Generalized Chiral Potts models and Hyperbolic Monopoles from Chern–Simons Theory

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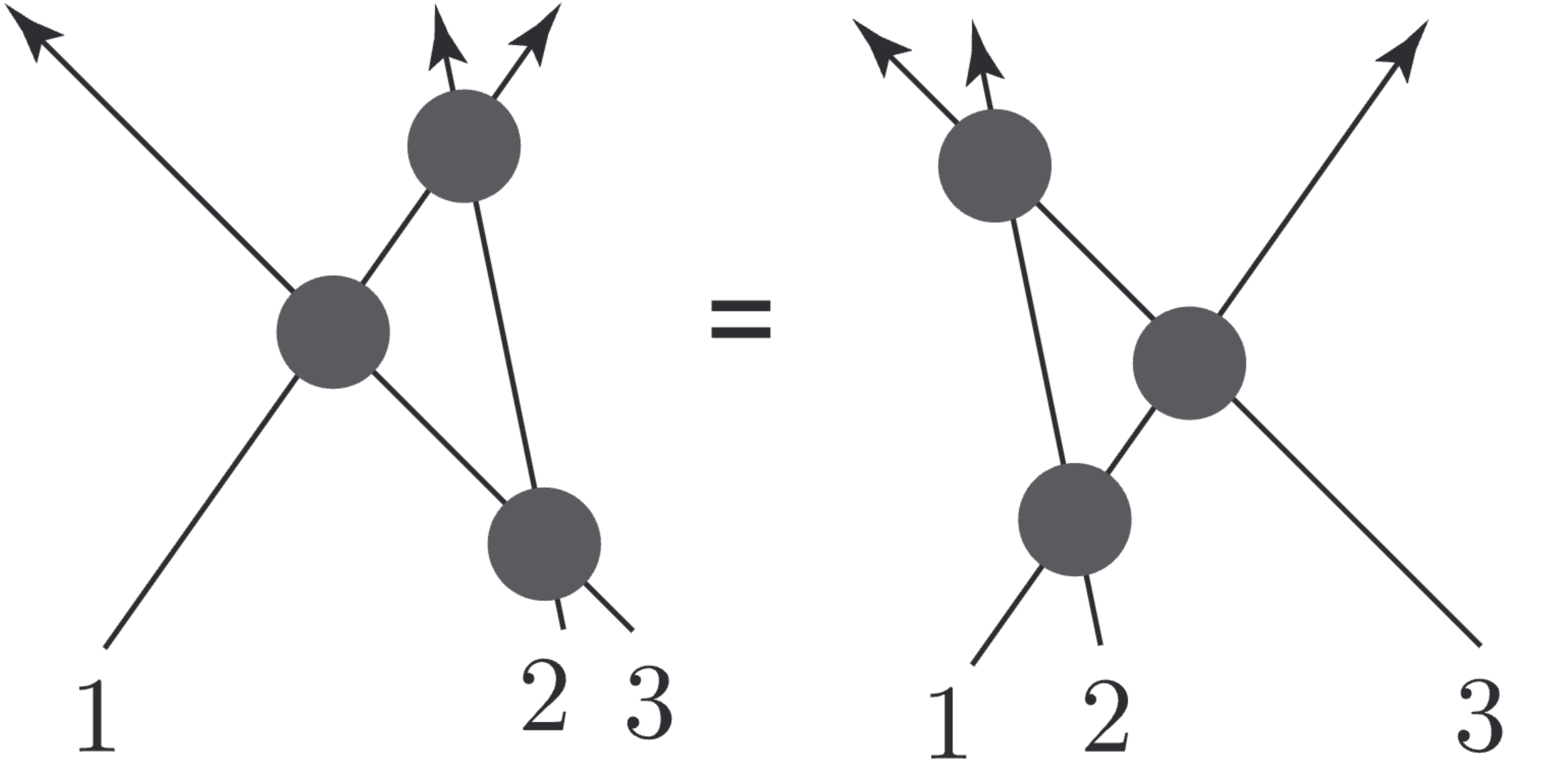
Sep. 11, 2025; in honor of Prof. Rodney J. Baxter


R.C.N.B.

Yang-Baxter : paradigm of integrable models



Breakthrough: Baxter's solution of 8v model

Eight-Vertex Model in Lattice Statistics

R. J. Baxter
Research School of Physical Sciences, The Australian National University, Canberra, A.C.T. 2600, Australia
(Received 25 February 1971)

The solution of the zero-field "eight-vertex" model is presented. This model includes the square lattice Ising, dimer, ice, F, and KDP models as special cases. It is found that in general the free energy has a branch-point singularity at a phase transition, with a continuously variable exponent.

It has been pointed out[1] that many of the previously solved two-dimensional lattice models, notably the Ising and "ice"-type models, can be regarded as special cases of a more general model. Adopting the arrow terminology used by Lieb,[3]

The method of attack on the problem was guided by some recent results[4] for an inhomogeneous system satisfying the "ice condition" ($d=0$), in which we observed that the Bethe *Ansatz* approach worked provided that the transfer matrices of any

8v model was difficult even for Feynman!

In particular, he hoped that tools for solving integrable models might be helpful for treating the soft part of QCD, the physics beyond the reach of renormalization-group improved perturbation theory. He wanted some students to study integrable models with him, to help him learn the subject. Well, he wanted students, and I had students, so we made an arrangement. Feynman and the students met once a week in his office, and those meetings would sometimes last all afternoon; a few times Feynman invited the students to dinner afterward.

