$T\bar{T}$ and $\sqrt{T\bar{T}}$ - deformations in 4d Chern-Simons Theory

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Paper to appear tomorrow !!

(arXiv: 2509.XXXX)

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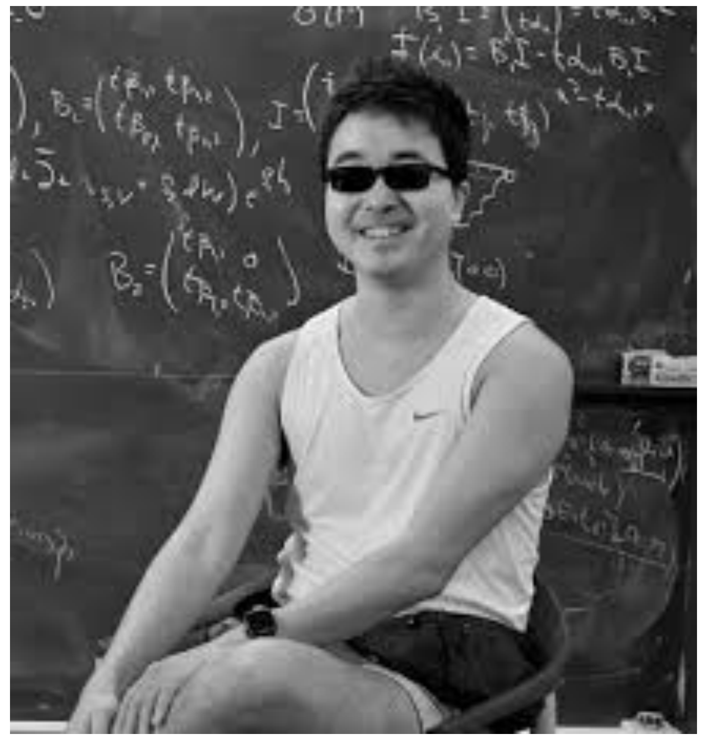
Jun-ichi Sakamoto



+ M.Y.

Motivations

Question

$T\bar{T}$, $\sqrt{T\bar{T}}$, ... deformations

$$\left(\begin{array}{c}\text{2d Integrable}\\ \text{QFT}\end{array}\right) \rightsquigarrow \left(\begin{array}{c}\text{2d Integrable}\\ \text{QFT}\end{array}\right)$$

$\lambda = 0$ $\qquad\qquad$ $\lambda \neq 0$

[Smirnov - Zamolodchikov, Cavaglià - Negro - Szécsényi - Tateo ('16)
Conti - Romano - Tateo, Ferko - Sfondrini - Smith - Tartaglino - Mazzucchelli ('22)]

Question

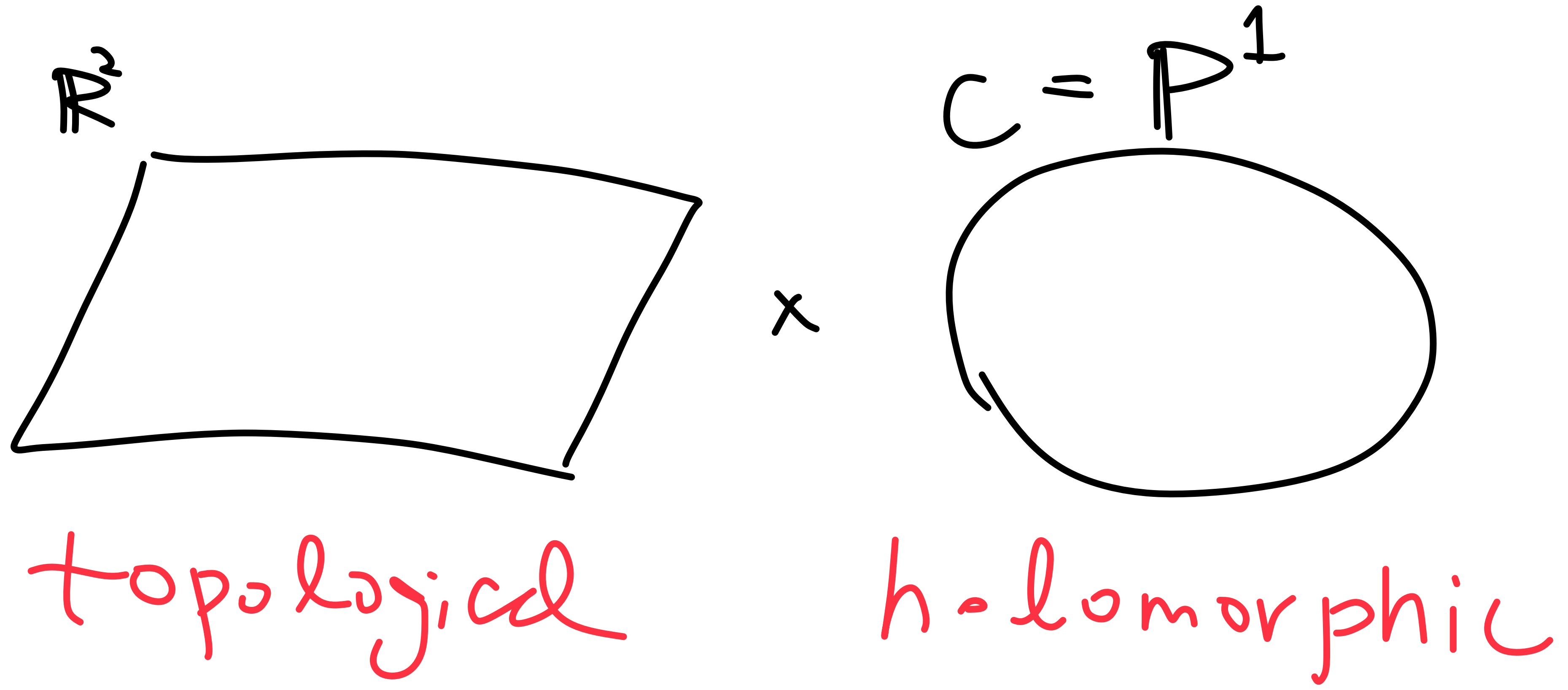
$$\begin{tikzcd}[column sep=huge, row sep=large]
(\text{4d CS + defect}) \arrow[r, squiggly, "\text{deformation}", "??"'] \arrow[d, squiggly, "\text{[Costello-MY '19]}"'] & (\text{4d CS + defect}) \arrow[d, squiggly] \\
\begin{pmatrix}\text{2d Integrable} \\ \text{QFT}\end{pmatrix} \arrow[r, squiggly, "\text{deformation}"'] & \begin{pmatrix}\text{2d Integrable} \\ \text{QFT}\end{pmatrix} \\
\lambda = 0 & \lambda \neq 0
\end{tikzcd}$$

$T\bar{T}$, $\sqrt{T\bar{T}}$, ...

