Topology in the Lattice Yang-Mills Theory

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Based on

R. Kitano, N. Yamada + MY
arXiv: 2010.08810 [hep-lat]

R. Kitano, N. Yamada, R. Matsudo + MY
arXiv: 2102.08784 [hep-lat]

R. Kitano, N. Yamada + MY
arXiv: 2403.10767, 2411.00375 [hep-lat]

cf. papers on axions

<Inflation>

K. Yonekura + MY
arXiv:1704.05852 [hep-th]

Y. Nomura, T. Watari + MY
arXiv:1706.08522 [hep-ph]

Y. Nomura + MY
arXiv:1711.10490 [hep-ph]

<Dark Energy>

M. Ibe, T.T. Yanagida + MY
arXiv:1811.04664 [hep-th]

MY
arXiv:1910.08691 [hep-ph]

Yesterday: lattice gauge theories.

Can we prove confinement ???

$\beta \sim \frac{1}{g^2}$ ← YM coupling

strong coupling

$\beta \to 0$

(confinement proven in general)

⟷

interpolate??

weak coupling

$\beta \to \infty$

(continuum limit)

(perturbation theory + resurgence ....)

※ not always smoothly connected

e.g. U(1) theory, SU(3) conformal window

When the system confines, we have a “trivial theory” below the gap

$\mu \ll \Lambda$

↖ dynamical scale

.. but the “trivial theory” can be a non-trivial TQFT, once we have a theta-angle

Gaiotto-Kapustin-Komargodski-Seiberg ‘17

Different topological sectors in 4d pure SU(N) YM: weighted by the θ-angle

Callan-Dashen-Gross '76

$$\begin{tikzcd}[column sep=small] \mathcal{L} \arrow[d, squiggly] & \supset \dfrac{\theta}{32\pi^2}\displaystyle\int \mathrm{Tr}\, F_{\mu\nu}\tilde{F}^{\mu\nu} = \theta \underbrace{Q}_{\substack{\rotatebox{90}{$\in$} \\ \mathbb{Z}}} \\ e^{i\mathcal{L}} & \supset e^{i\theta Q} \end{tikzcd}$$

- $\theta \sim \theta + 2\pi$
- $\cancel{CP}$ unless $\theta = 0, \pi$!