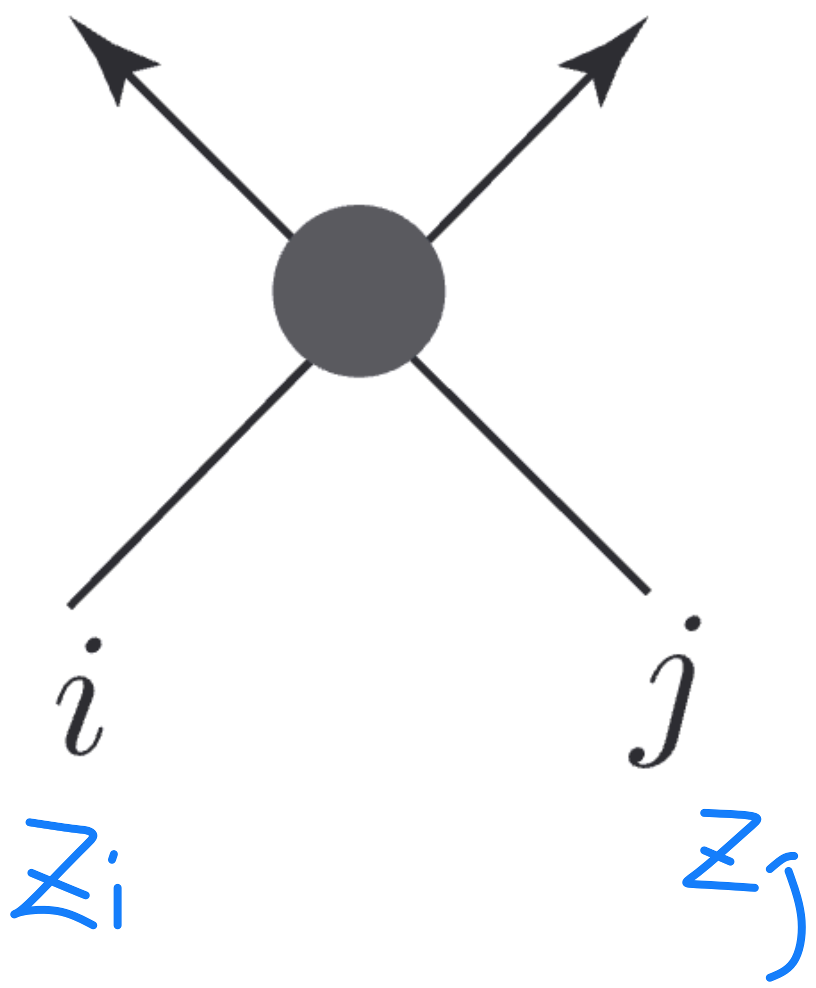
Feynman told the students "We gotta know how to solve every problem that has been solved," and he urged them to solve the problems on their own because "What I cannot create I do not understand." To get things started he described the six-vertex model, and told everyone to solve it without looking up any references [85]. That went on for weeks, without notable progress, until Feynman triumphantly unveiled his own solution. The next challenge was the eight-vertex model, but the students never solved that one, and neither did Feynman!

"Beyond Baxter?"

genus $g=1$ $\rightsquigarrow$ genus $g>1$?

elliptic functions higher-genus theta functions?