4d CS

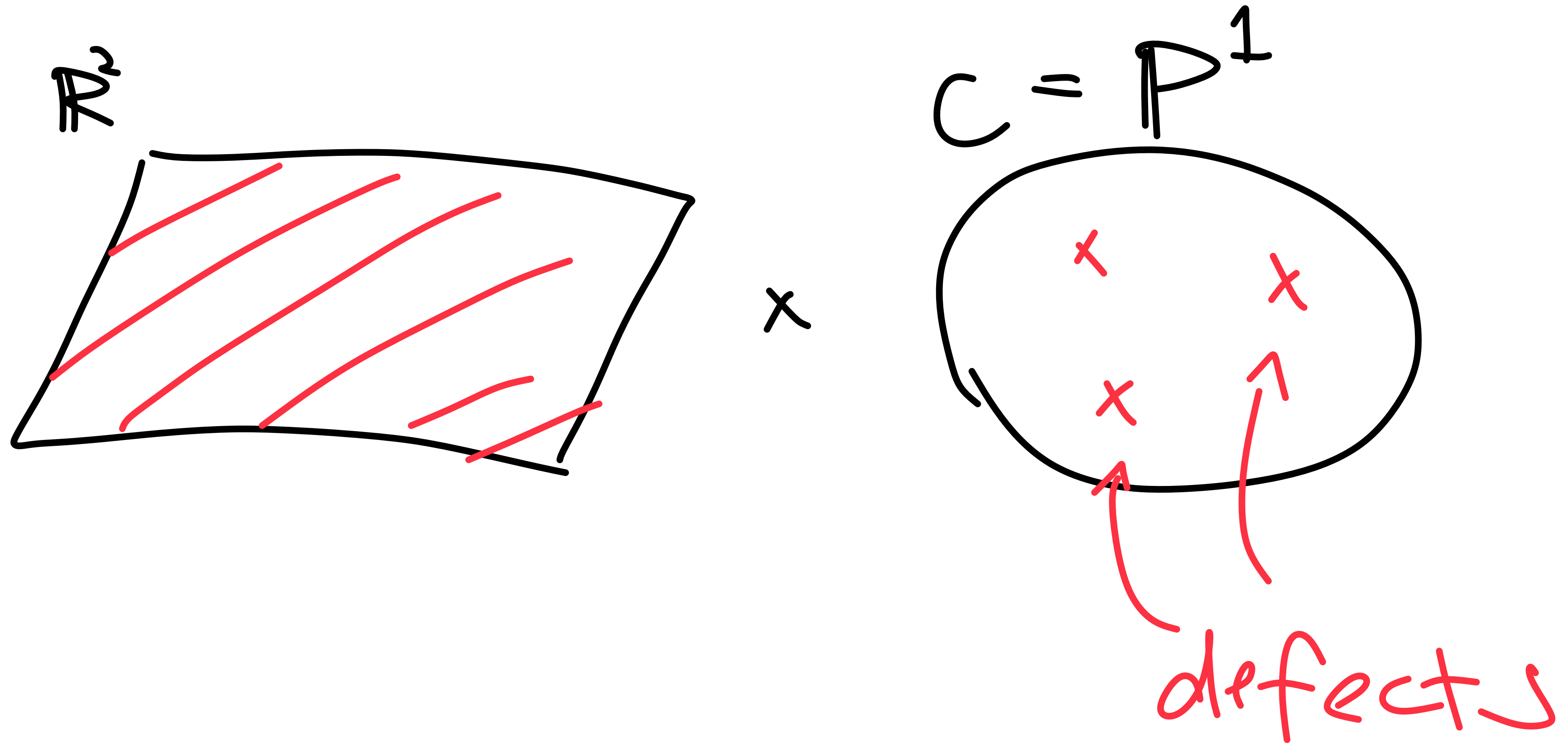
Costello – Witten – MY ('17, '18)
Costello – MY ('19)

$$S = \frac{1}{2\pi\hbar}\int_{\mathbb{R}^2 \times C} \omega \wedge \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right)$$

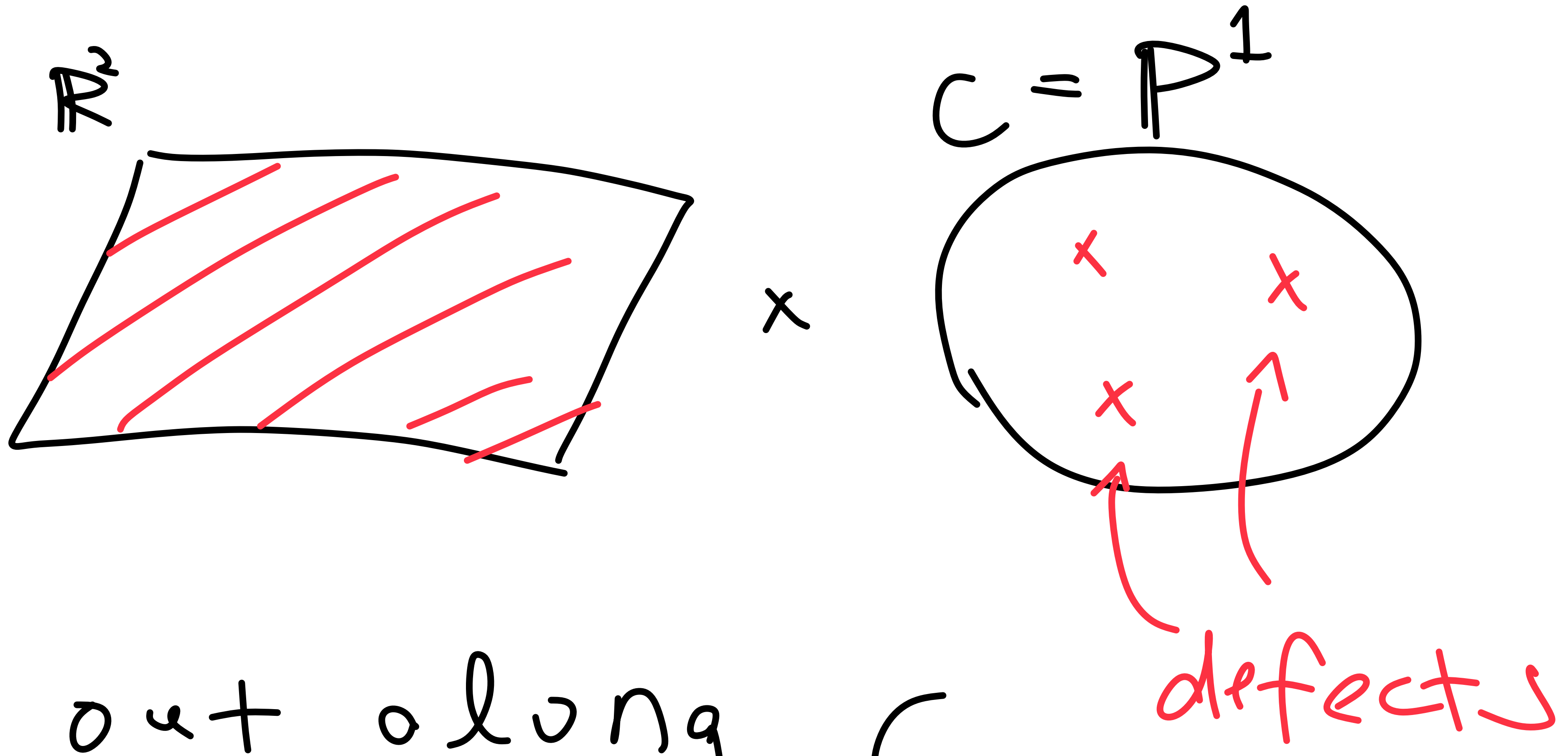


ω : one-form on C

$$S = \frac{1}{2\pi\hbar}\int_{\mathbb{R}^2 \times C} \omega \wedge \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right)$$

