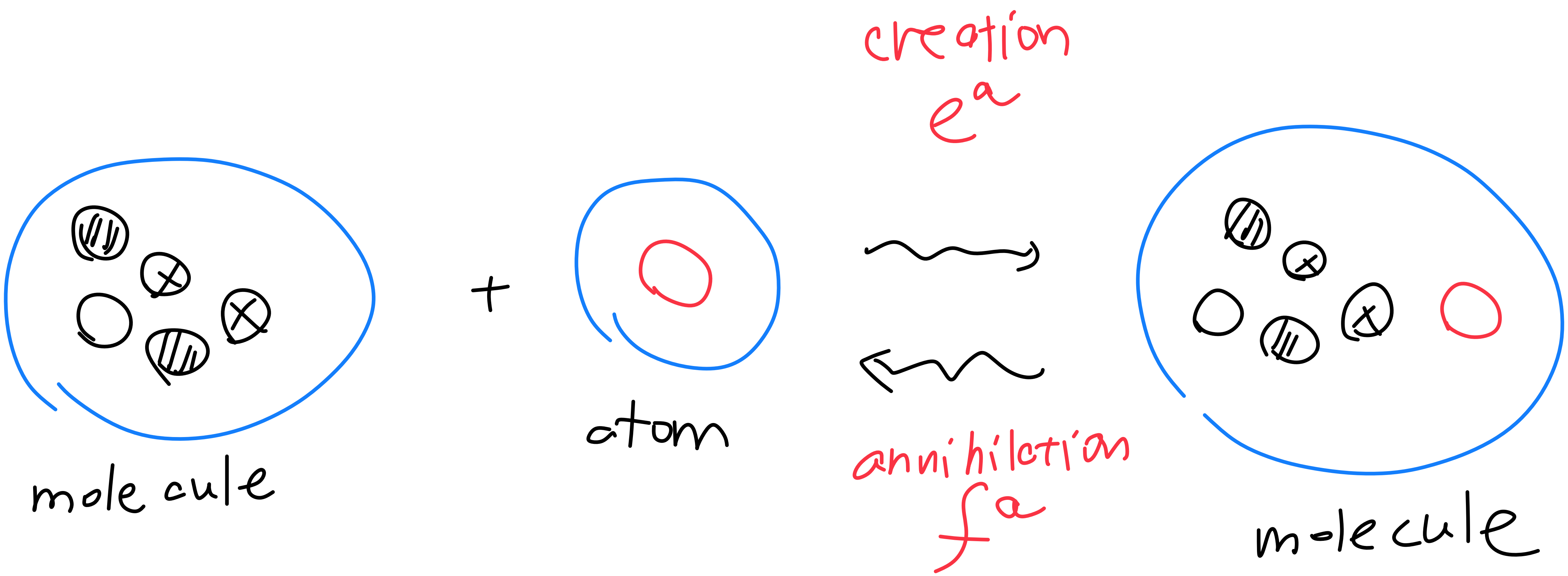
$\#\left(\begin{array}{c}a\text{-color}\\ \text{atoms}\end{array}\right) \to |K(a)|$

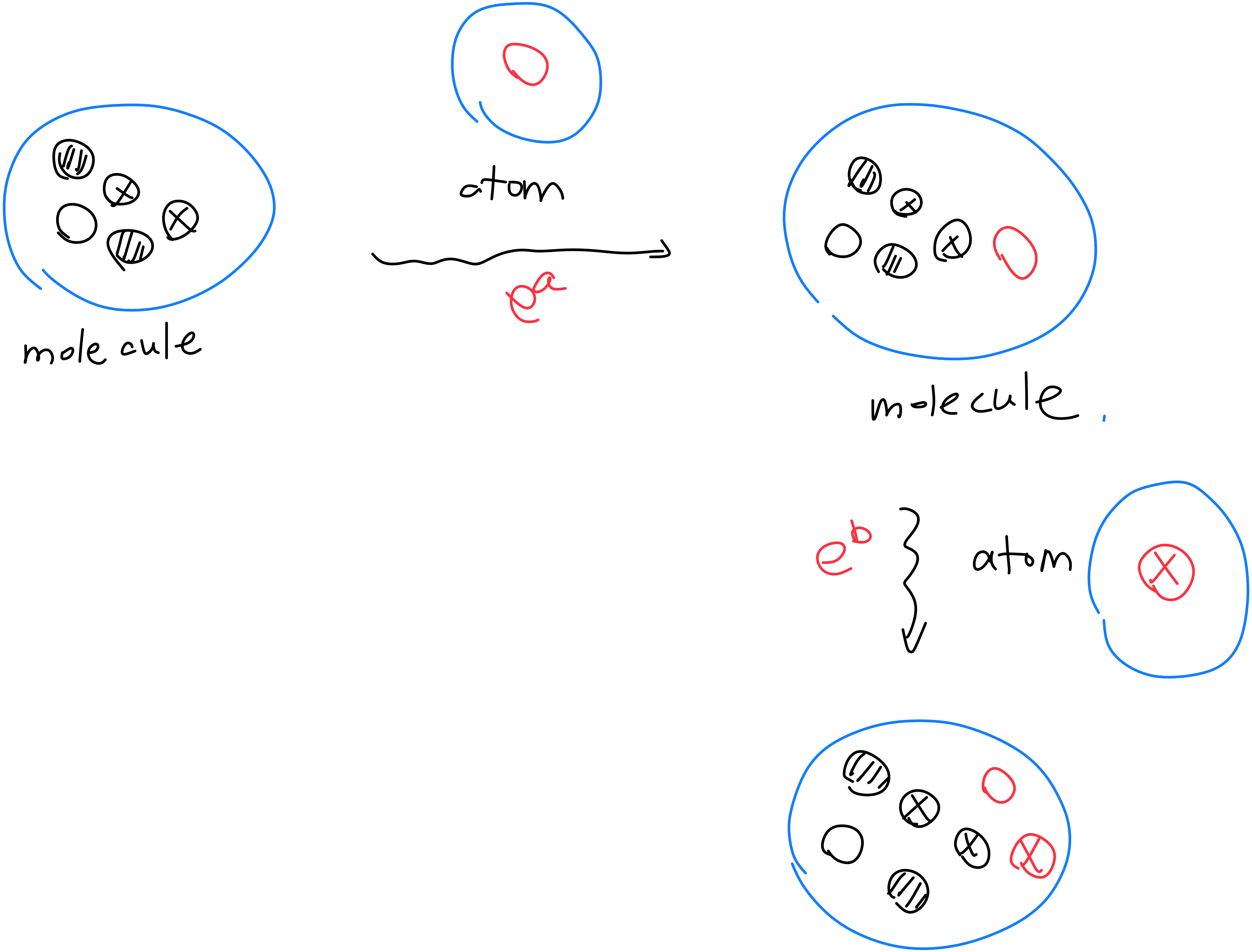
$$Z(q_1 \cdots q_{|Q_0|}) = \sum_{K} \prod_{a:\text{vertex}} q_a^{|K(a)|}$$