Consider 4d SU(N) pure YM theory + θ-angle

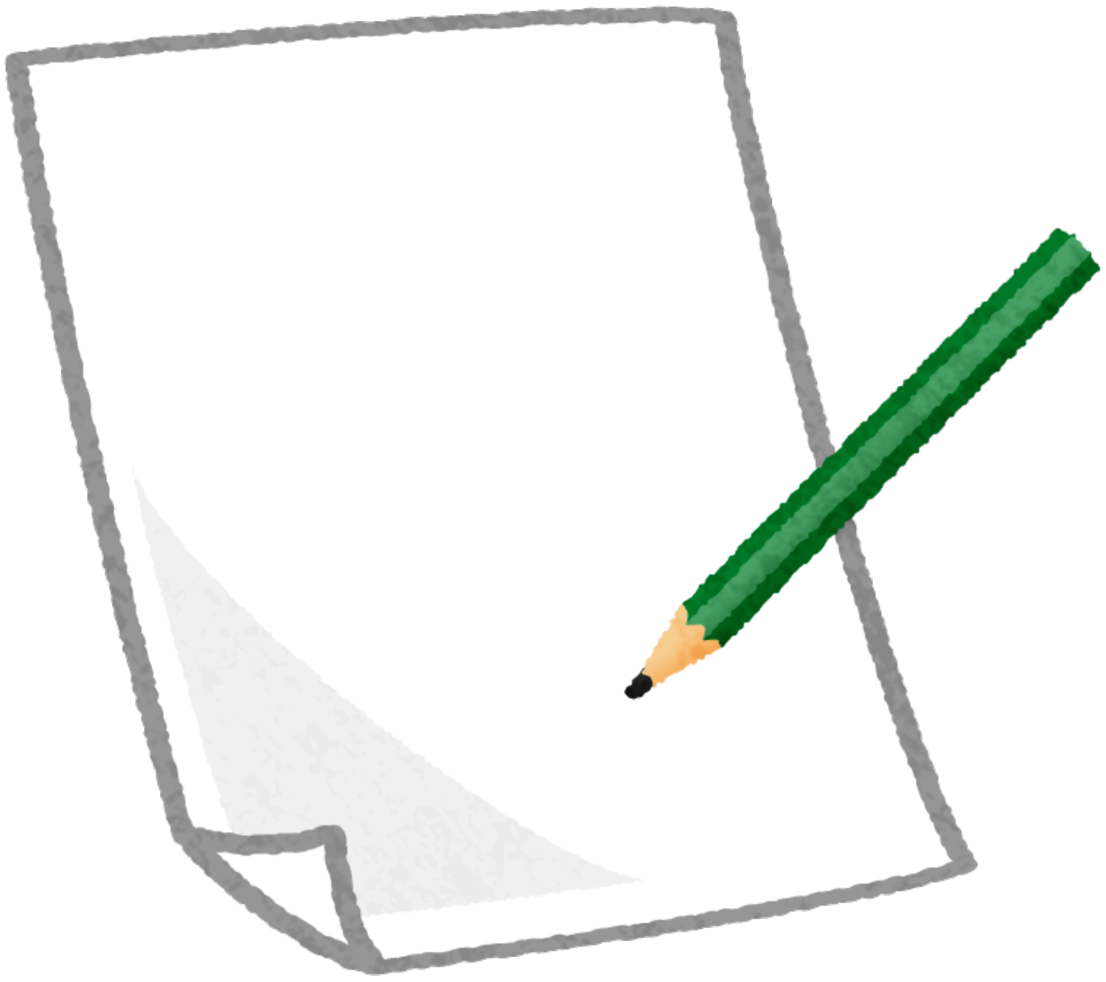
Q: Free Energy $F(\theta) = -\frac{1}{V}\ln\frac{Z(\theta)}{Z(0)}$

as a function of θ ?

Q: Fate of CP-sym. @ $\theta = \pi$?

↕ cf. Gaiotto-Kapustin-
-Komargodski-Seiberg ‘17

Q: confinement, i.e. $\mathbb{Z}_N$ 1-form sym.



Instanton Analysis

t' Hooft '76

$$F(\theta) \sim \int^{\rho\to\infty:\,\mathrm{IR}} \frac{d\rho}{\rho^5}\, e^{-\frac{8\pi^2}{g^2}} (\mu\rho)^{\frac{11N}{3}} (1-\cos\theta) + \cdots\cdots$$

(multi instanton)

※ 2π - periodic, CP preserved

Instanton Analysis

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(multi instanton)

※ 2π - periodic, CP preserved

※ Not correct in general ! (for $T \ll T_c$)

divergence as $\rho \to \infty$; IR problem

Large N

't Hooft '73,⋯, Witten '80

$$\mathcal{L} \sim \frac{1}{g^2} \mathrm{Tr}\, F\wedge *F + \theta\, \mathrm{Tr}\, F\wedge F$$

"$\hbar \sim \frac{1}{N} \ll 1$"

$$\sim \frac{1}{N^{-1}} \left(\underbrace{\frac{1}{g^2 N}}_{\text{fixed}} \mathrm{Tr}\, F\wedge *F + \underbrace{\frac{\theta}{N}}_{\text{fixed}} \mathrm{Tr}\, F\wedge F \right)$$