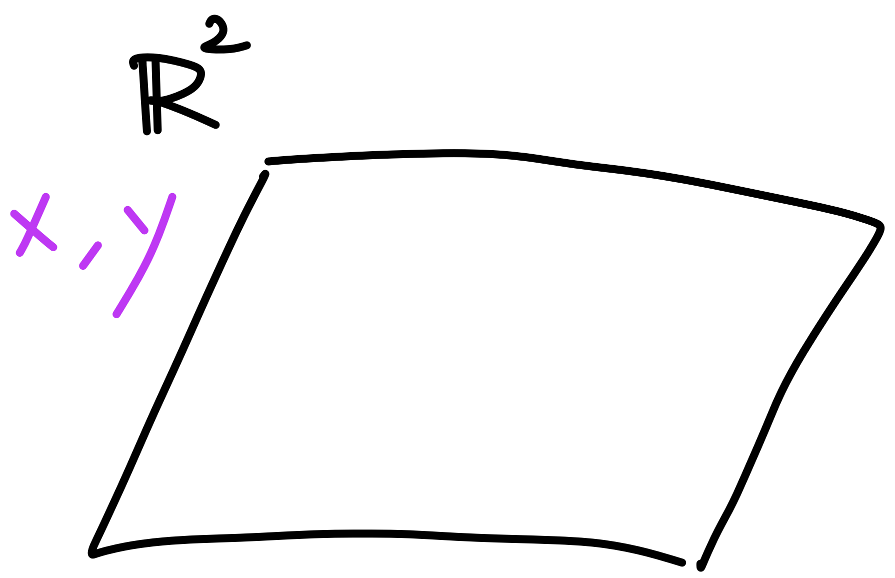


$$S = \frac{1}{2\pi\hbar}\int_{\mathbb{R}^2 \times C} \omega \wedge \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right)$$

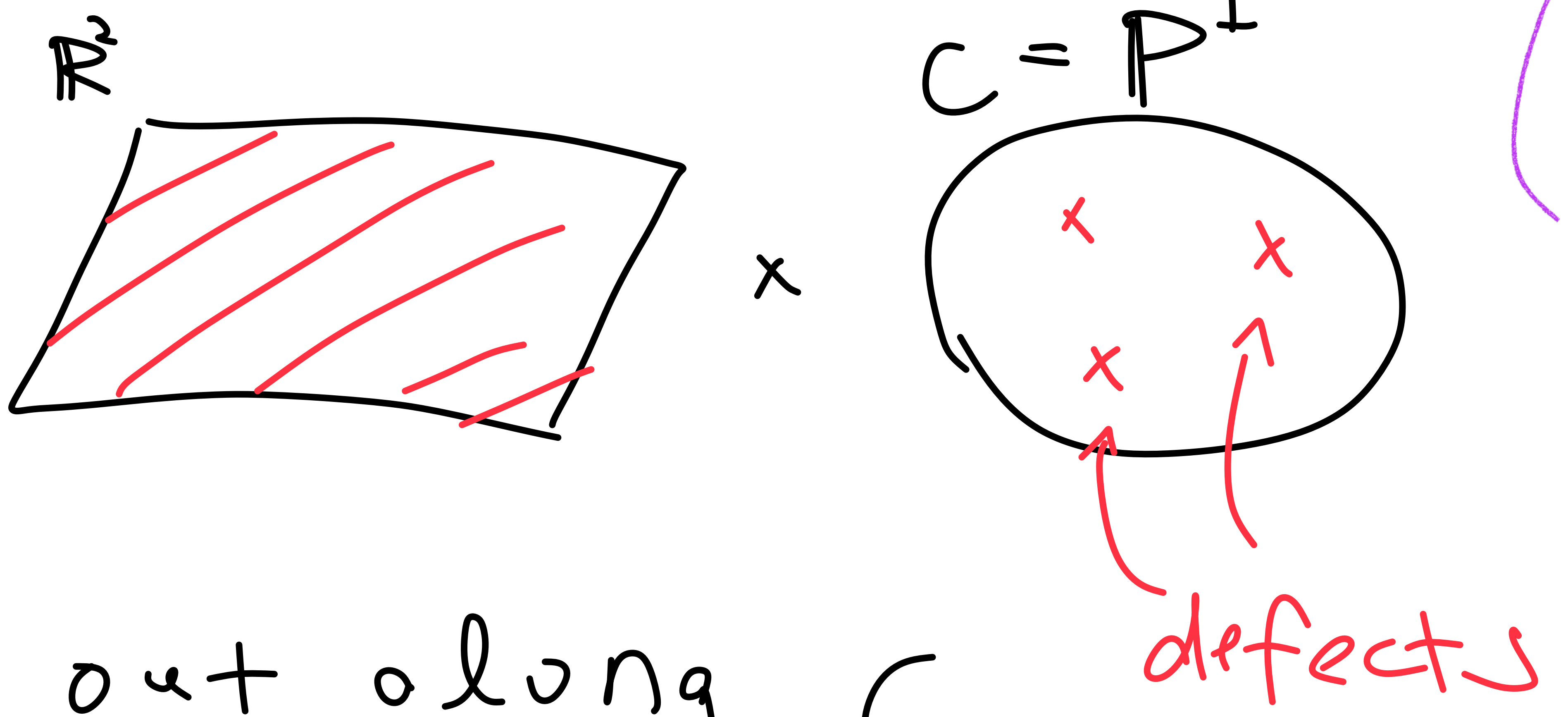


integrate out along C

2d IFT



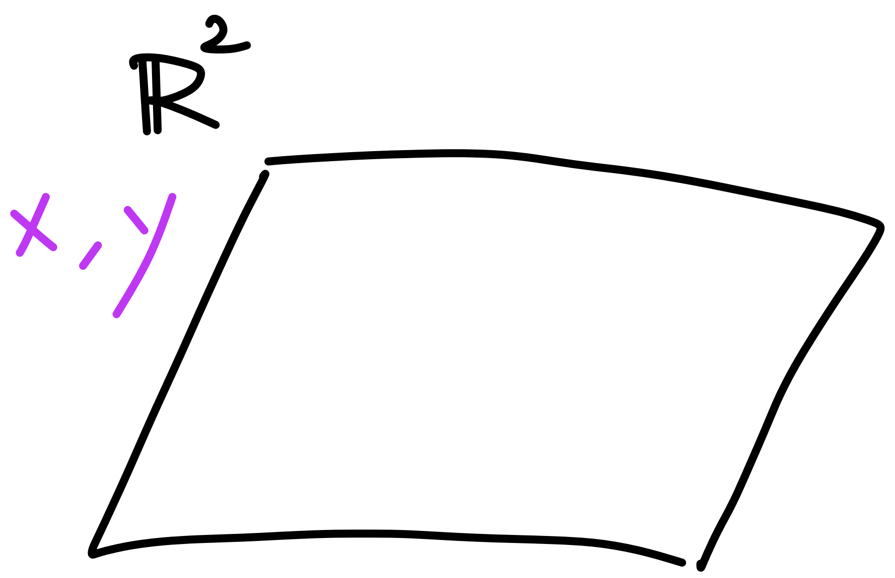
$$S = \frac{1}{2\pi\hbar} \int_{\mathbb{R}^2 \times C} \omega \wedge \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right)$$



$$\left(F = dA + A \wedge A = 0 \right.$$
[e.o.m.]