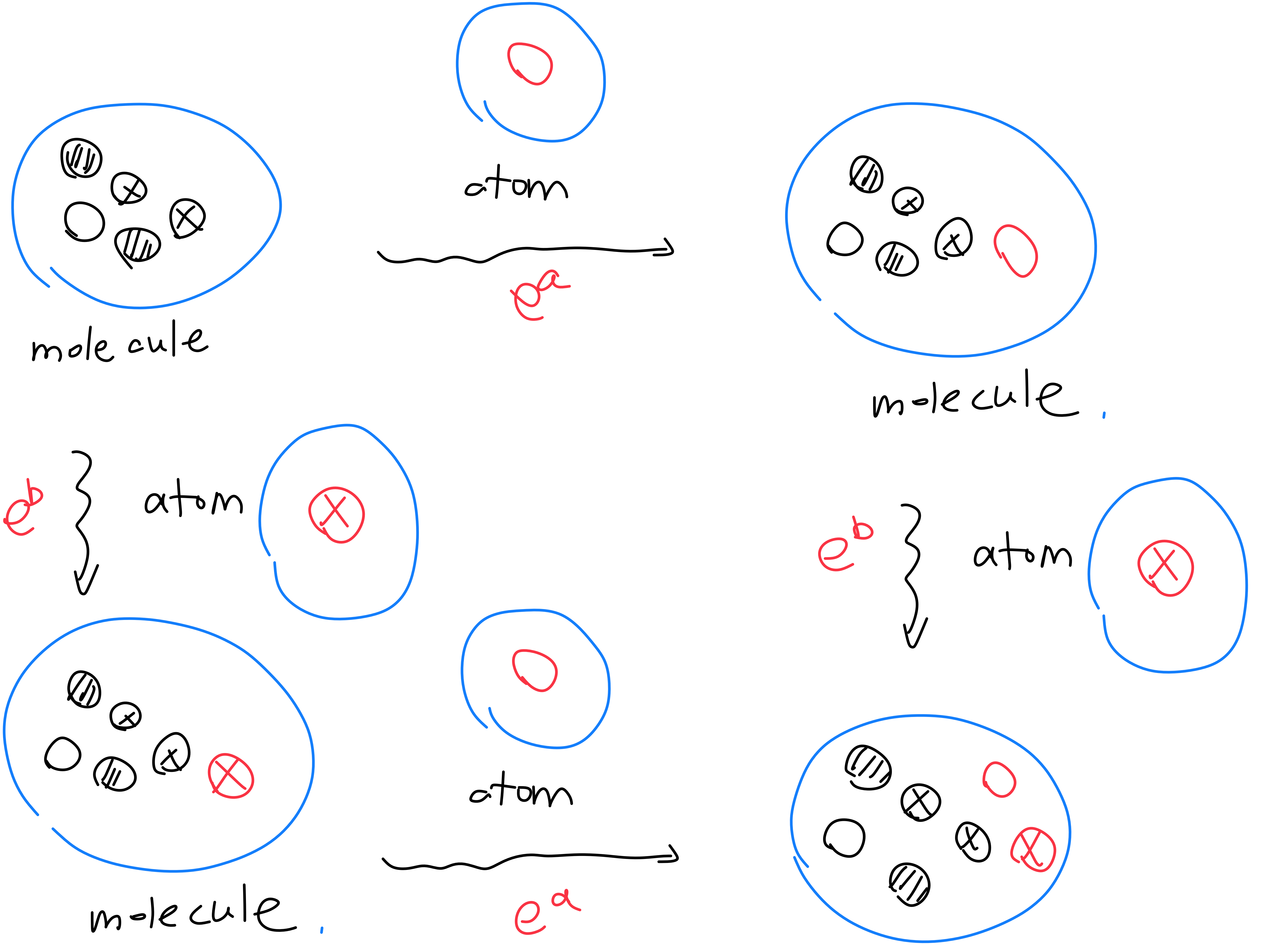
Q: Can we become

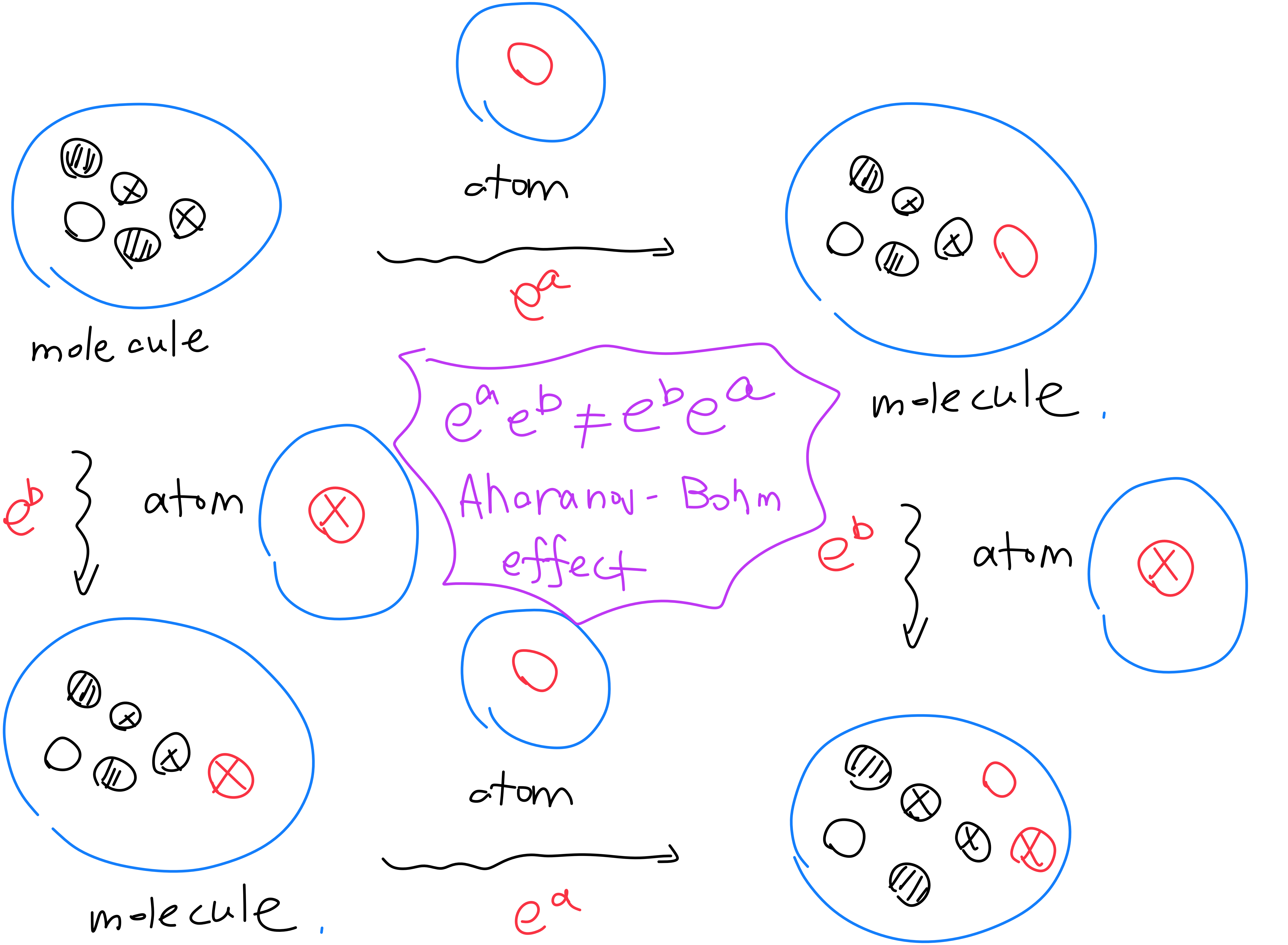
"Quantum - Gravity Experimentalist"

and create / annihilate atoms?




molecule
atom
e^a
molecule
e^b
atom


atom
molecule
e^a
molecule
e^b
atom
atom
molecule
e^a
e^b
atom


atom
e^a
molecule
molecule.
e^b
atom
$e^a e^b \neq e^b e^a$
Aharanov - Bohm
effect
e^b
atom
atom
e^a
molecule.

[Li-MY, Galakhov-MY '20]

Quiver Yangian

& their cousins, e.g.

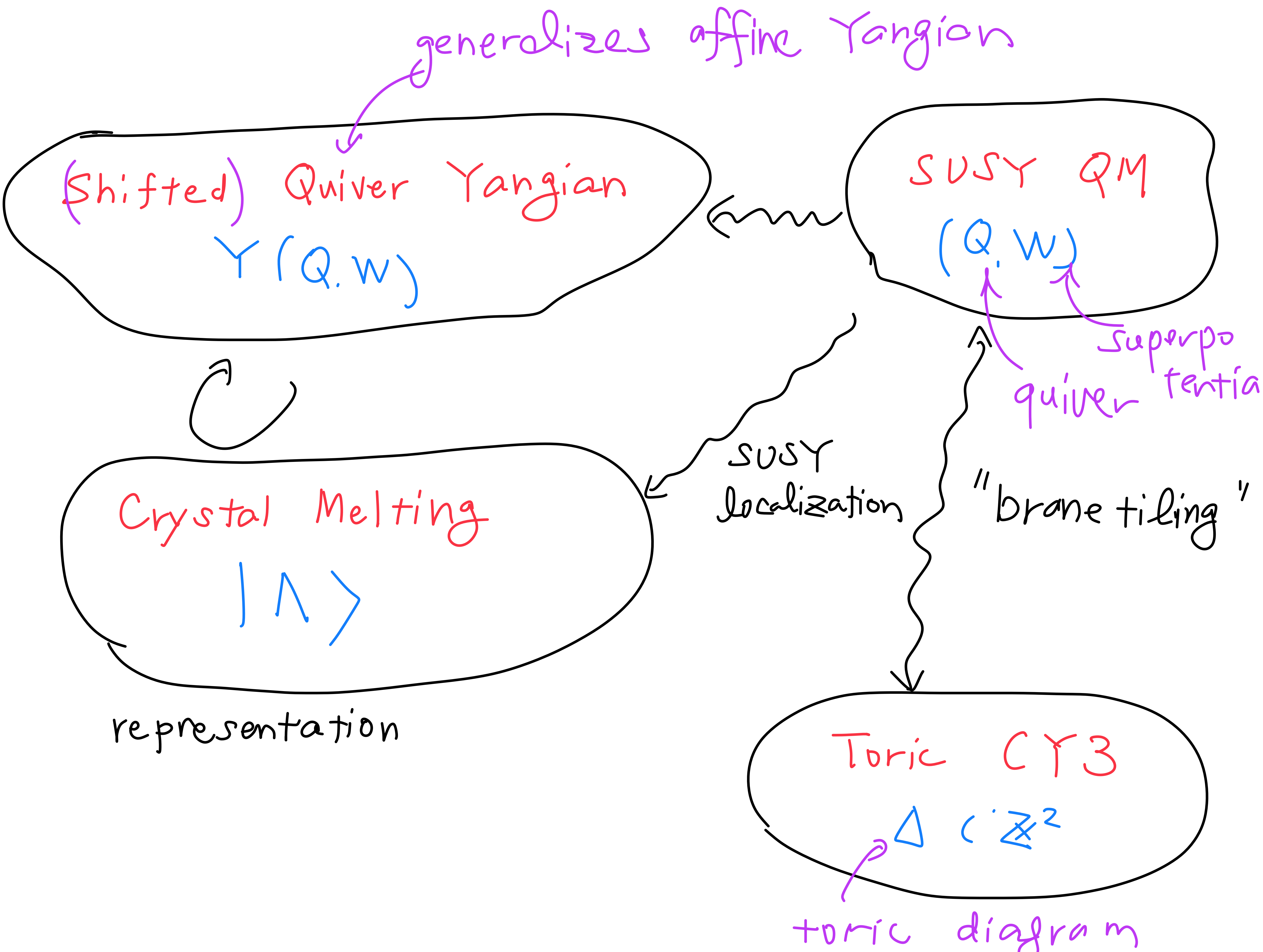
Quiver Quantum Toroidal Algebra

Quiver Elliptic Algebra

geometry combinatorics algebra

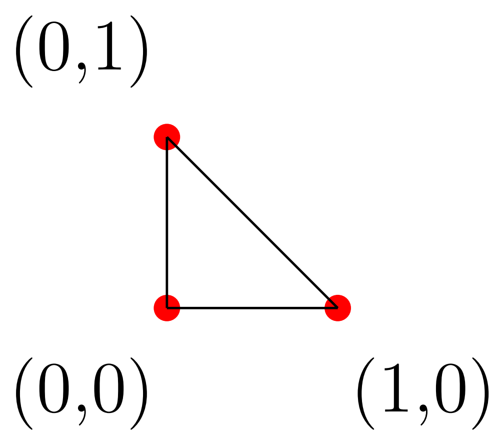
$$Z_{BPS} = Z_{crystal} = \underbrace{\chi_{QY}}_{\text{character of a rep. of QY}}$$

Crystal / Quiver Yangian Practicalities

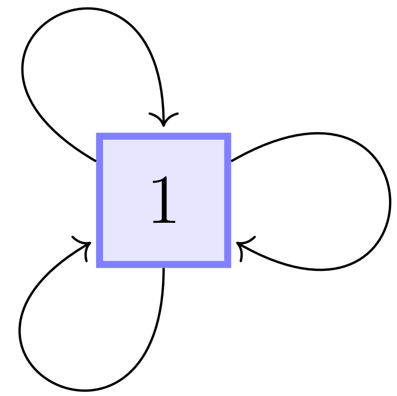


generalizes affine Yangian
(Shifted) Quiver Yangian
$Y(Q,W)$
SUSY QM
(Q,W)
superpotential
quiver
SUSY localization
"brane tiling"
Crystal Melting
$|\Lambda\rangle$
representation
Toric CY3
$\Delta \subset \mathbb{Z}^2$
toric diagram

toric diagram



quiver Q



superpotential

$$+\left(\begin{array}{l} W = \mathrm{Tr}\, xyz \\ \quad - \mathrm{Tr}\, xzy \end{array}\right)$$