rapidity difference property

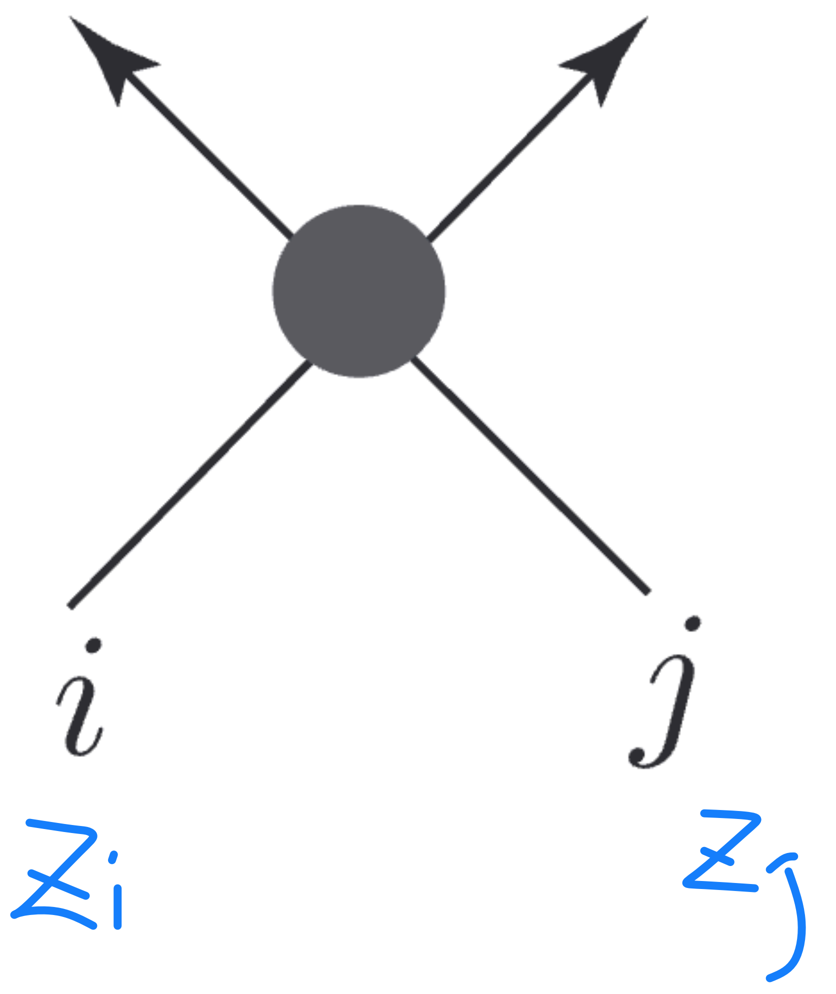


$$= R_{ij}(z_i, z_j) = R_{ij}\underbrace{(z_i - z_j)}$$

ill-defined

for $g > 1$

rapidity difference property



$$= R_{ij}(z_i, z_j) = R_{ij}(z_i - z_j)$$

quasi-classicality

$$R_{\hbar}(z_i, z_j) = I + \hbar \underbrace{r(z_i, z_j)}_{\text{classical r-matrix}} + O(\hbar^2)$$

classification [Belavin – Drinfeld ('82) revisited by Costello – Witten – MY ('17)]

rapidity difference property

+ quasi - classicality (+ α)

$\Rightarrow$ (rational, trigonometric, elliptic
$z \in \mathbb{C}$ $\quad$ $z \in \mathbb{C}^\times$ $\quad$ $z \in E$)

Difficult to go "beyond Baxter"!

"Baxter beyond Baxter"

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NEW SOLUTIONS OF THE STAR–TRIANGLE RELATIONS FOR THE CHIRAL POTTS MODEL

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Communicated by A.A. Maradudin

We present new explicit N-state solutions of the star–triangle relations for a nearest-neighbour two-spin interaction model. The solutions include families with real and positive Boltzmann weights. They are given in terms of two rapidities associated with two lines, which cross through each edge. The rapidities are 4-vectors, restricted to lie on the intersection of two Fermat surfaces. The usual difference property is not present.

also [Au-Yang, McCoy, Perk, Tang, Yan]

chiral Potts model

* No rapidity difference property

$$R(z, z') \neq R(z - z')$$

* No quasi-classical limit

$$R(z) \neq I + \hbar\, r(z) + O(\hbar^2)$$

* genus > 1 spectral curve

CP spectral curve

$$\left\{\begin{aligned} a^N + k' b^N &= k d^N \\ k' a^N + b^N &= k c^N \\ k a^N + k' c^N &= d^N \\ k b^N + k' d^N &= c^N \end{aligned}\right\} \subset \mathbb{P}^3_{a,b,c,d}$$

$$(k^2 + k'^2 = 1)$$

$g = N^2(N-2) + 1$: mysterious

Chiral Potts = loner ?

The Stony Brook researchers found their integrable three-state chiral Potts model had a genus of 10. At that time, string theorists were struggling with equations of genus 2. "For even looking at genus 10, we were ridiculed," Perk recalls.

recollections by Jacques Perk

[Physics today, 2005]

modern viewpoint: CP in a family

Kels
MY

Bazhanov
Sergeev

Kashiwara
Miwa

Fateev
Zamolodchikov

Faddeev
Volkov

chiral
Potts

Ising

4d $\mathcal{N}=1$
quiver
gauge theory

[MY ('12, '13)]

"Gauge/YBE"

saddle pt analysis

$\rightsquigarrow$ CP spectral curve

Surprising connection

with hyperbolic monopoles

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MAGNETIC MONOPOLES AND THE YANG-BAXTER EQUATIONS

Michael Atiyah
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ABSTRACT

A comparison is made between the new solutions of the Yang-Baxter equations, arising from curves of higher genus, and magnetic monopoles of higher charge. It is shown that essentially the same algebraic curves arise in both cases, and this leads to speculations about possibly more general solutions of the Yang-Baxter equations.

CP spectral curve

$$\left\{\begin{aligned} a^N + k' b^N &= k d^N \\ k' a^N + b^N &= k c^N \\ k a^N + k' c^N &= d^N \\ k b^N + k' d^N &= c^N \end{aligned}\right\} \Big/ \mathbb{Z}_N =: \Sigma_N$$