integrate out along C

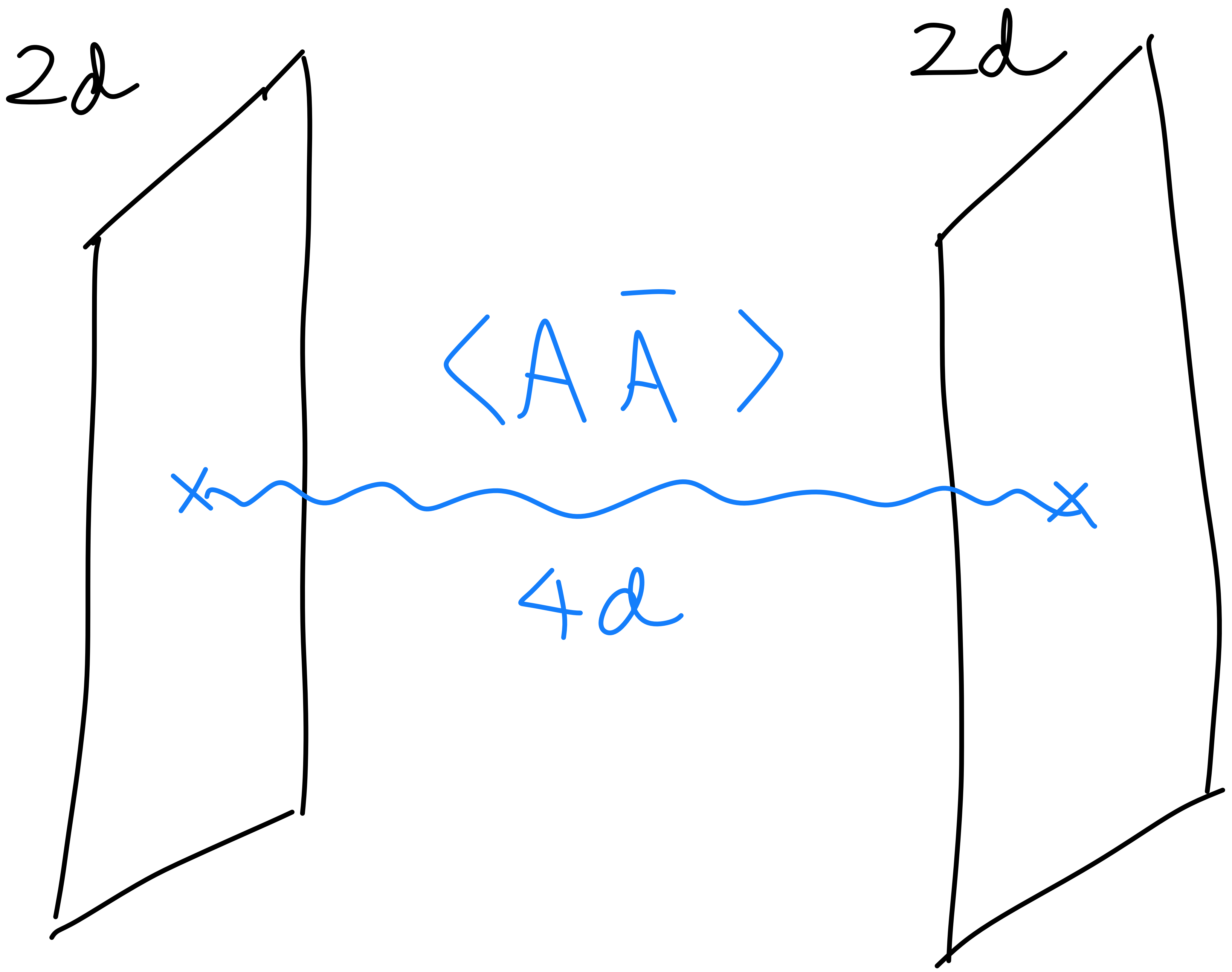
2d IFT



$$\mathcal{L} = A_x(z)\, dx + A_y(z)\, dy \quad (A_{\bar{z}} = 0)$$

$$d\mathcal{L} + \mathcal{L} \wedge \mathcal{L} = 0 \quad \text{[Lax eqn]}$$

e.g. chiral / anti-chiral order defect



4-Fermi interaction

$$J\bar{J} = (\psi\psi)(\bar{\psi}\bar{\psi})$$

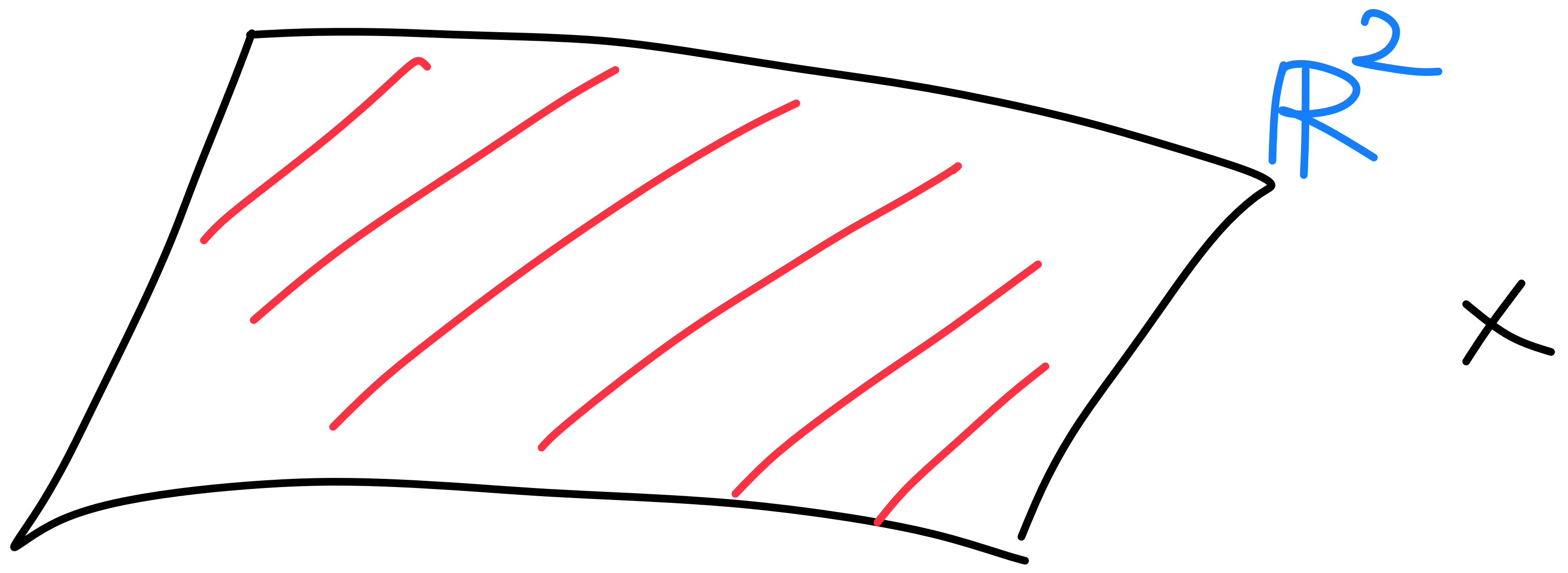
$$\psi\bar{\partial}\psi + \psi\bar{A}\psi$$