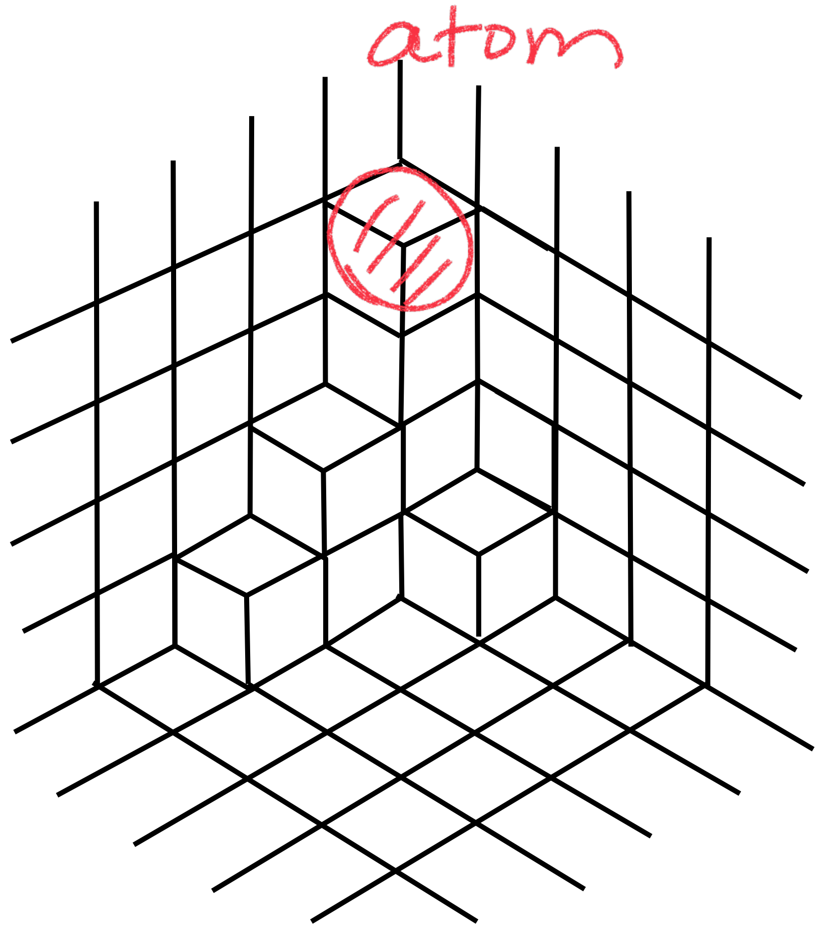
Path algebra

$$\mathbb{C}\langle x, y, z\rangle / (\partial W)$$

$$(\partial W) = (xy - yx,\ yz - zy,\ zx - xz)$$

$$\mathbb{C}\langle x, y, z\rangle / (\partial W) = \mathbb{C}[x, y, z]$$

crystal melting



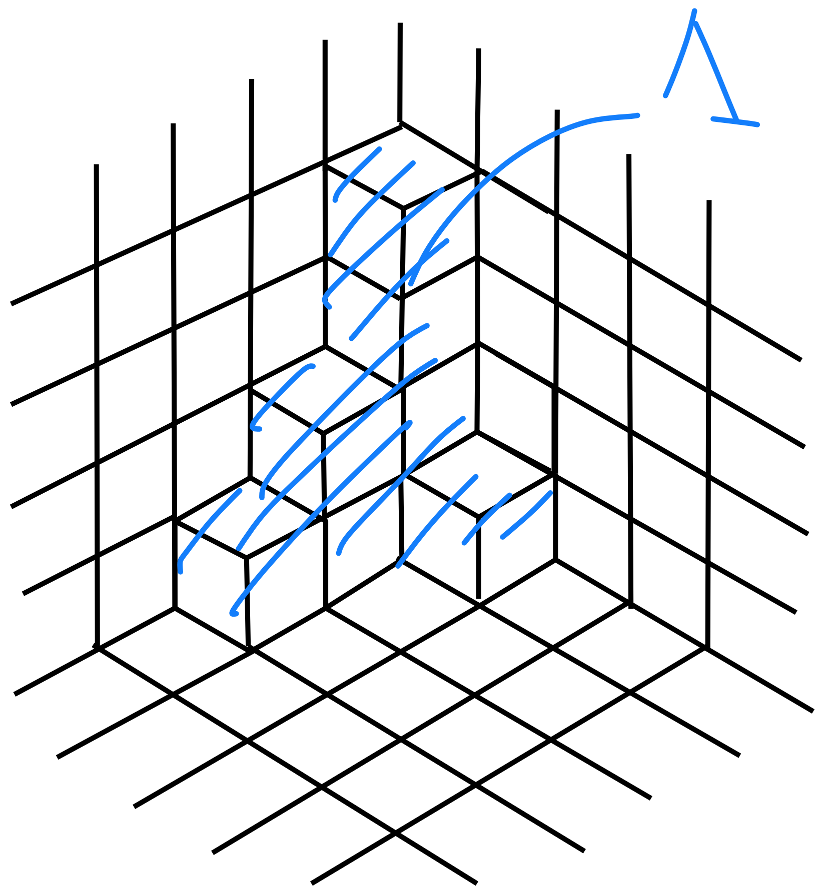
- "atom" at location (i, j, k):

$$x^i y^j z^k \in \mathbb{C}[x, y, z]$$
$$\shortparallel$$
$$\mathbb{C}\langle x, y, z\rangle / (\partial W)$$

- atom = element of

$$\mathbb{C}Q/\partial W$$

crystal melting



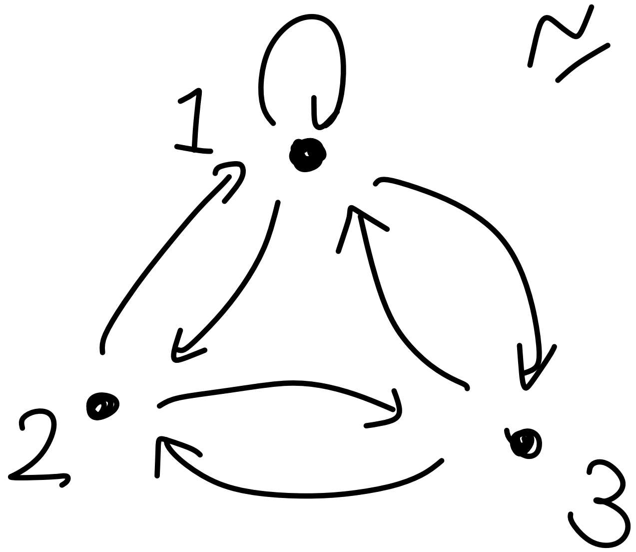
Λ^c (complement of Λ):

ideal of the path alg.

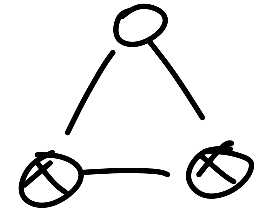
$$\left(\begin{array}{c} I_\Lambda \subset \mathbb{C}[x, y, z] \\ \shortparallel \\ \mathrm{Span}\{x^i y^j z^k \mid (i,j,k) \notin \Lambda\} \\ \\ x \cdot I_\Lambda,\ y \cdot I_\Lambda,\ z \cdot I_\Lambda \subset I_\Lambda \end{array} \right)$$

We have an associated SQM

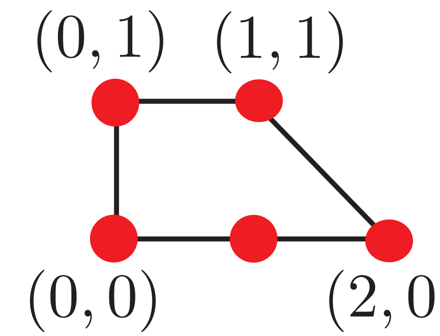
Q



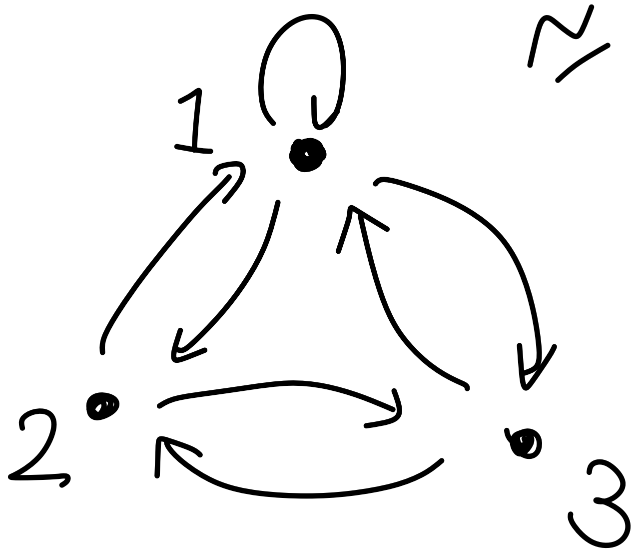
$\cong$



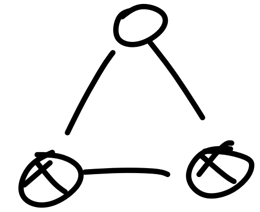
Dynkin



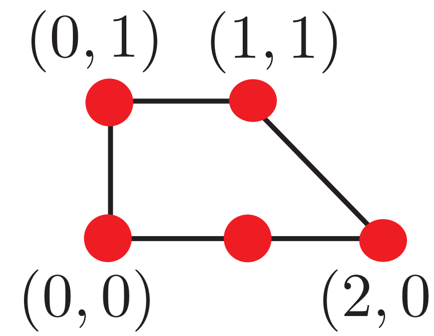
We have an associated SQM



$\simeq$



Dynkin



X_{ij}

$i \bullet \longrightarrow \bullet j$

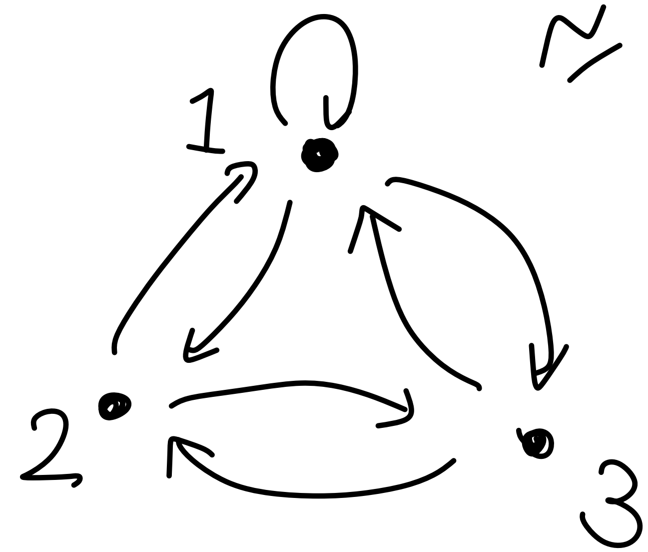
Q

W

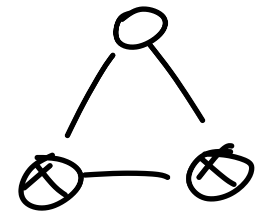
$$W = \mathrm{Tr}\big(X_{11} X_{12} X_{21} - X_{11} X_{13} X_{31} - X_{12} X_{21} X_{23} X_{32} + X_{23} X_{31} X_{13} X_{23} \big)$$

We have an associated SQM

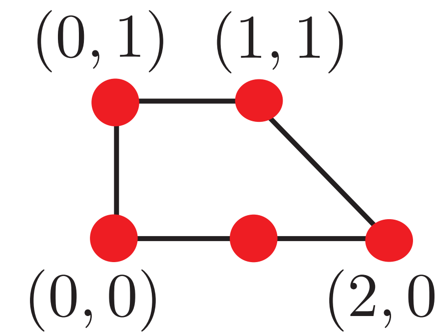
Q



$\cong$



Dynkin



$i \xrightarrow{X_{ij}} j$

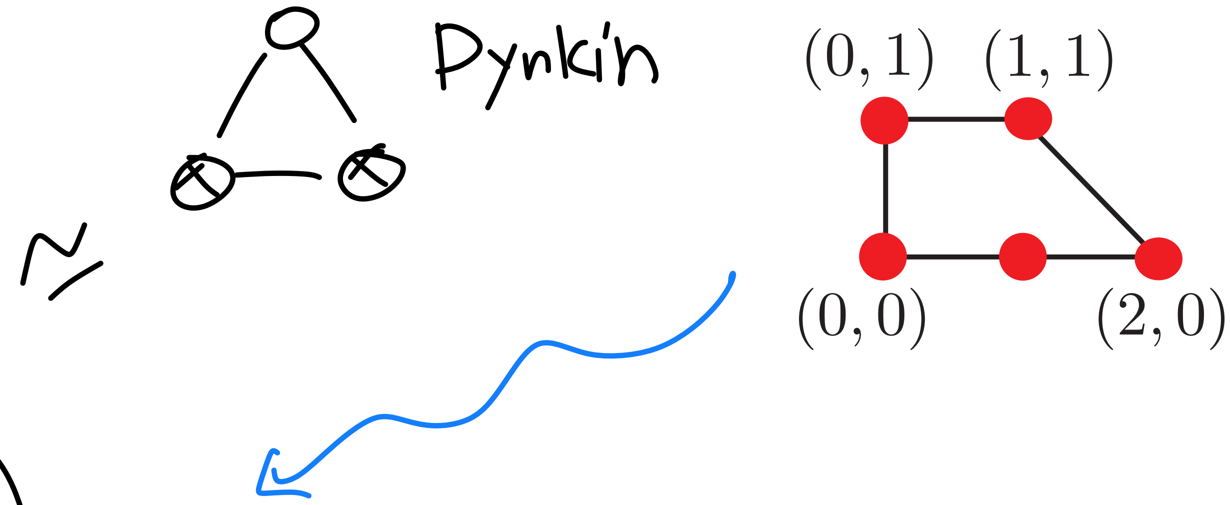
W

$$W = \mathrm{Tr}\left(X_{11} X_{12} X_{21} - X_{11} X_{13} X_{31} - X_{12} X_{21} X_{23} X_{32} + X_{23} X_{31} X_{13} X_{23} \right)$$

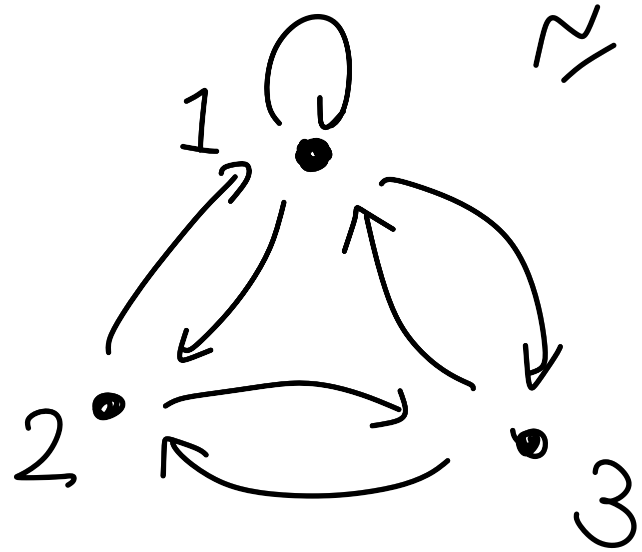
$\mathbb{C}Q/(\partial W)$: path algebra (non-commutative in general)