in $\mathbb{P}^3(a,b,c,d)$

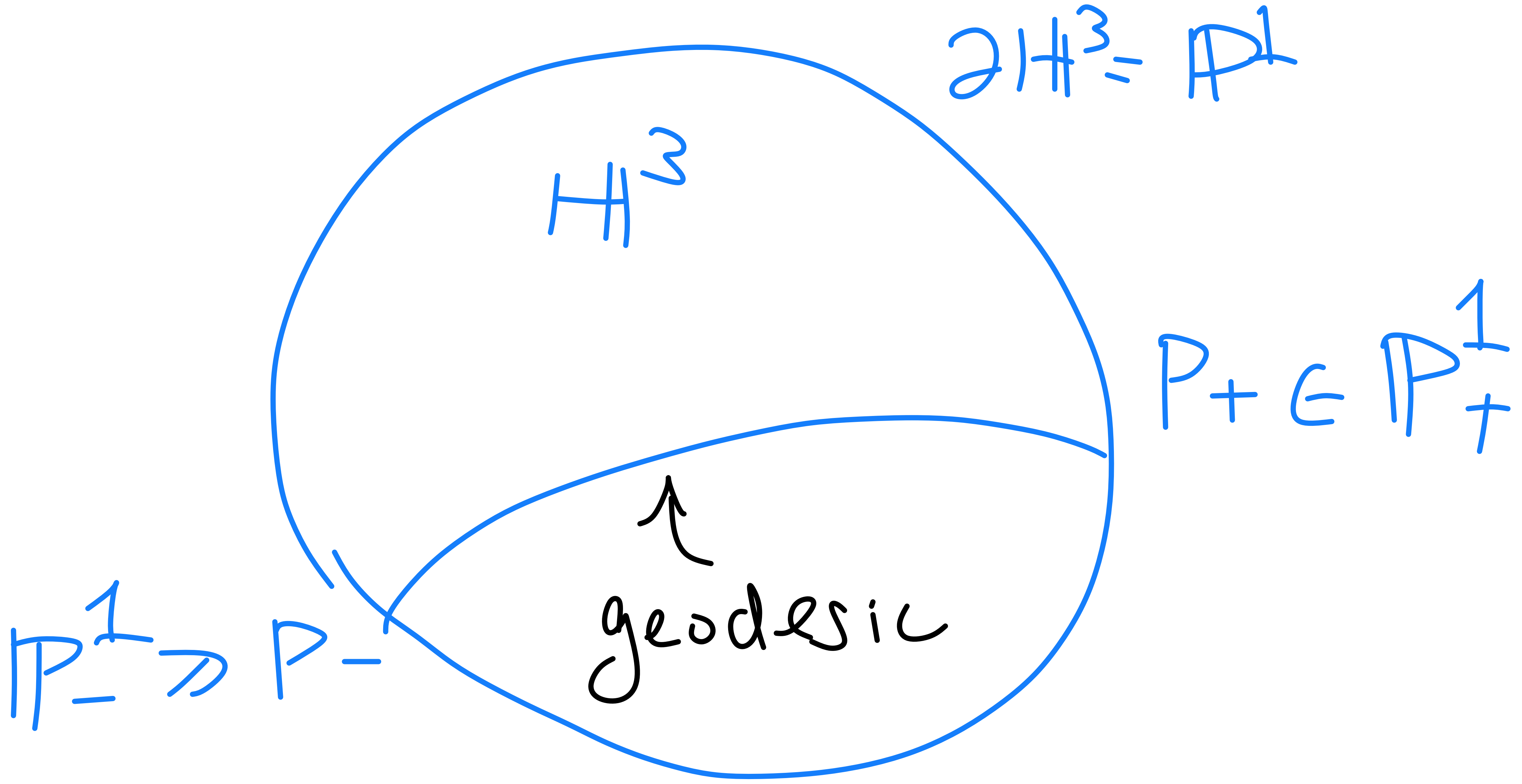
$$\left[(a,b,c,d) \sim (\omega a, b, c, \omega d) \quad \omega^N = 1 \right]$$

$$\Sigma_N \subset \frac{\mathbb{P}^3 - (\mathbb{P}^1_+ \cup \mathbb{P}^1_-)}{\mathbb{C}^\times} \simeq \mathbb{P}^1_+ \times \mathbb{P}^1_-$$

Hyperbolic Monopole

Twistor correspondence [Atiyah, Ward, Hitchin, ...]:

monopoles $\longrightarrow$ hol. bundle on twistor space.




Atiyah


Hitchin

spectral curve $\Sigma_N' \subset \mathbb{P}^1_+ \times \mathbb{P}^1_-$ ← mini-twistor

- [Atiyah & Murray ('91)] pointed out identification

$$\Sigma_N \longleftrightarrow \Sigma_{N'} \quad \text{for} \quad G = SU(2)$$

chiral Potts $\qquad$ hyperbolic monopole

New Results in [Moosavian - MY - Zhou]

(after $\gtrsim 30$ years ...)


Moosavian


Zhou

1: Generalization to $G = SU(N)$

2: Framework for conceptual explanation

Generalized Chiral Potts

[Bazhanov - Kashaev - Mangazeev - Stroganov]

A_1 : $\mathbb{Z}_N$ spin $\rightsquigarrow$ A_n : $(n-1)$ - copies of $\mathbb{Z}_N$ spins

spectral curve :

$$A_{n-1} : \; g = N^{2(n-1)}\left((n-1)N - n\right) + 1$$

spectral curve:

$$[z_1^+, z_1^-, \cdots, z_n^+, z_n^-] \in \mathbb{P}^{2n-1}$$

$$\boxed{\tilde{\Sigma}_{N,n} : \quad \begin{pmatrix} z_i^{+N} \\ z_i^{-N} \end{pmatrix} = K_{ij} \begin{pmatrix} z_j^{+N} \\ z_j^{-N} \end{pmatrix} \quad i, j = 1, \ldots, n}$$

$$K_{ii} = I, \quad K_{ij} K_{jk} K_{ki} = I, \quad K_{ij} = K_{ji}^{-1}$$

Imitate Atiyah ...