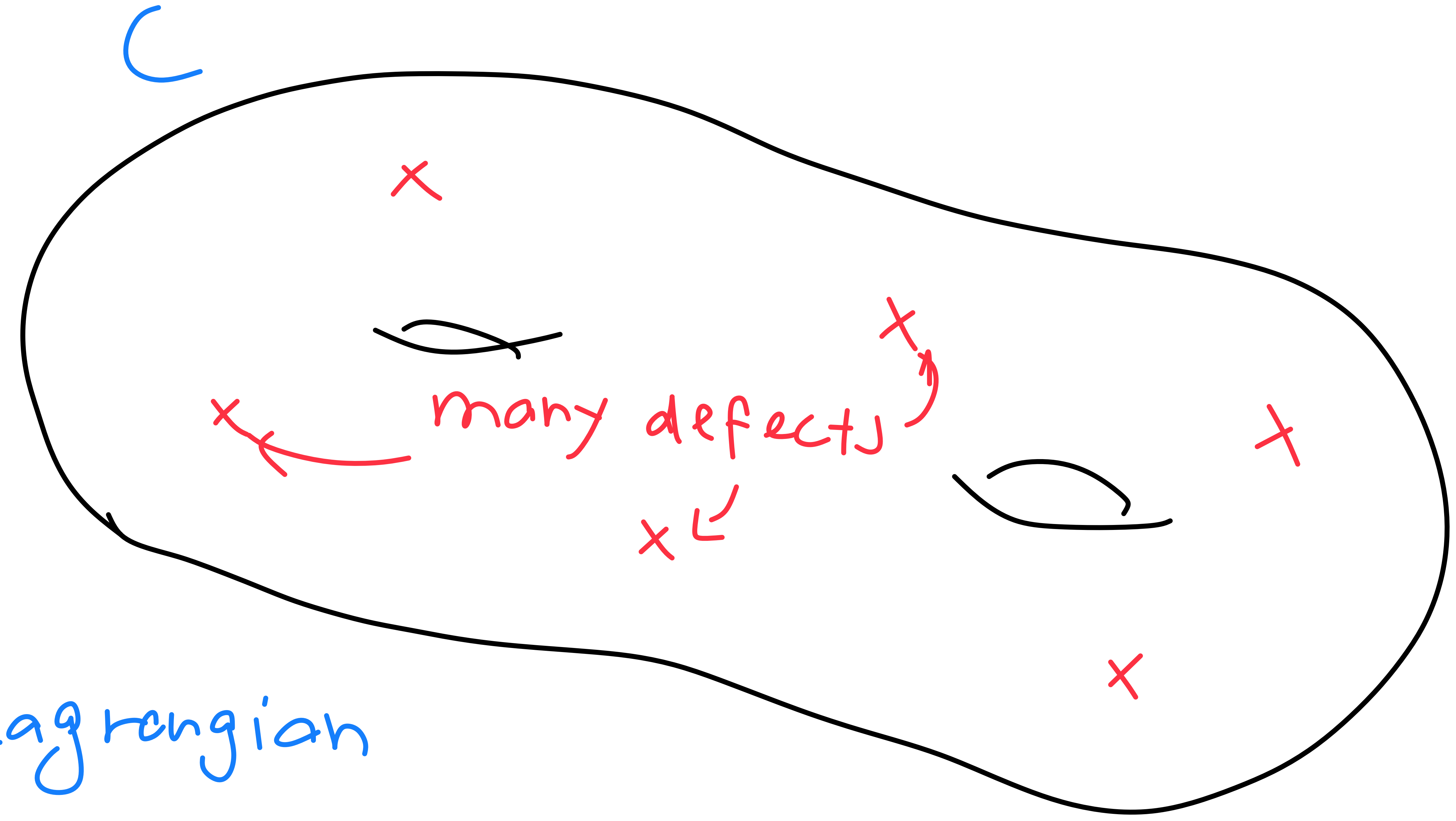
free chiral

$$\bar{\psi}\partial\bar{\psi} + \bar{\psi}A\bar{\psi}$$

free anti-chiral

The discussion generalizes





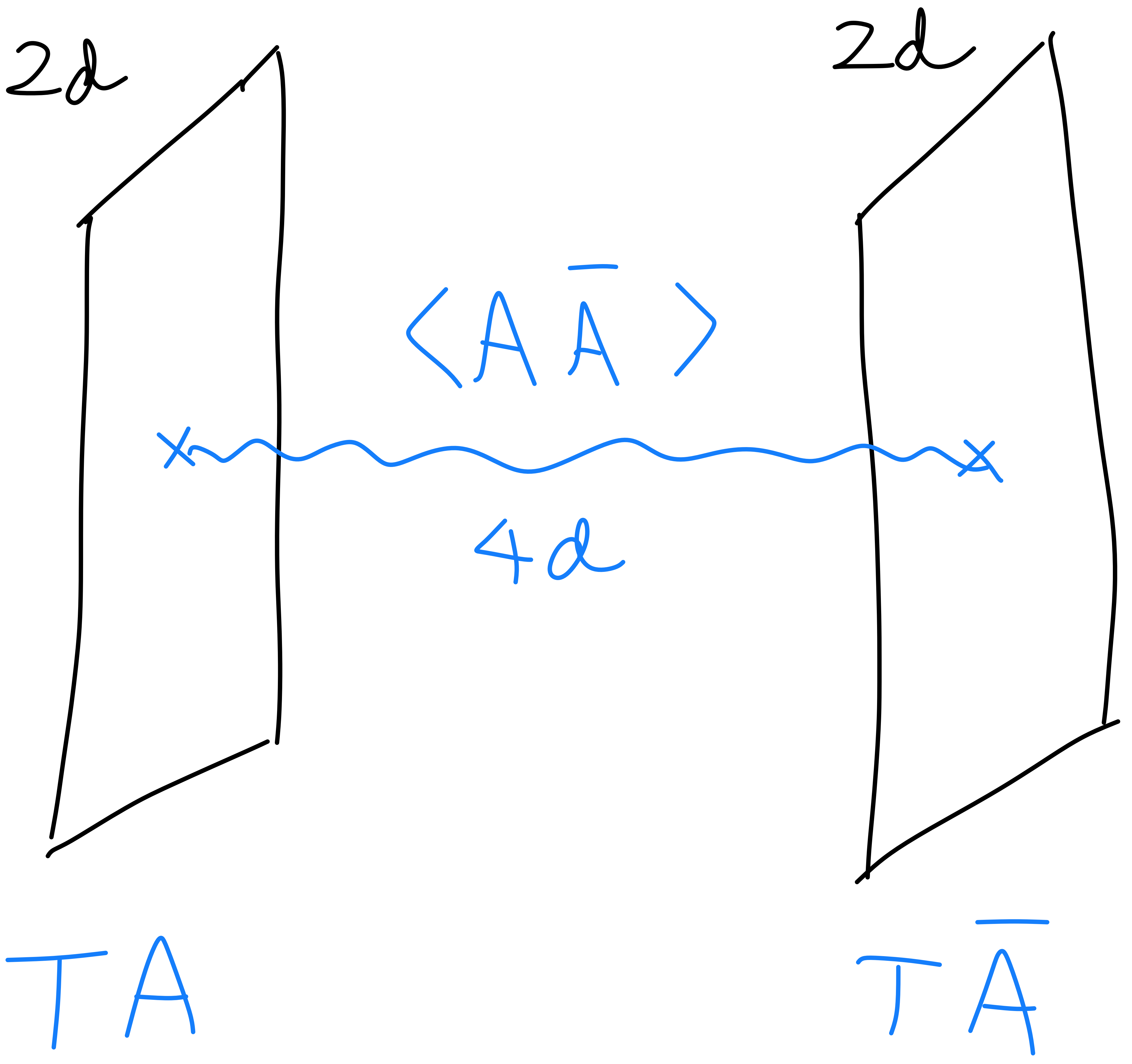
- Order defect : 2D Lagrangian
- disorder defect : singularities of ω