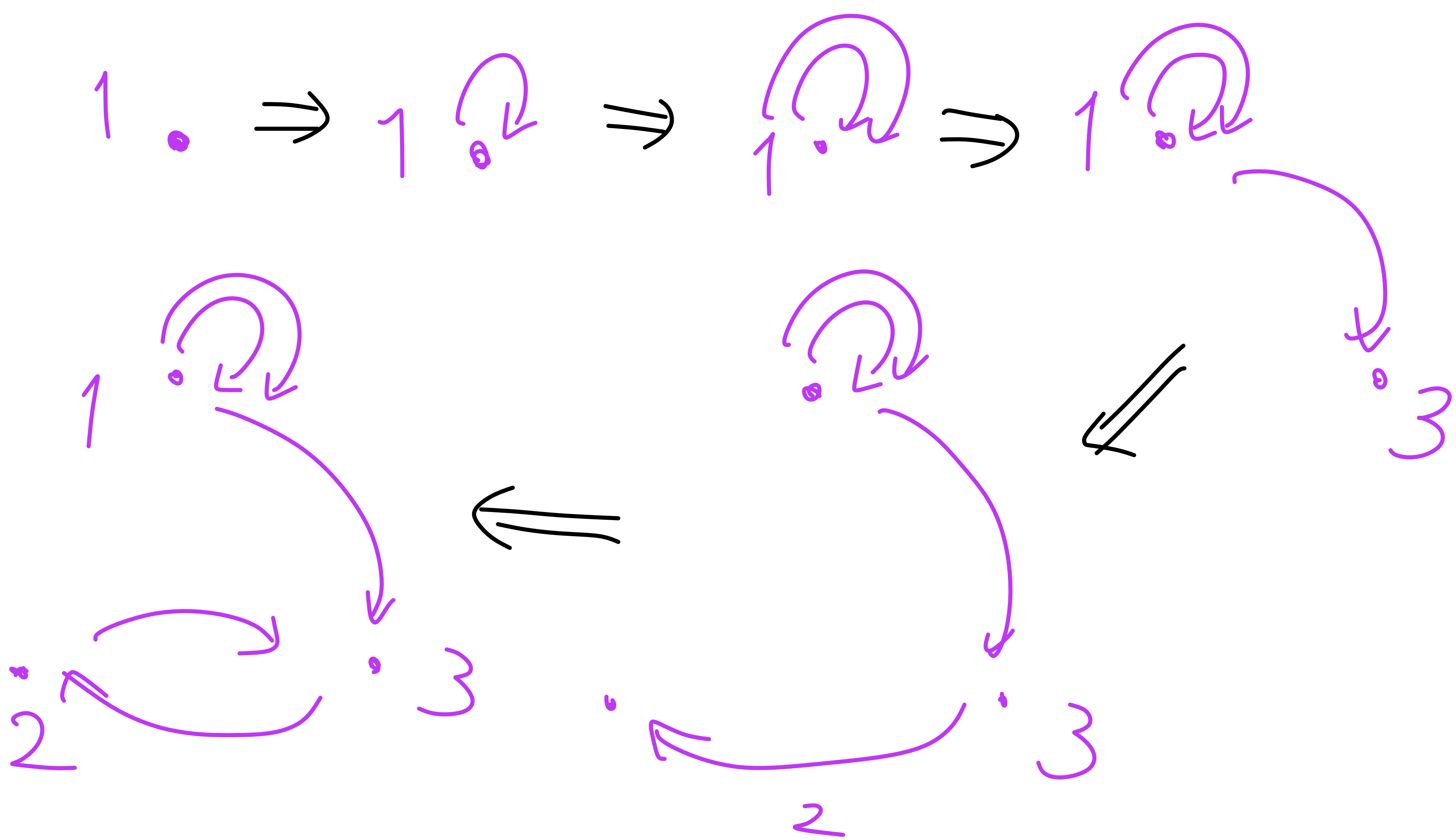
atom in the crystal

We have an associated SQM

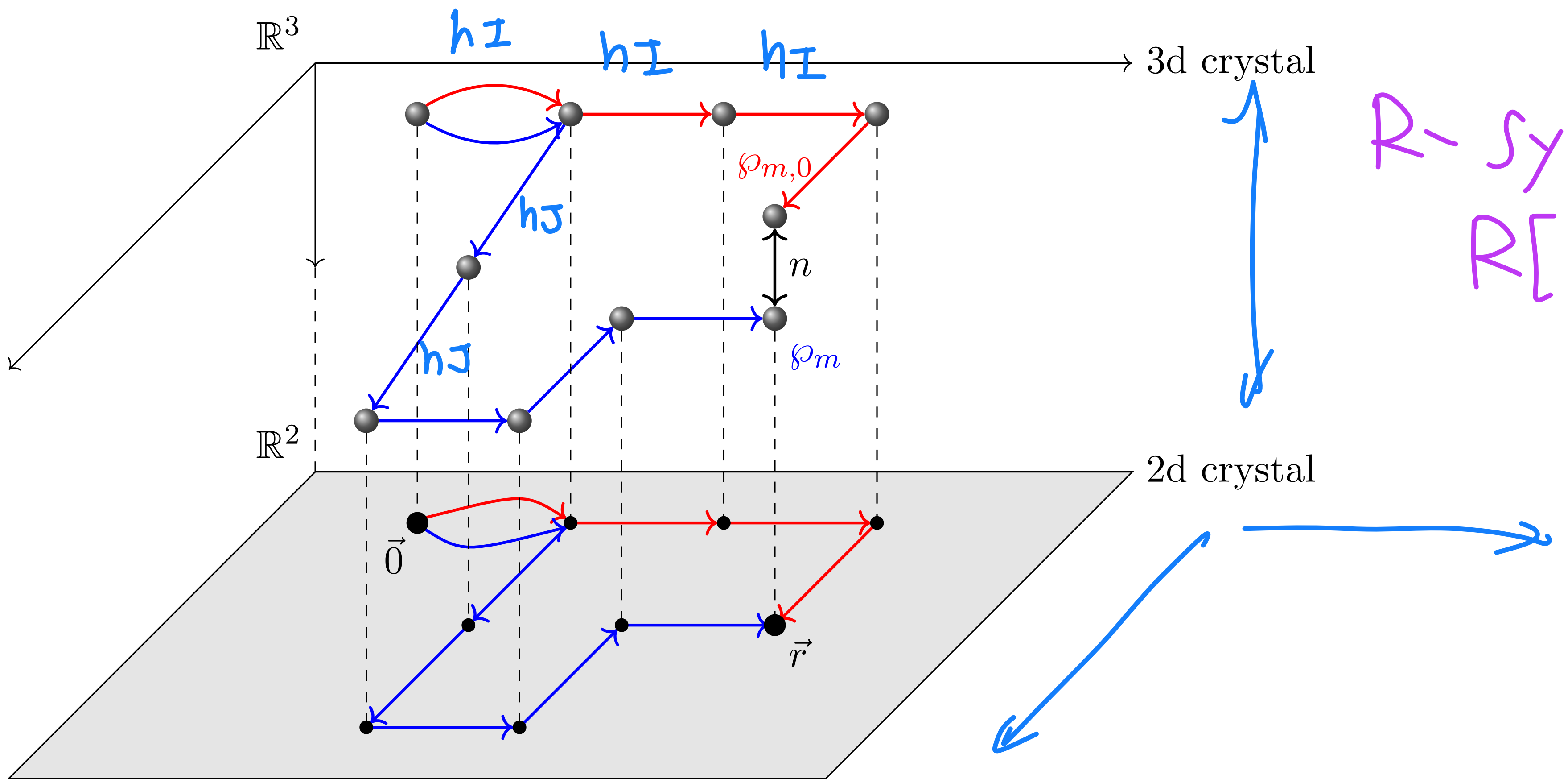


Q





We can place the atoms in 3D according to their symmetry charges (equivariant parameters corresponding to toric isometries)



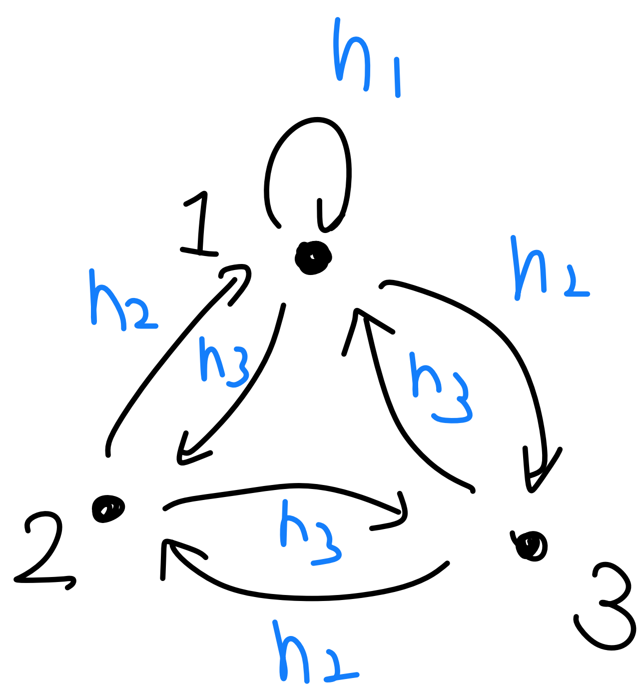
R-sym

$R[W] = 2$

flavor sym

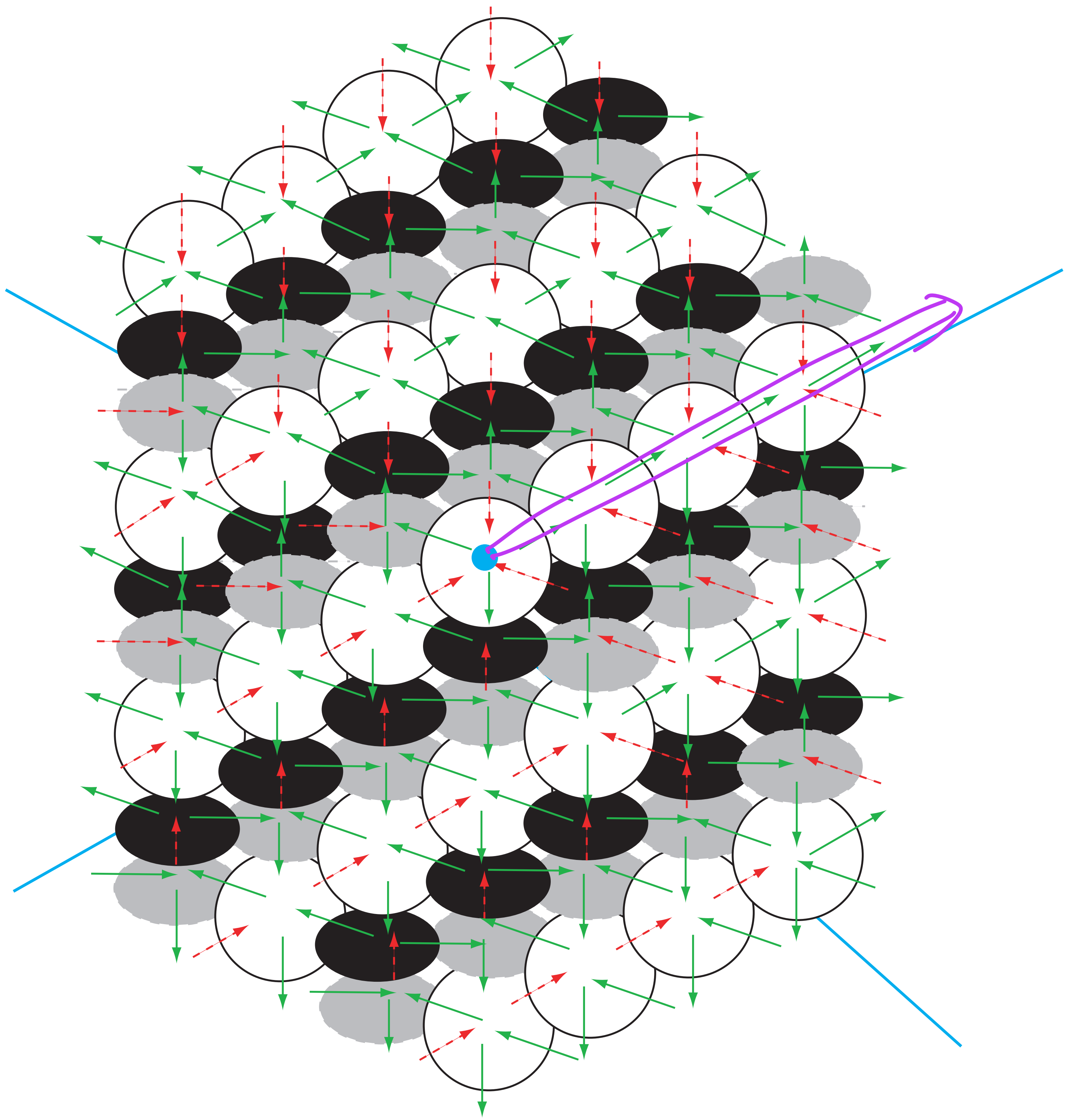
$F[W] = 0$

e.g.