Large N

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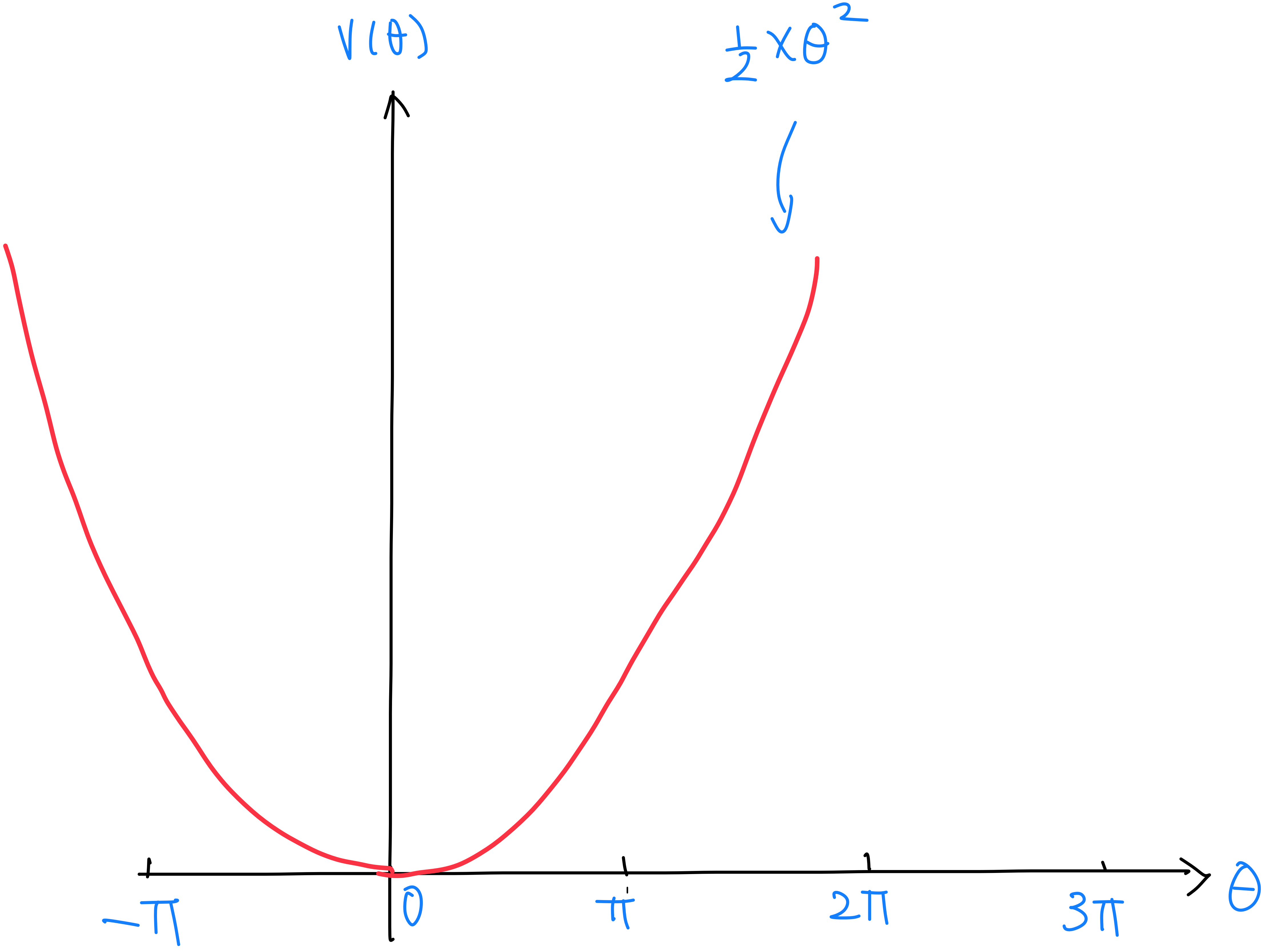
$$\sim \frac{1}{N^{-1}} \left(\underbrace{\frac{1}{g^2 N}}_{\text{fixed}} \mathrm{Tr}\, F \wedge * F + \underbrace{\frac{\theta}{N}}_{\text{fixed}} \mathrm{Tr}\, F \wedge F \right)$$

$$E(\theta) = N^2 f\left(\frac{\theta}{N}\right) = \frac{1}{2} \chi \theta^2 \left(1 + b_2 \theta^2 + \cdots\right)$$