- Some deformations, e.g. Yang-Baxter & λ-deformations realized here e.g. [Delduc - Lacroix - Magro - Vicedo ('20)]

$T\bar{T}$ in 4d CS?

$J\bar{J} \to T\bar{T}$?

[V. Py ('22)]



$T\bar{T}$ - deformation

$T\bar{T}$

$A = SL_2$ CS
$\parallel$
gravity

Questions

- $T\bar{T}$ - deformation : non-linear in λ

$$\frac{d}{d\lambda} L^{(\lambda)}_{PCM} = \det\left(T^{(\lambda)}_{\mu\nu}\right)$$

- $T\bar{T}$ - deformation : not the only integrable deformation

  e.g. $\sqrt{T\bar{T}}$ - deformation [e.g. Borsato - Ferko - Sfondrini ('22)]

- Extension to general IQFT.

- Gravity should be (at least) 4D, not 3D

We take a different approach

in [Tateo–Sakamoto–MY]

open

(6d CS)

$\downarrow$ T-duality

(4d CS)

top./hol

$\downarrow$

(2d IQFT)

open | closed

$$
\begin{tikzcd}
(\text{6d CS}) \arrow[d, "\text{T-duality}"] & (\text{6d KS gravity}) \arrow[l, squiggly] \arrow[d, "\text{complex str. deformation}", "\text{T-duality}"'] \\
(\text{4d CS}) \arrow[d] & (\text{4d Kähler/KS gravity}) \arrow[l, squiggly] \arrow[d] \\
(\text{2d IQFT}) & (\text{deformation}) \arrow[l, squiggly]
\end{tikzcd}
$$

top./hol

$$\begin{tikzcd}
(\text{4d CS})_{\text{top./hol}} \arrow[d] & (\text{4d Kähler/KS gravity})_{g} \arrow[l, squiggly, "\Phi"'] \arrow[d] \\
(\text{2d IQFT}) & (\text{deformation}) \arrow[l, squiggly]
\end{tikzcd}$$

1. Solve gravity e.o.m.

$$g_{\mu\nu}[\Phi] = g_{\mu\nu}[T_{\mu\nu}(\Phi)]$$

$$\begin{tikzcd} (\text{4d top./hol CS}) \arrow[d] & (\text{4d Kähler/KS gravity})\ g \arrow[l, squiggly, "\Phi"'] \arrow[d] \\ (\text{2d IQFT}) & (\text{deformation}) \arrow[l, squiggly] \end{tikzcd}$$

2. consider curved bg $g_{\mu\nu}[\Phi]$

3. coordinate transf.

Φ-dependent

1. Solve gravity e.o.m.

$$g_{\mu\nu}[\Phi] = g_{\mu\nu}[T_{\mu\nu}(\Phi)]$$

(4d CS) $\xleftarrow{\Phi}$ (4d Kähler/KS gravity) g

top./hol

(2d IQFT) $\longleftarrow$ (deformation)

Indeed, $T\bar{T}$ - deformations in 2D arise from dynamical / field-dependent transformations

[Dubovsky - Gorbenko - Mirbabyi ('17)
Conti - Negro - Tateo ('18)]

We can apply the same idea already in 4D

- 4D / 2D in curved spaces
- $T\bar{T}$ / $\sqrt{T\bar{T}}$ for degenerate $\mathcal{E}$ - model

$T\bar{T}$ from Dynamical Coordinate Transformation

[Conti - Negro - Tateo ('18), Chen - Hou - Tian ('21)]

$T\bar{T} \leftarrow$ dynamical coordinate transformation

$$ds^2 = -dx^+ dx^-$$

$$\begin{pmatrix} dx^{+\prime} \\ dx^{-\prime} \end{pmatrix} = \begin{pmatrix} 1 + \lambda T^{(\lambda)}_{+-} & -\lambda T^{(\lambda)}_{--} \\ -\lambda T^{(\lambda)}_{++} & 1 + \lambda T^{(\lambda)}_{+-} \end{pmatrix} \begin{pmatrix} dx^+ \\ dx^- \end{pmatrix}$$

"Linearizes" the deformation

$$S^{(\lambda)} = \int \mathcal{L}^{(\lambda)}(x)\, dx^+ \wedge dx^-$$

$$= \int \left(\mathcal{L}^{(0)}(x) - \lambda \det\left(T^{(0)}\right) \right) dx^{+\prime} \wedge dx^{-\prime}$$

In general coordinate transf. is harmless in

4D bulk $\leadsto$ top. along 2D

$$\mathcal{L} = \mathcal{L}_+ dx^+ + \mathcal{L}_- dx^- = \mathcal{L}_{+'} dx^{+'} + \mathcal{L}_{-'} dx^{-'}$$

In general coordinate transf. is harmless in

4D bulk $\leadsto$ top. along 2D

$$\mathcal{L} = \mathcal{L}_+ dx^+ + \mathcal{L}_- dx^- = \mathcal{L}_{+'} dx^{+'} + \mathcal{L}_{-'} dx^{-'}$$

The situation is non-trivial on the defects

esp. for field-dependent ones

(a priori no guarantee that integrability be preserved)

The dynamical coordinate transformation was derived originally for sine-Gordon models

However, the logic should apply much more generally, e.g. to degenerate $\mathcal{E}$-models [Klimcik-Severa ('95, '96)]