$h_1 + h_2 + h_3 = 0$

for flavor sym.



1

2

3

Quiver Yangian

$Y(Q, W)$

Generators

(z : spectral parameter)

$$e^{(a)}(z) \equiv \sum_{n=0}^{+\infty} \frac{e_n^{(a)}}{z^{n+1}}, \qquad \psi^{(a)}(z) \equiv \sum_{n=-\infty}^{+\infty} \frac{\psi_n^{(a)}}{z^{n+1}}, \qquad f^{(a)}(z) \equiv \sum_{n=0}^{+\infty} \frac{f_n^{(a)}}{z^{n+1}},$$

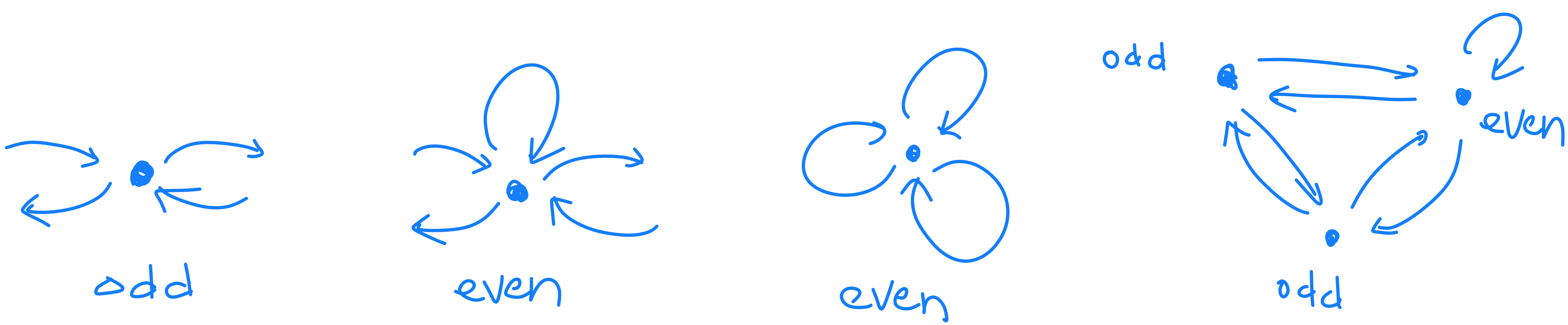
$n = -k$

"k-shifted Quiver Yangian"

a : quiver vertex

$\mathbb{Z}_2$-grading

$$|a| = \begin{cases} 0 & (\exists \text{ edge } I \text{ s.t. } I \text{ starts and ends at } a) \\ 1 & (\text{otherwise}) \end{cases}$$



Relations

$Y(Q, W)$

$$\psi^{(a)}(z)\,\psi^{(b)}(w) = \psi^{(b)}(w)\,\psi^{(a)}(z)\,,$$
$$\psi^{(a)}(z)\,e^{(b)}(w) \simeq \varphi^{b\Rightarrow a}(\Delta)\,e^{(b)}(w)\,\psi^{(a)}(z)\,,$$
$$e^{(a)}(z)\,e^{(b)}(w) \sim (-1)^{|a||b|}\varphi^{b\Rightarrow a}(\Delta)\,e^{(b)}(w)\,e^{(a)}(z)\,,$$
$$\psi^{(a)}(z)\,f^{(b)}(w) \simeq \varphi^{b\Rightarrow a}(\Delta)^{-1}\,f^{(b)}(w)\,\psi^{(a)}(z)\,,$$
$$f^{(a)}(z)\,f^{(b)}(w) \sim (-1)^{|a||b|}\varphi^{b\Rightarrow a}(\Delta)^{-1}\,f^{(b)}(w)\,f^{(a)}(z)\,,$$
$$\left[e^{(a)}(z), f^{(b)}(w)\right\} \sim -\delta^{a,b}\,\frac{\psi^{(a)}(z)-\psi^{(b)}(w)}{z-w}\,, \qquad (\Delta = z-w)$$

"$\simeq$" means equality up to $z^n w^{m\geq 0}$ terms

"$\sim$" means equality up to $z^{n\geq 0} w^m$ and $z^n w^{m\geq 0}$ terms

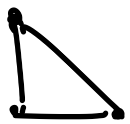
bonding factor

$$\varphi^{a\Rightarrow b}(u) \equiv \frac{\prod_{I\in\{b\to a\}}(u+h_I)}{\prod_{I\in\{a\to b\}}(u-h_I)}$$

charge for edge

edge

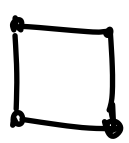
* $\mathbb{C}^3 \rightsquigarrow Q =$  $\rightsquigarrow Y(\widehat{\mathfrak{gl}}_1)$



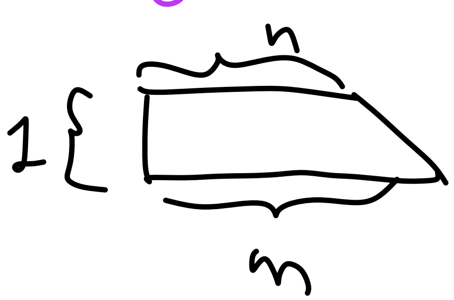
$W = \mathrm{Tr}(XYZ - XZY)$

[Miki; Ding-Iohara;···
Tsymbaulik; Prochazka;
Gaberdiel, Gopakumar, Li, Peng,···]

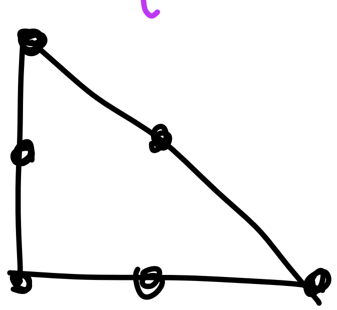
* conifold $\rightsquigarrow Y(\widehat{\mathfrak{gl}}_{1|1})$



* $xy = z^n w^m \rightsquigarrow Y(\widehat{\mathfrak{gl}}_{m|n})$ [Bezerra-Mukhin ('19)]



* $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2) \rightsquigarrow Y(\widehat{D(2,1,\alpha)})$ [Noshita-Watanabe ('21)]

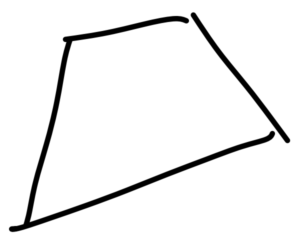


$Y(\hat{\mathfrak{g}})$ for $\left\{ \begin{array}{l} \text{non-chiral quiver} \\ \text{toric CY3 w.o. 4-cycle} \end{array} \right.$

$\left\{ \begin{array}{l} \text{chiral quiver} \\ \text{toric } CY_3 \text{ w/ cpt 4-cycle} \end{array} \right.$

* general toric CY_3 $\rightsquigarrow$ $Y(Q, W)$

has no "$\mathfrak{g}$"



$\Delta \subset \mathbb{Z}^2$

new algebra beyond $Y(\mathfrak{g})$, $Y(\hat{\mathfrak{g}})$!

Representations from Crystal Melting

cf. earlier developments on quantum toroidal algebras (Ding-Iohara-Miki) and affine Yangians by [Feigin, Jimbo, Miwa, Mukhin; Tsymbaulik; Prochazka; Rapcak; Gaberdiel, Gopakumar; Li, Peng,⋯]

Representation by crystal melting [Li-MY '20], inspired by [FFJMM] and [Prochazka]

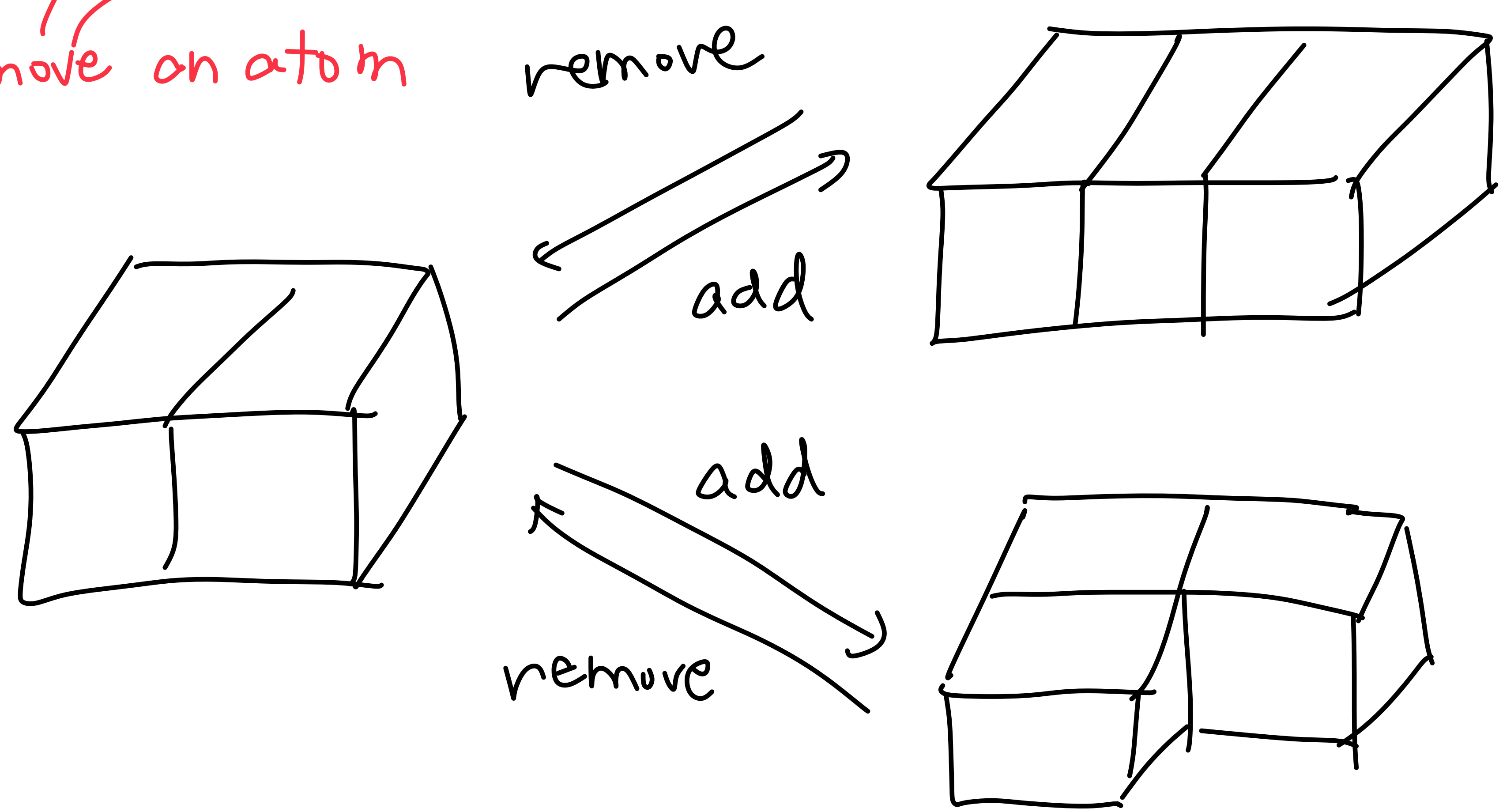
crystal

$$\psi^{(a)}(z)|\mathrm{K}\rangle = \Psi^{(a)}_{\mathrm{K}}(z)|\mathrm{K}\rangle ,$$