$$\Sigma_{N,n} = \tilde{\Sigma}_{N,n} \,/\, (\mathbb{Z}_N)^{n-1}$$

free action

$$[z_1^+, z_1^-, \dots, z_n^+, z_n^-]$$

$$\mapsto [\omega^{k_1} z_1^+, \omega^{k_1} z_1^-, \dots, \omega^{k_{n-1}} z_{n-1}^+, \omega^{k_{n-1}} z_{n-1}^+, z_n^+, z_n^-]$$

$$(\omega^n = 1)$$

"Decompose A_n into copies of A_1"

$$\mathbb{P}^{2n-1} \qquad\qquad [z_1^+, z_1^-, \dots, z_n^+, z_n^-]$$

$$\downarrow \qquad\qquad \downarrow$$

$$\underbrace{\mathbb{P}^1 \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1}_{n} \qquad\qquad ([z_1^+, z_1^-], \dots, [z_n^+, z_n^-])$$

$$\downarrow \qquad\qquad \downarrow$$

$$S_i \subset \mathbb{P}^1_i \times \mathbb{P}^1_{i+1} \qquad\qquad ([z_i^+, z_i^-], [z_{i+1}^+, z_{i+1}^-])$$

$$\underbrace{\text{match with } SU(n) \text{ monopole data}}$$

$(n-1)$ curves $\quad \underbrace{S_i \subset \mathbb{P}^1 \times \mathbb{P}^1}_{\text{bidegree } (N,\, N)}$