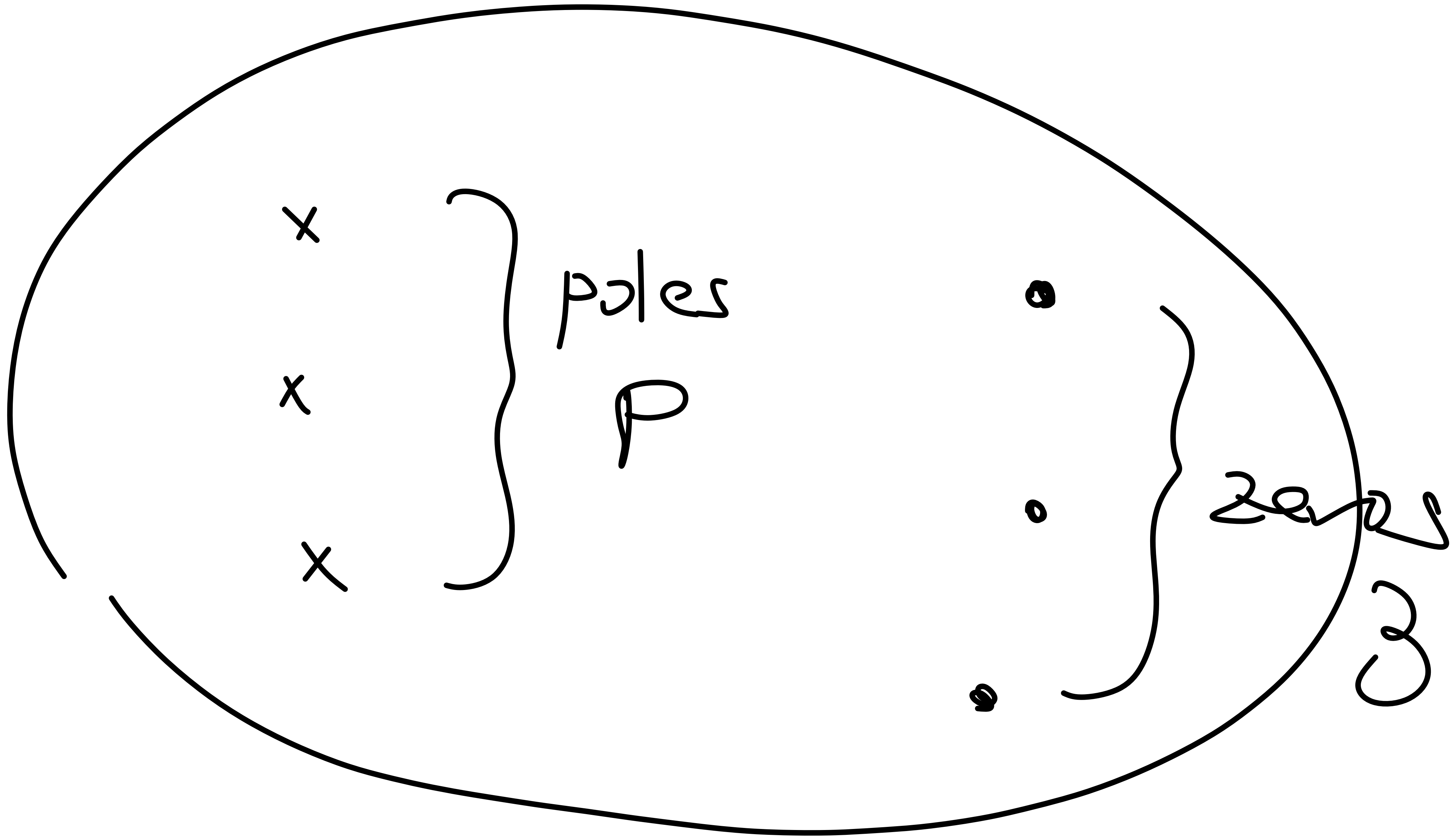
Digression on deg $\mathcal{E}$-model

[Benini - Schenkel - Vicedo ('20)
Lacroix - Vicedo ('20), Liniado - Vicedo ('23)]

Choose one-form ω to have

set of poles $\mathfrak{P}$, zeros $\mathfrak{Z}$

$$\omega = \left(\sum_{x \in \mathfrak{P}} \sum_{q=0}^{n_x - 1} \frac{l_q^x}{(z-x)^{p+1}} - \sum_{q=1}^{n_\infty - 1} l_q^\infty z^{p-1} \right) dz$$



Need to choose boundary conditions:

poles to ω $\rightsquigarrow$ we need expansion of A up to some order

$$\mathfrak{g} = T_x G \longrightarrow \underbrace{T_x J^n G}_{\text{jet-bundle}}$$

$$u \longmapsto u \otimes \varepsilon_x^p$$

$$\mathcal{D} := \text{"} \prod_{x \in P} \left(\mathfrak{g} \otimes \mathbb{R}[\varepsilon_x] / (\varepsilon_x^{n_x}) \right) \text{"}$$

w/ bilinear pairing $\langle\!\langle -, - \rangle\!\rangle_{\mathcal{D}}$

In variation of action $\omega \wedge (dA + A \wedge A) = 0$

we need $\int_{\Sigma \times C} \underbrace{d\omega}_{\text{only at poles of } \omega} \wedge \mathrm{Tr}(A \wedge \delta A)$

$j^* : C^\infty(\Sigma \times C) \to C^\infty(\Sigma \times p)$

$$\int_{\Sigma \times p} \langle\langle j^* A, j^* \delta A \rangle\rangle_{\mathfrak{d}} = 0$$