$$e^{(a)}(z)|\mathrm{K}\rangle = \sum_{\boxed{a}\in \mathrm{Add}(\mathrm{K})} \frac{E^{(a)}(\mathrm{K}\to\mathrm{K}+\boxed{a})}{z-h(\boxed{a})}|\mathrm{K}+\boxed{a}\rangle ,$$

$$f^{(a)}(z)|\mathrm{K}\rangle = \sum_{\boxed{a}\in \mathrm{Rem}(\mathrm{K})} \frac{F^{(a)}(\mathrm{K}\to\mathrm{K}-\boxed{a})}{z-h(\boxed{a})}|\mathrm{K}-\boxed{a}\rangle ,$$

add/remove an atom



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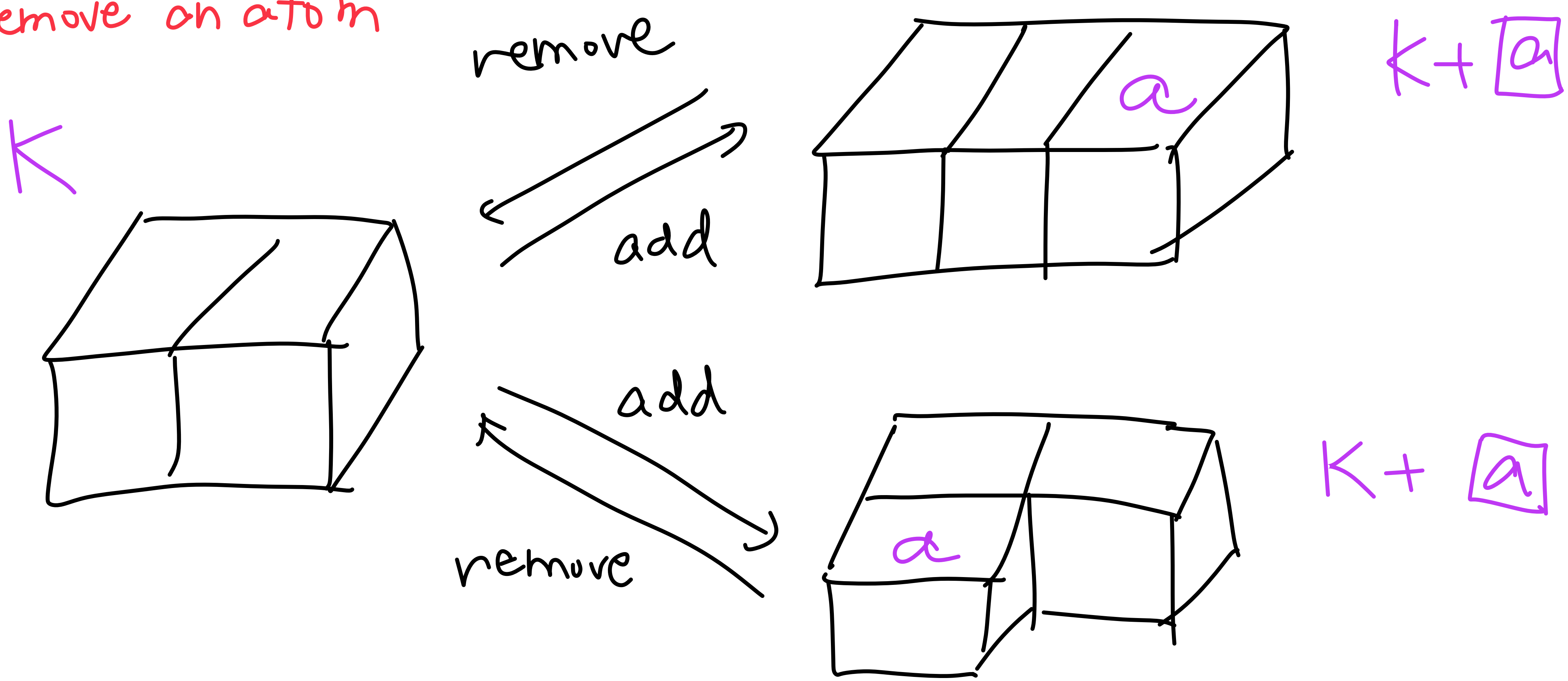
poles for atom $\boxed{a}$

add/remove an atom



Representation by crystal melting [Li-MY '20], inspired by [FFJMM] and [Prochazka]

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poles for atom $\boxed{a}$

$$h(\boxed{a}) \equiv \sum_{I\,\in\,\mathrm{path}[\mathfrak{o}\to\boxed{a}]} h_I\ .$$

$\Psi^{(a)}_{\mathrm{K}}$:
$$\Psi^{(a)}_{\mathrm{K}}(u) = \psi^{(a)}_0(z)\prod_{b\in Q_0}\prod_{\boxed{b}\,\in \mathrm{K}} \varphi^{b\Rightarrow a}(u-h(\boxed{b}))\ ,$$

$$\varphi^{a\Rightarrow b}(u) \equiv \frac{\prod_{I\in\{b\to a\}}(u+h_I)}{\prod_{I\in\{a\to b\}}(u-h_I)}$$

$E^{(a)}/F^{(a)}$:
$$E^{(a)}/F^{(a)} = \sqrt{\pm \operatorname*{Res}_{u=h(\boxed{a})} \Psi^{(a)}_{\mathrm{K}}(u)}$$

Representation by crystal melting [Li-MY '20], inspired by [FFJMM] and [Prochazka]

$$
\begin{aligned}
\psi^{(a)}(z)|\mathrm{K}\rangle &= \Psi^{(a)}_{\mathrm{K}}(z)|\mathrm{K}\rangle\,, \\
e^{(a)}(z)|\mathrm{K}\rangle &= \sum_{\boxed{a}\in \mathrm{Add}(\mathrm{K})} \frac{E^{(a)}(\mathrm{K}\to\mathrm{K}+\boxed{a})}{z-h(\boxed{a})}|\mathrm{K}+\boxed{a}\rangle\,, \\
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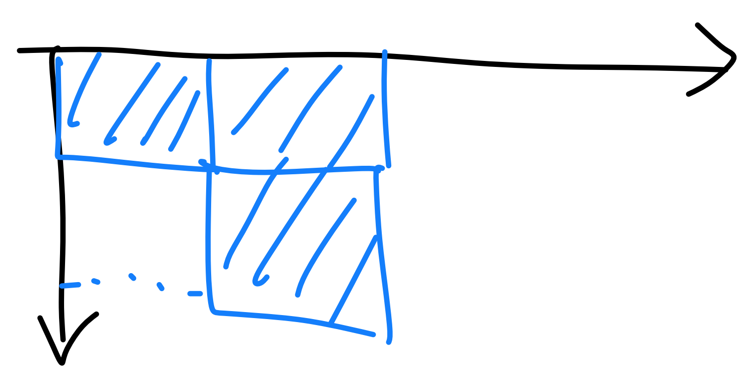
\end{aligned}
$$

poles for atom $\boxed{a}$

$E^{(a)}/F^{(a)}$: $E^{(a)}/F^{(a)} = \sqrt{\pm \operatorname*{Res}_{u=h(\boxed{a})} \Psi^{(a)}_{K}(u)}$

※ Residue = 0 when crystal "not allowed"

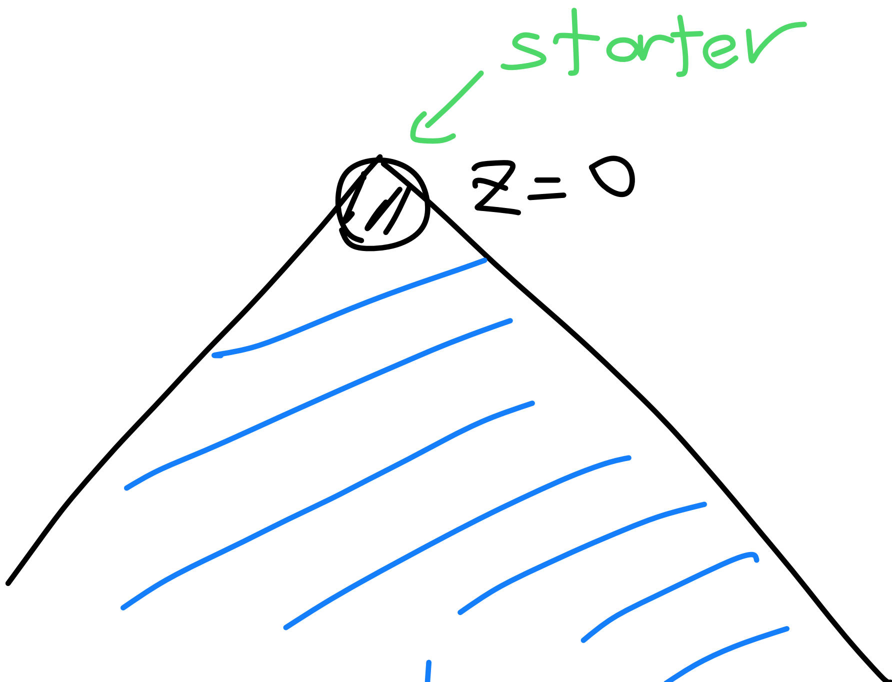
e.g.



$$\psi^{(a)}(z)|\emptyset\rangle = \underbrace{\psi_0^{(a)}(z)}_{\text{vacuum charge function}}|\emptyset\rangle$$

[Galakhov-Li-MY '21]