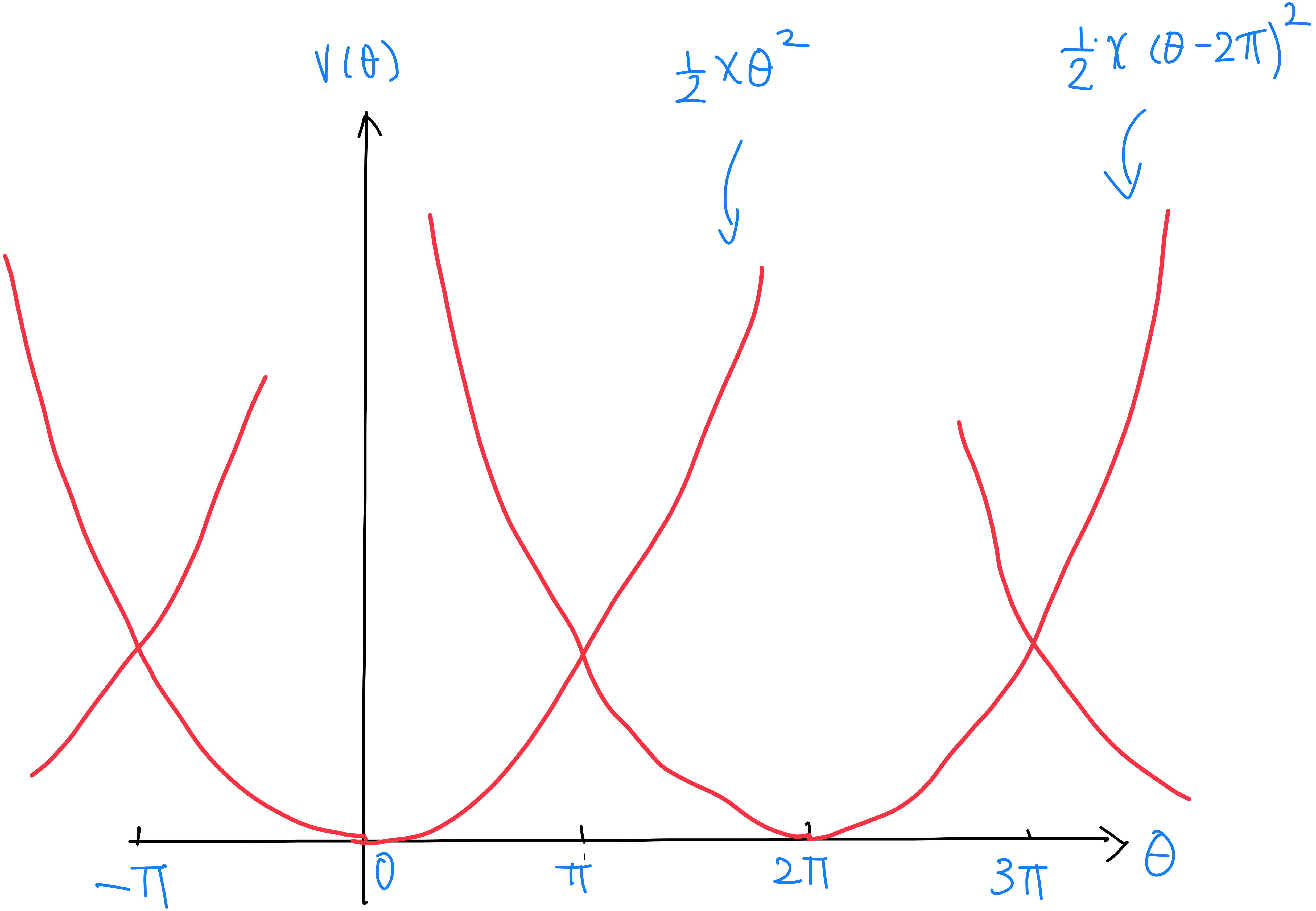
$$\chi = \chi^{(0)} + O\left(\frac{1}{N^2}\right) \rightarrow \chi^0$$

$$b_{2n} = \frac{b_{2n}^{(0)}}{N^{2n}} + O\left(\frac{1}{N^{2n+2}}\right) \rightarrow 0$$