satisfied by $A|_p \in \mathfrak{k} \subset \mathfrak{d}$

maximally isotropic

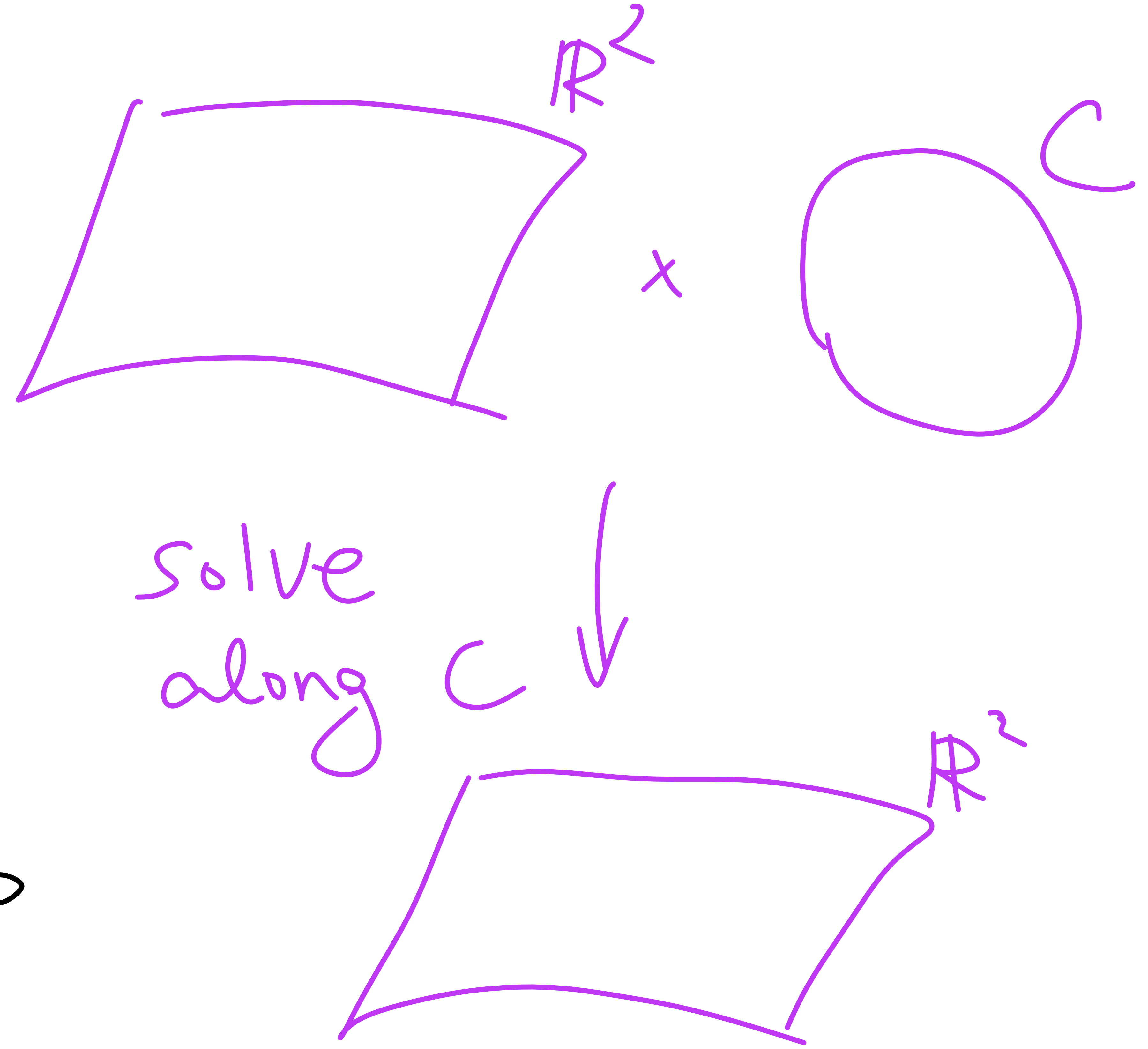
When we solve e.o.m. along C

it is useful to make explicit the DOF at poles

h : DOF at the pole of ω

$$\uparrow$$

$$C^\infty\Big(\underbrace{\Sigma \times P}_{\text{defined @ poles of } \omega},\ \underbrace{\mathbb{D}}_{\text{takes value in the } \underbrace{\text{defect group}}_{\text{group at poles}}}\Big)$$



h : edge mode defect.

$$S_{\text{defect}}[A,h] = -\frac{1}{2}\int_{\Sigma\times p}\langle\!\langle h^{-1}dh, j^*A\rangle\!\rangle_{\mathfrak{d}}$$

$$+\frac{1}{6}\int_{\Sigma\times p\times[0,1]}\langle\!\langle \hat{h}^{-1}d\hat{h}, [\hat{h}^{-1}d\hat{h}, \hat{h}^{-1}d\hat{h}]\rangle\!\rangle_{\mathfrak{d}}$$

$$\left[A \mapsto uAu^{-1} - du\,u^{-1},\ h \mapsto h(j^*u)^{-1}\right]$$

Degenerate $\mathcal{E}$-model [Klimcik - Severa ('95, '96)]

$$\mathfrak{h} \in \mathbb{F} \backslash \underbrace{\mathbb{D}} / \mathbb{K}$$

"gauge transf"

diagonal embedding

defect algebra

maximally isotropic from b.c

Degenerate $\mathcal{E}$-model

$$h \in \mathbb{F} \backslash \underbrace{\mathbb{D}} / \mathbb{K}$$

we can write down e.g. Lax

$$\mathcal{L}_\pm(h) = \underbrace{j_*}_{\mathfrak{d} \to \text{function on the whole } C}\left(P_h^\pm\left(\underbrace{h^{-1}\partial_\pm h}_{\mathfrak{d}}\right)\right)$$

$\Big(P_h^\pm : \mathfrak{d} \to \mathfrak{D}$
projectors $\Big)$

[Liniado-Vicedo ('23)]

Now ready to apply dynamical coord. transformations . . . .

$$\delta S[\mathcal{L}^{(\lambda)}(h), h]$$

$$= \frac{1}{4}\int_{\Sigma\times P} \overbrace{\delta\gamma^{*(\lambda)\mu\nu} \langle\!\langle h^{-1}\partial_\mu h, (P_h^+ - P_h^-)(h^{-1}\partial_\nu h)\rangle\!\rangle_{\mathfrak{D}}}^{\text{new contribution}}$$

$$+ \int_{\Sigma\times P} \underbrace{\langle\!\langle u, d(j^*\mathcal{L}^{(\lambda)}) + \frac{1}{2}[j^*\mathcal{L}^{(\lambda)}, j^*\mathcal{L}^{(\lambda)}]\rangle\!\rangle_{\mathfrak{D}}}_{\text{flat connection}}$$

$(h \to e^u h)$

$$\delta S\left[\mathcal{L}^{(\lambda)}(h),\, h\right]$$

$$= \frac{1}{4}\int_{\Sigma\times P} \underbrace{\delta\gamma^{*(\lambda)\mu\nu}\left\langle\!\left\langle h^{-1}\partial_\mu h,\, \left(P_h^{+} - P_h^{-}\right)\left(h^{-1}\partial_\nu h\right)\right\rangle\!\right\rangle_{\mathcal{D}}}_{=\;4\lambda\,\delta\left(\det T^{(\lambda)}\right)}$$

! (appendix of our paper)

This can be cancelled by adding $-\lambda\left(\det T^{(\lambda)}\right)$ to the action

We thus arrive at

dynamical coord. transformction

$$S^{(\lambda)} = \int L^{(\lambda)}(x)\, dx^+ \wedge dx^-$$

$$= \int \Big(L^{(0)}(x) \underbrace{- \lambda \det\big(T^{(0)}\big)}_{\text{Needed to preserve integrability}} \Big)\, dx^{+\prime} \wedge dx^{-\prime}$$

Similar discussion applies to

- $\sqrt{T\bar{T}}$ - deformation
- $T\bar{T}$ combined w/ $\sqrt{T\bar{T}}$

In general involves

both complex / Kähler deformation

Summary

- $T\bar{T}/\sqrt{T\bar{T}}$ deformation : dynamical coord. transf.

in 4d CS + disorder defect.
on $\mathbb{P}^1$

$\Downarrow$

degenerate $\mathcal{E}$-model

Questions

- $T\bar{T}$ - deformation ; non-linear in λ

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  e.g. $\sqrt{T\bar{T}}$ - deformation

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Question:

Can we systematically classify all integrable deformations?

(Questions for closed-string sector
Kähler / BCOV

"deformation as gravity")