vacuum charge function $\longleftrightarrow$ representation

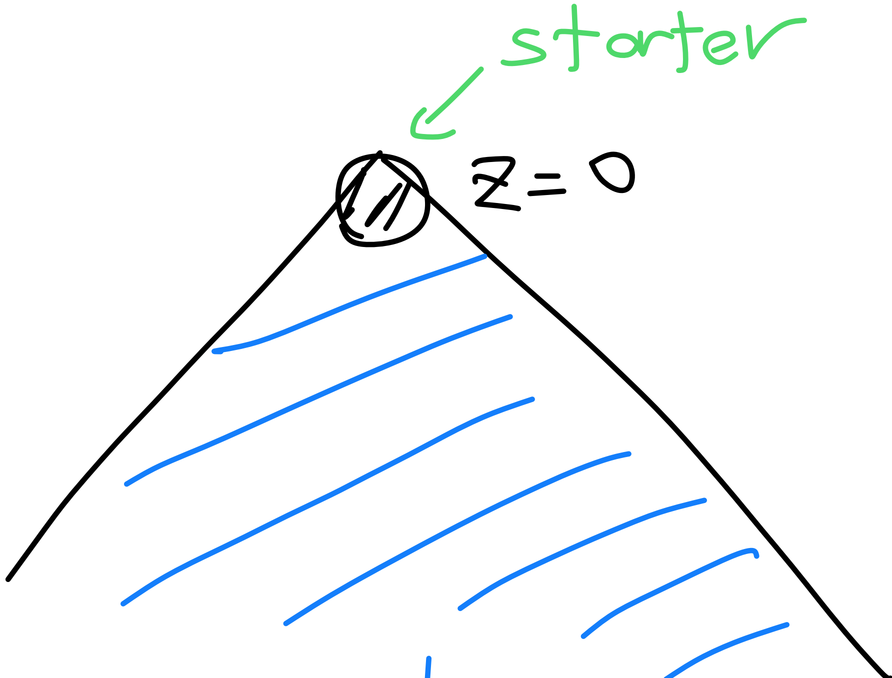


$\psi_0 \sim \frac{1}{z}$

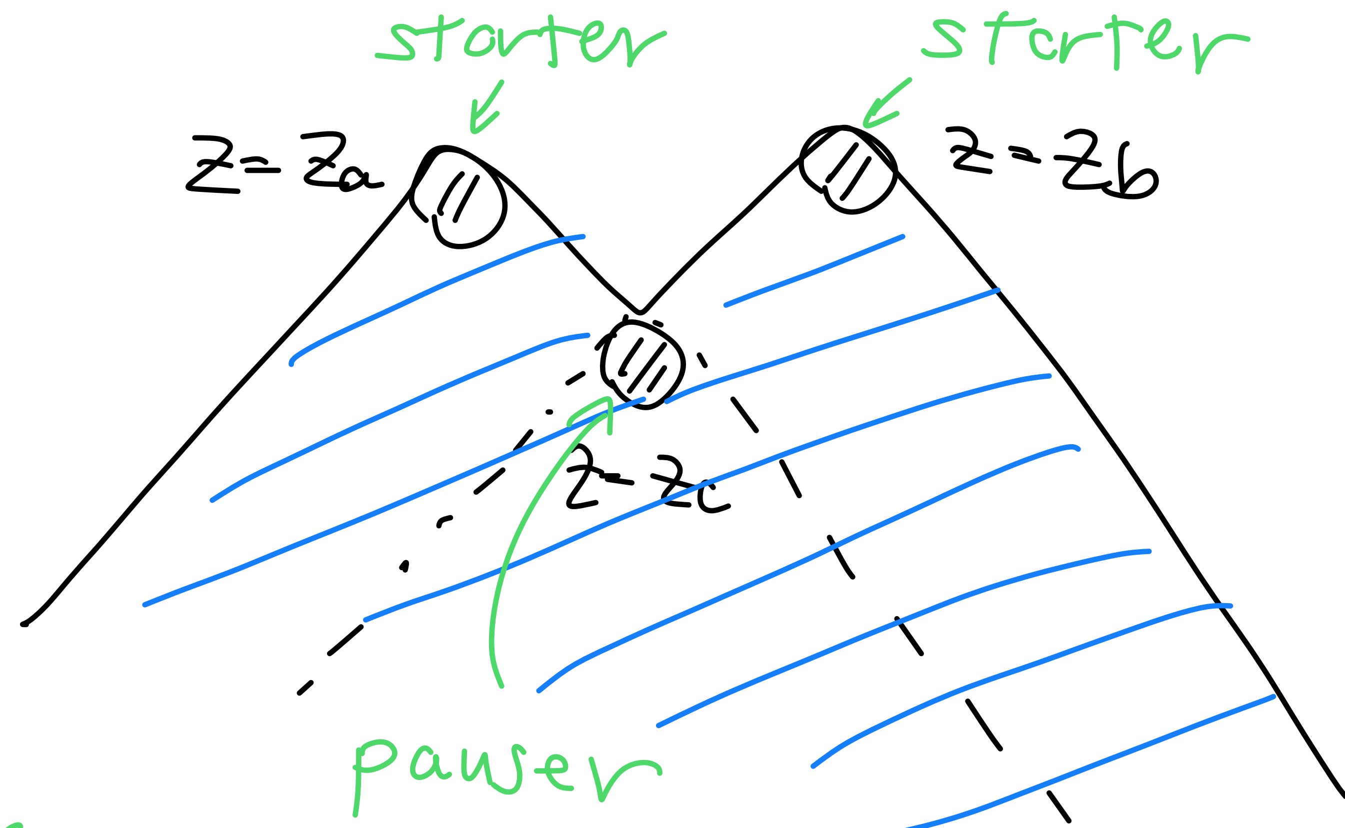
$$\psi^{(a)}(z)\,|\emptyset\rangle = \underbrace{\psi_0^{(a)}(z)}\,|\emptyset\rangle$$

[Galakhov-Li-MY '21]

vacuum charge function $\longleftrightarrow$ representation

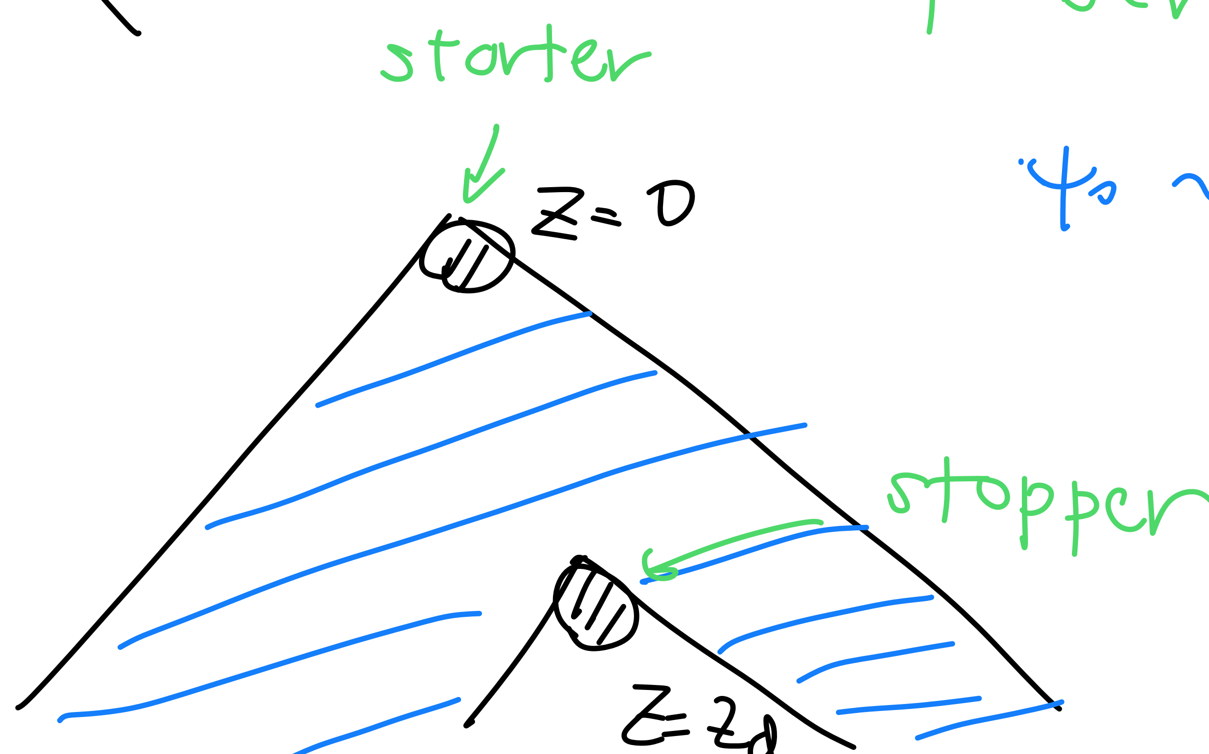


$$\psi_0 \sim \frac{1}{z}$$



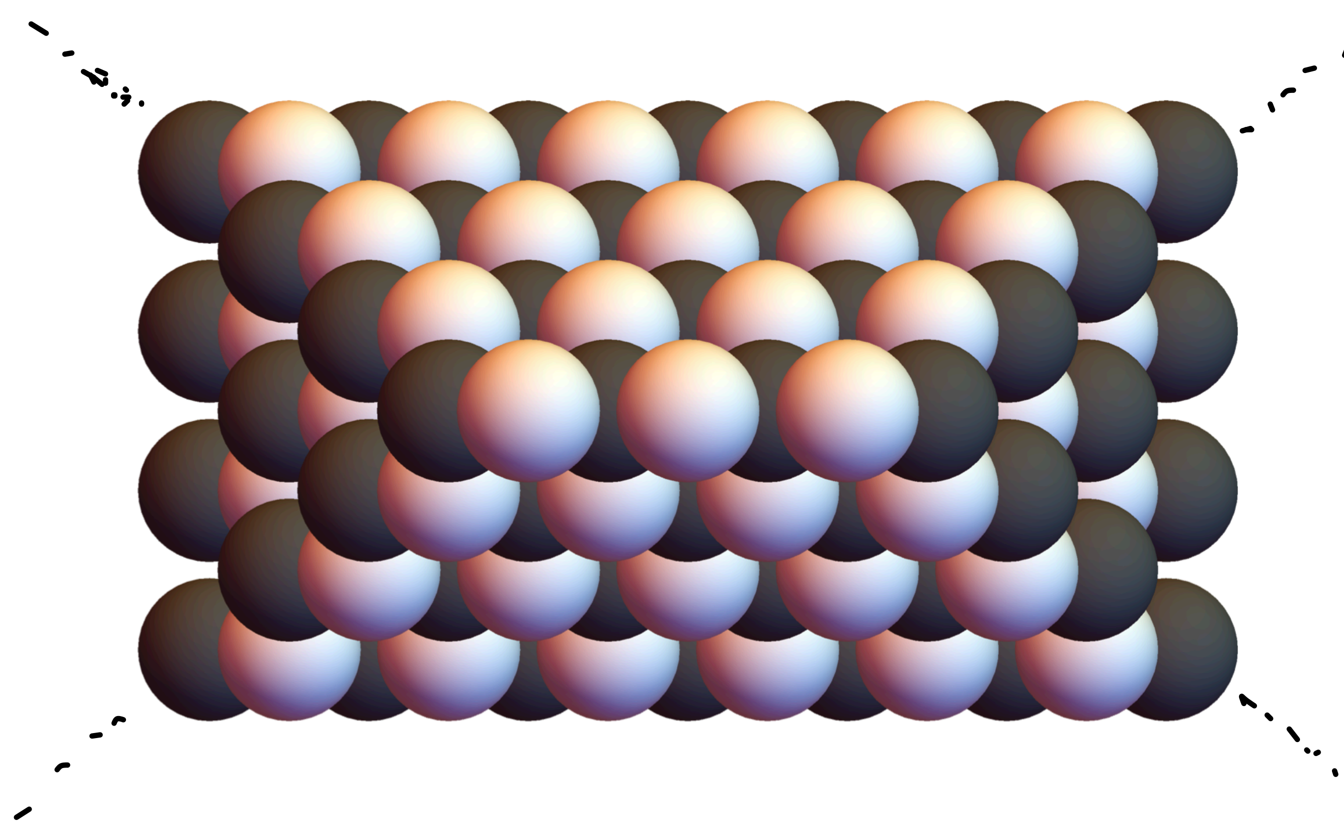
$$\psi_0 \sim \frac{z-z_c}{(z-z_a)(z-z_b)}$$