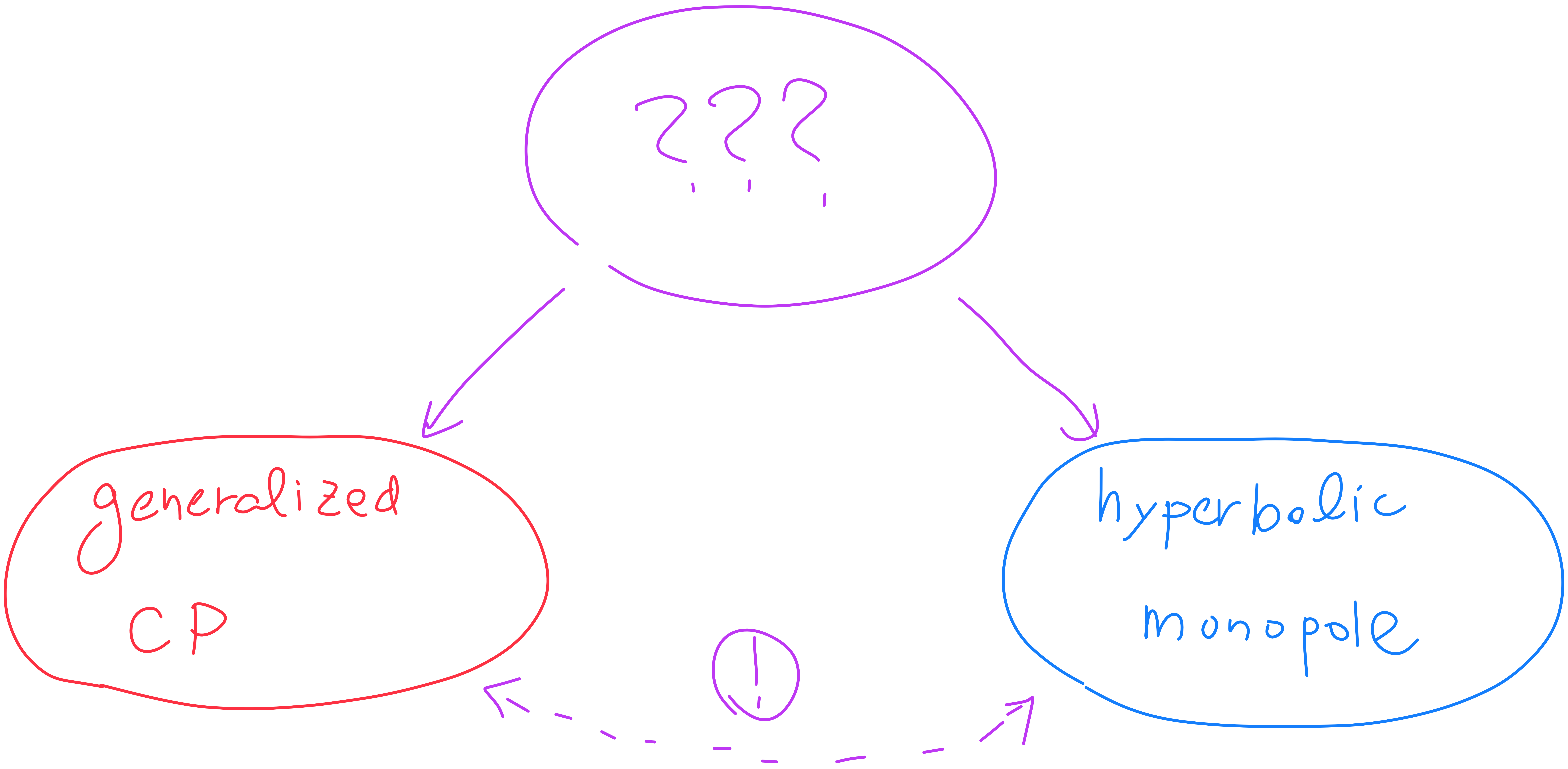
$\updownarrow$

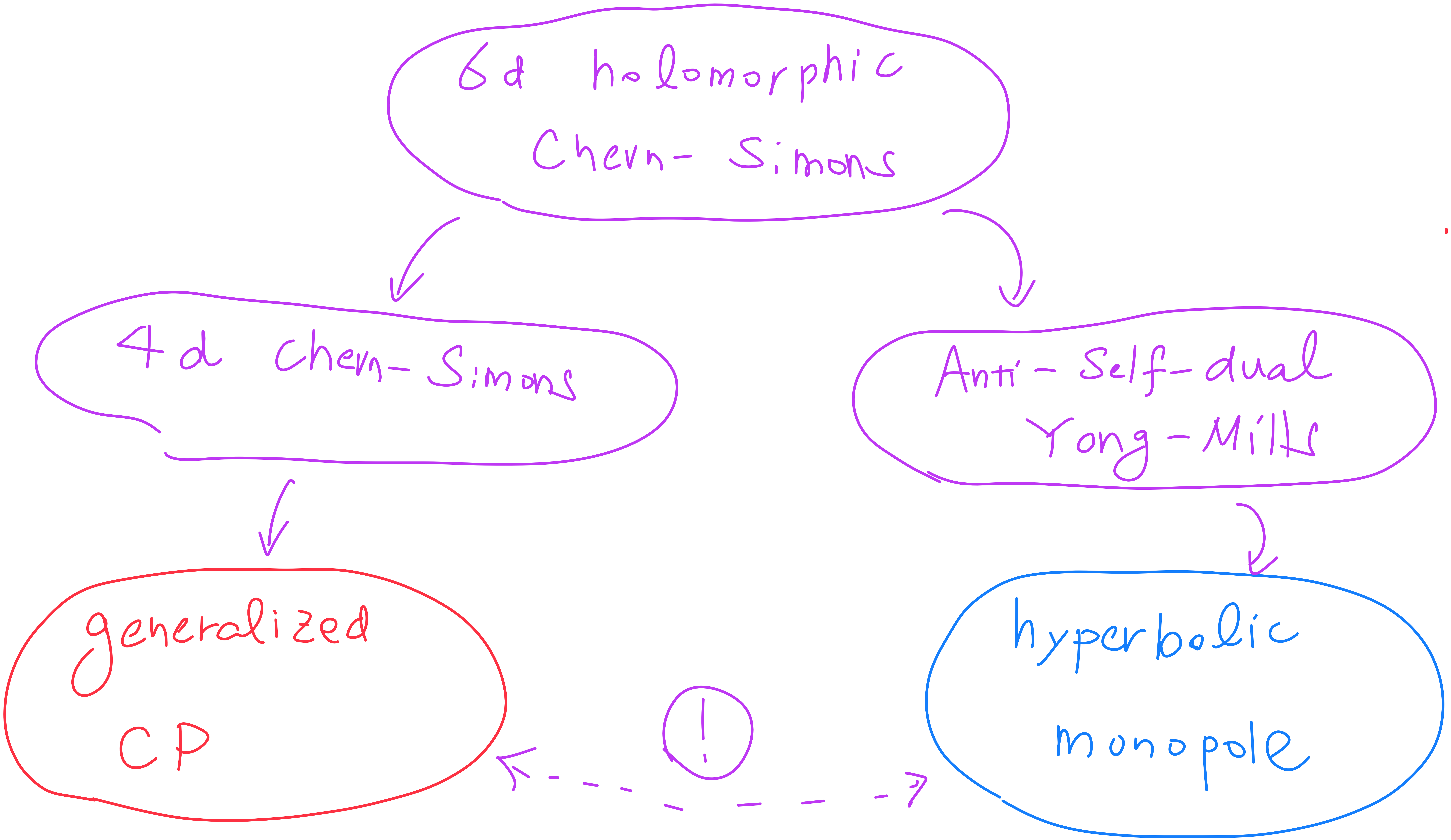
$S_i \cap S_{i+1}$: $2N^2 = N^2 + N^2$ points

involution

$(n-1)$ simple roots of $SU(n)$

Explanation?


???
generalized CP
hyperbolic monopole
!


6d holomorphic Chern-Simons
4d Chern-Simons
Anti-Self-dual Yang-Mills
generalized CP
!
hyperbolic monopole

4d ASDYM

$F = *F$ on $\mathbb{R}^4$

$S^1 \Big\downarrow \quad \mathbb{R}^4 - \mathbb{R}^2 \simeq \mathbb{H}^3 \times S^1$

$$\left(dr^2 + r^2 d\theta^2 + d\vec{x}^2 = r^2 \left[\frac{(dr)^2 + (d\vec{x})^2}{r^2} + (d\theta)^2 \right] \right)$$

3d hyperbolic monopole

$F = * d\phi$ on $\mathbb{H}^3$

4d ASDYM

$F = *F$ on $\mathbb{R}^4$

twistor

$\mathbb{P}^3$

$$\Big\downarrow S^1$$

3d hyperbolic monopole

$F = * d\phi$ on $\mathbb{H}^3$

mini-twistor

$$\left(\mathbb{P}^3 - (\mathbb{P}^1 \cup \mathbb{P}^1)\right) / \mathbb{C}^\times$$

$$\cong \mathbb{P}^1_+ \times \mathbb{P}^1_+$$

6d hol. CS on $\mathbb{P}^3 - \mathbb{P}^1 \simeq \mathbb{R}^4 \times \mathbb{P}^1_S$

$\mathbb{P}^1_S \Big\downarrow$ [Costello Bittelson - Skinner]

flat connection

4d ASDYM (twistor space $\mathbb{P}^3$)

Lax eqn.

$S^1 \Big\downarrow$ $\mathbb{R}^4 - \mathbb{R}^2 \simeq S^1 \times \mathbb{H}^3$

$\mathbb{C}^\times$

3d hyperbolic monopole $\mathbb{H}^3$

(mini-twistor space $\mathbb{P}^1 \times \mathbb{P}^1$)

6d hol. CS

$\mathbb{P}^3 - \mathbb{P}^1 \simeq \mathbb{R}^4 \times \mathbb{P}^1$

$\mathbb{R}^4 \to \mathbb{R}^2 \times T^2$

[MY]

$\mathbb{P}^1$

4d top/hol. CS

$\underbrace{\mathbb{R}^2}_{\text{top.}} \times \underbrace{\mathbb{P}^1}_{\text{hol.}}$

4d ASD YM $\mathbb{R}^4$

S^1

3D hyperbolic monopole H^3

4d top./hol Chern-Simons

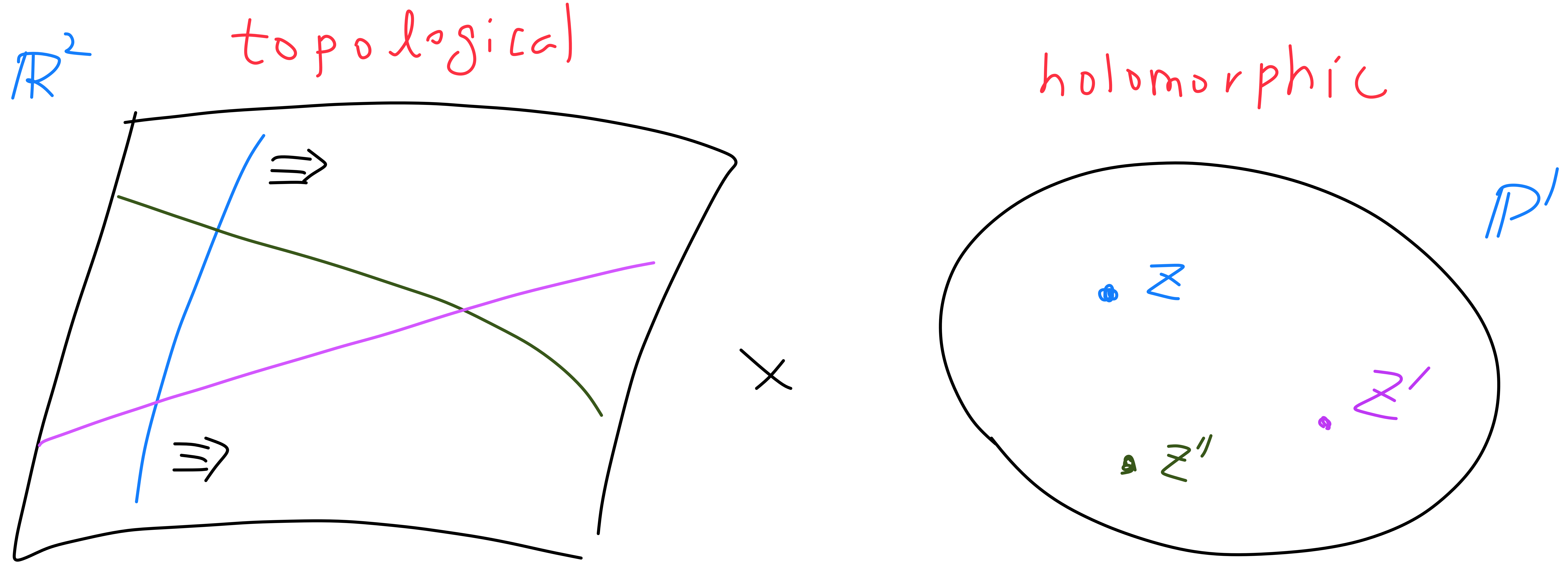
$$S = \frac{1}{2\pi\hbar}\int_{\mathbb{R}^2\times\mathbb{P}^1_z} dz \wedge \left(\mathrm{Tr}\; A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right)$$