$$\psi_0 \sim \frac{z-z_d}{z}$$

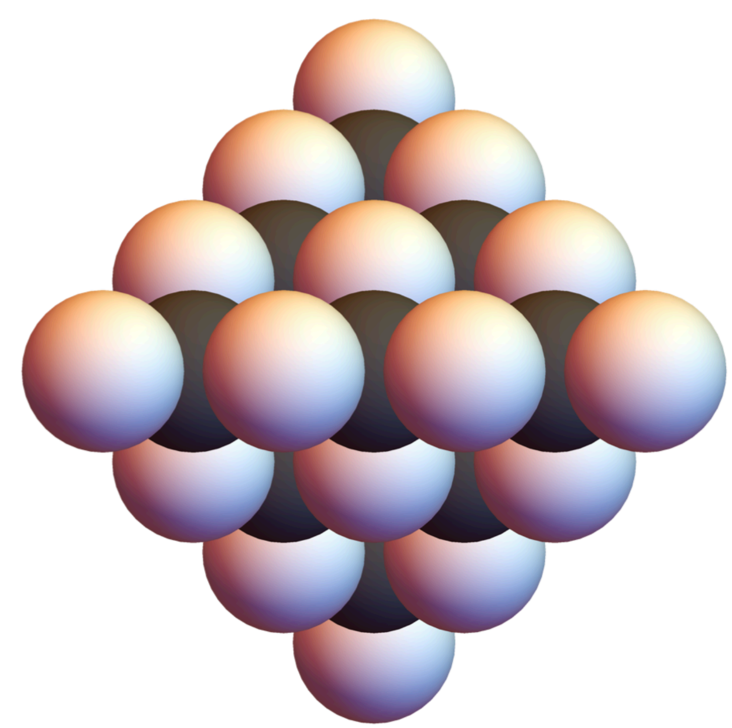


We can obtain rather general reps by using starter / pauser / stoppers

e.g. open / closed BPS state counting and their wall crossings



conifold : ∞-chamber

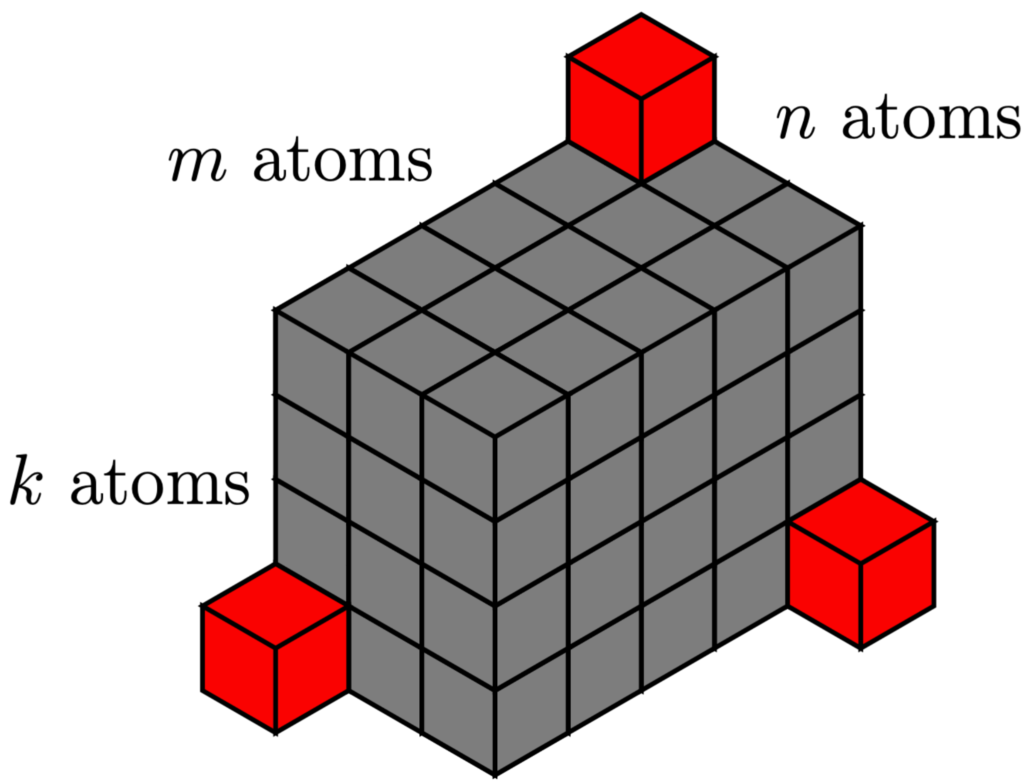


conifold : finite chamber

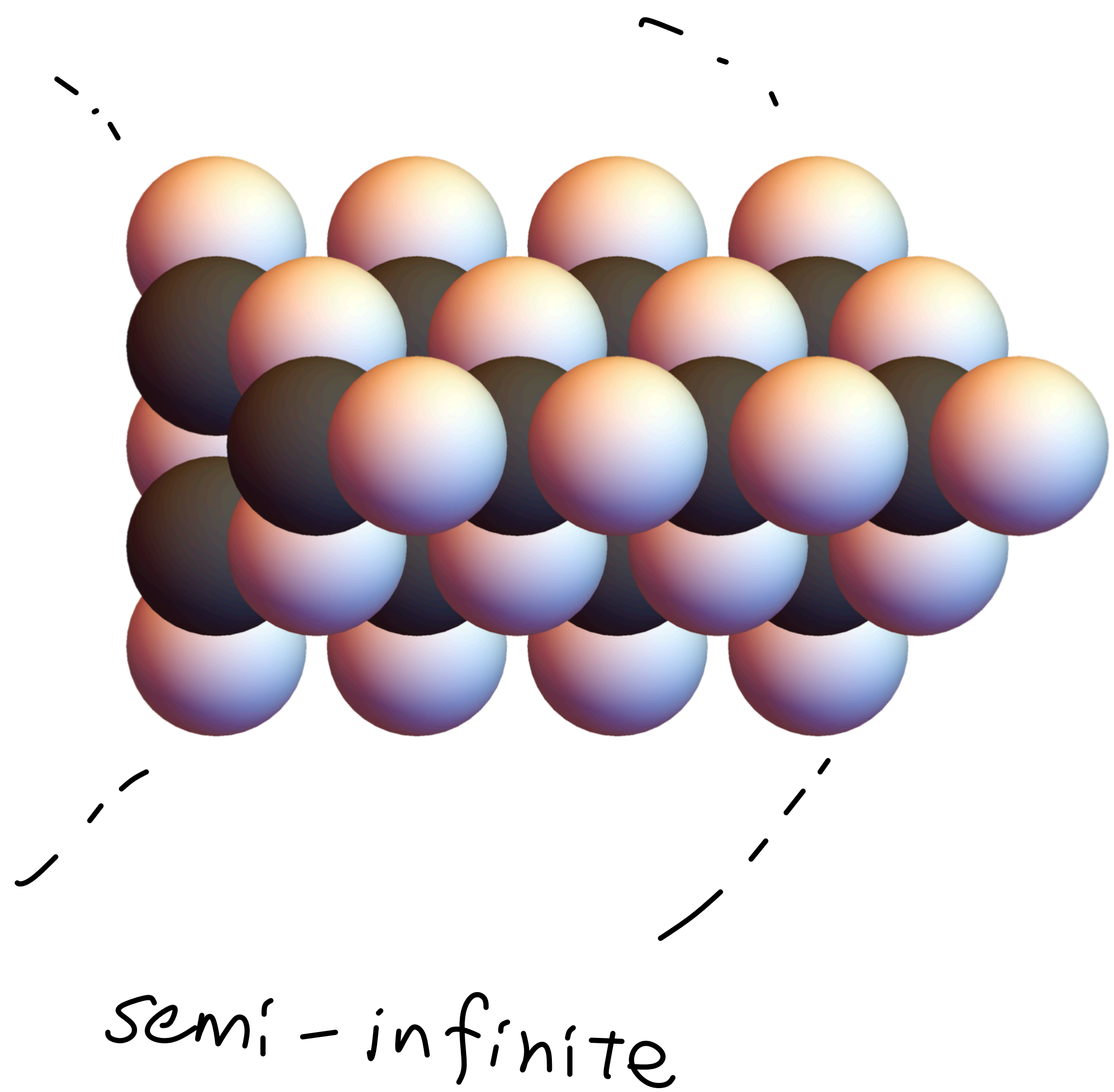
[Nagao-Nakajima; Jafferis-Moore; Chuang-Jafferis,⋯ '08]

Some representations have no known CY_3/geometry counterparts

$Y(\hat{gl}_1)$ $\mathbb{C}^3$-like

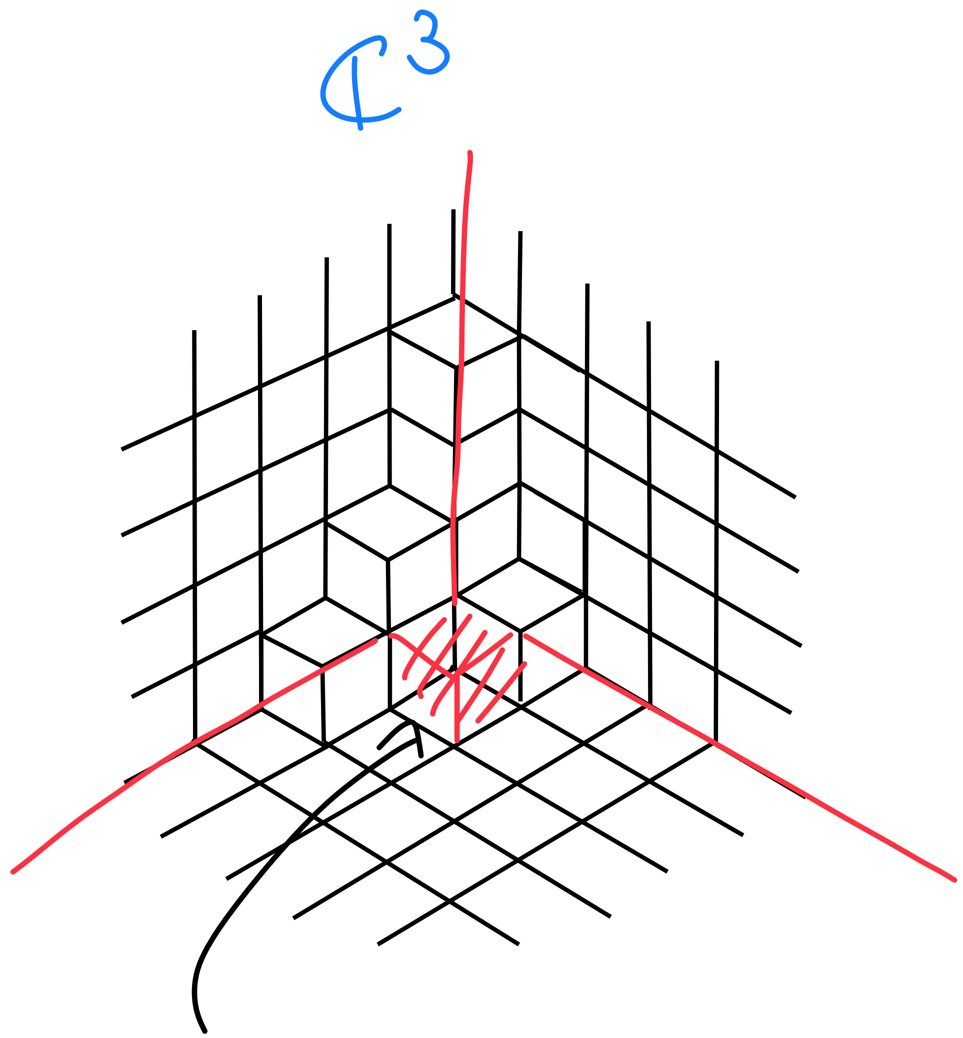


$Y(\hat{gl}_{1|1})$ conifold-like



[Galakhov-Li-MY '21]

example of truncation

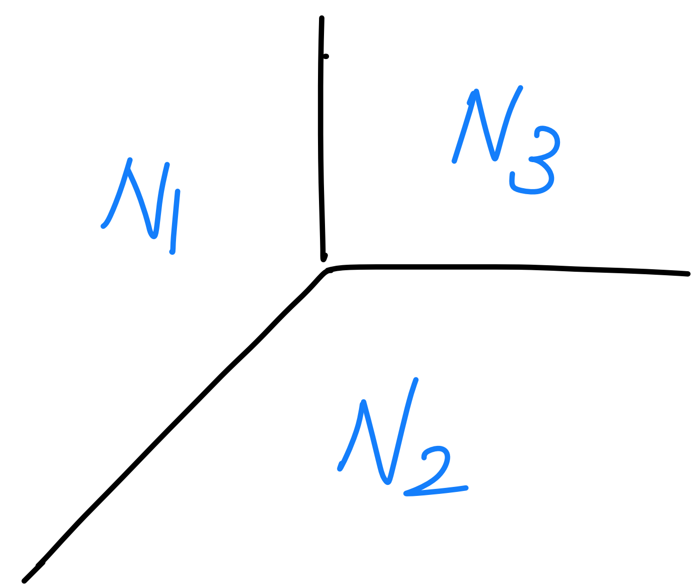


pit: location of null state

@ (N_1, N_2, N_3)

There is a corresponding truncation of the algebra studied by [Gaiotto-Rapcak]

$$Y(\hat{\mathfrak{gl}}_1) \to Y_{N_1, N_2, N_3}$$



D-branes wrapping divisors

(framing of quiver)

Summary

$$
\begin{tikzcd}
\begin{array}{c}\text{Toric } CY_3 \\ + \text{ D-branes}\end{array} \arrow[r] \arrow[d] & Z_{BPS} \arrow[d, equal] \\
\begin{array}{c}\text{Quiver QM} \\ (Q, W)\end{array} \arrow[r] \arrow[d] & Z_{crystal} \arrow[d, equal] \\
\begin{array}{c}\text{Quiver Yangian} \\ Y(Q, W)\end{array} \arrow[r, "\text{rep.}"] & \chi_{QY}
\end{tikzcd}
$$

* More general than toric cases

but not always ....

CY 4-fold [Bao-MY ('24))]
More general quivers
[Bao-MY (To Appear)]

* More studies needed in math/phys

* Q: General Lessons about

Quantum Geometries?