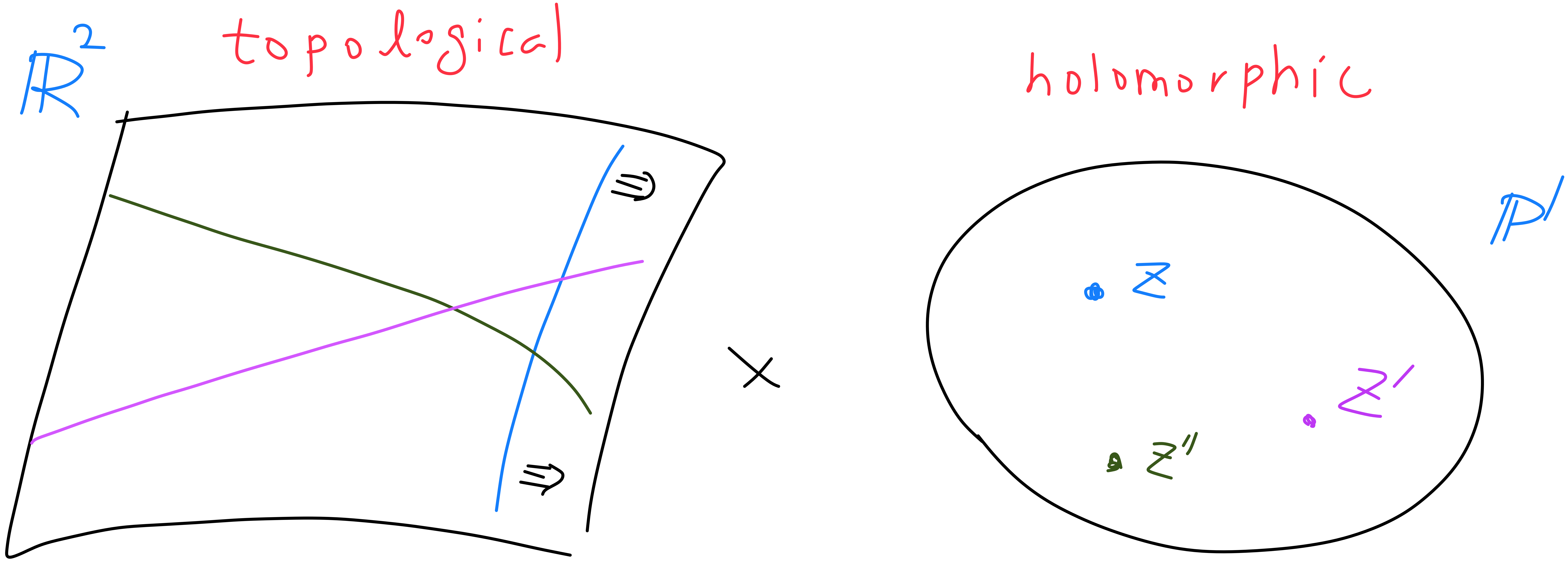
$\downarrow$ [Costello-Witten-MY ('17, '18)]

Costello

Witten

Integrable Models


$\mathbb{R}^2$
topological
$\times$
holomorphic
$\mathbb{P}^1$
z
z'
z''



topological invariance $\Longrightarrow$ YBE

$$
\begin{tikzcd}[column sep=small]
& \text{6d hol. CS} \quad \mathbb{P}^3 \arrow[dl, bend right, "\mathbb{R}^2 \to T^2"'] \arrow[dr, bend left, "\mathbb{P}^1 \quad (\mathbb{P}^3 - \mathbb{P}^1 \simeq \mathbb{R}^4 \times \mathbb{P}^1)"] & \\
\begin{array}{c}\text{4d top/hol. CS} \\ \underbrace{\mathbb{R}^2}_{\text{top.}} \times \underbrace{\mathbb{P}^1}_{\text{hol.}}\end{array} \arrow[d, Rightarrow, "??"'] & & \text{4d ASD YM} \quad \mathbb{R}^4 \arrow[d, "S^1 \quad (\mathbb{R}^4 - \mathbb{R}^2 \simeq S^1 \times H^3)"] \\
\text{Chiral Potts} \arrow[rr, dashed, leftrightarrow] & & \text{3D hyperbolic monopole} \quad H^3
\end{tikzcd}
$$

Chiral Potts from 4d Chern-Simons?

We seem to have $\mathbb{P}^1$ as a spectral curve

but CP is NOT rational

Chiral Potts from 4d Chern-Simons?

We seem to have $\mathbb{P}^1$ as a spectral curve

but CP is NOT rational

※ Actually, $\mathbb{P}^1$ with singularities / defects

Chiral Potts from 4d Chern-Simons?

Our proposal:

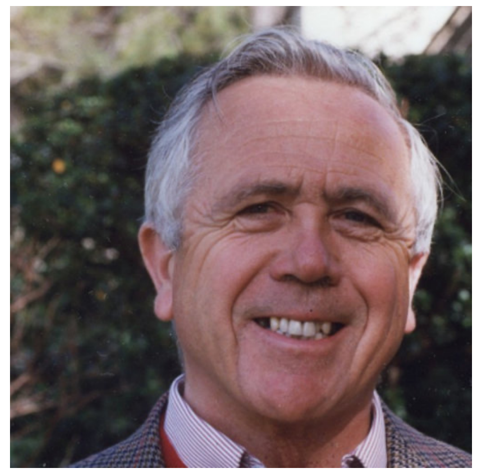
N-fold cover

$$\left(\begin{array}{c} \Sigma_{cp} \\ \pi \big\downarrow \\ \mathbb{P}^1 \end{array} \qquad \begin{array}{l} \omega_{\Sigma_{cp}} = \pi^* \omega_{\mathbb{P}^1} \\[2em] \omega_{\mathbb{P}^1} = \dfrac{dz}{\sqrt[N]{(z^N - z_1)(z^N - z_2)}} \end{array} \right)$$

branch cuts

$$\begin{tikzcd}
 & \text{6d hol. CS on } \mathbb{P}^3 \arrow[dl, bend right] \arrow[dr, bend left] & \\
\text{4d CS on } \Sigma_{cp} \arrow[d, "\pi"] \arrow[d, phantom, shift right=8ex, "\parallel"] & & \text{4d ASDYM} \arrow[d, bend left] \\
\text{4d CS on } \mathbb{P}^1 \text{ with branch pts} \arrow[dr, leftrightarrow, bend right] & & \text{3D hyperbolic monopole} \arrow[dl, leftrightarrow, dashed] \\
 & \text{chiral Potts model} &
\end{tikzcd}$$

Summary



- 8v model

- Chiral Potts model
- Hyperbolic monopole

4d CS

6d CS