※ NOT 2π-periodic

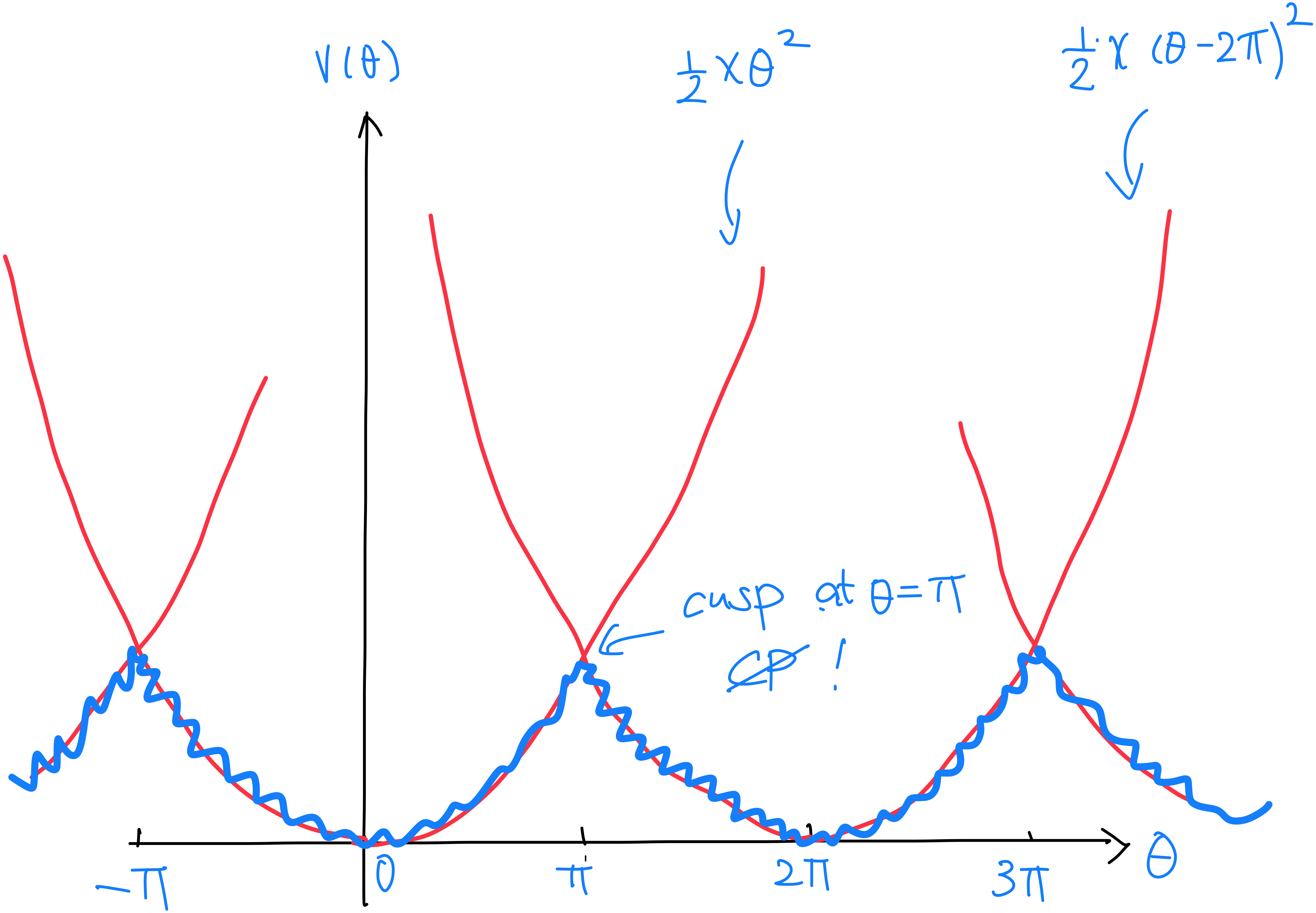


V(θ)
½ χθ²
-π
0
π
2π
3π
θ



multiple branches

Witten '80; cf. Dashen '71



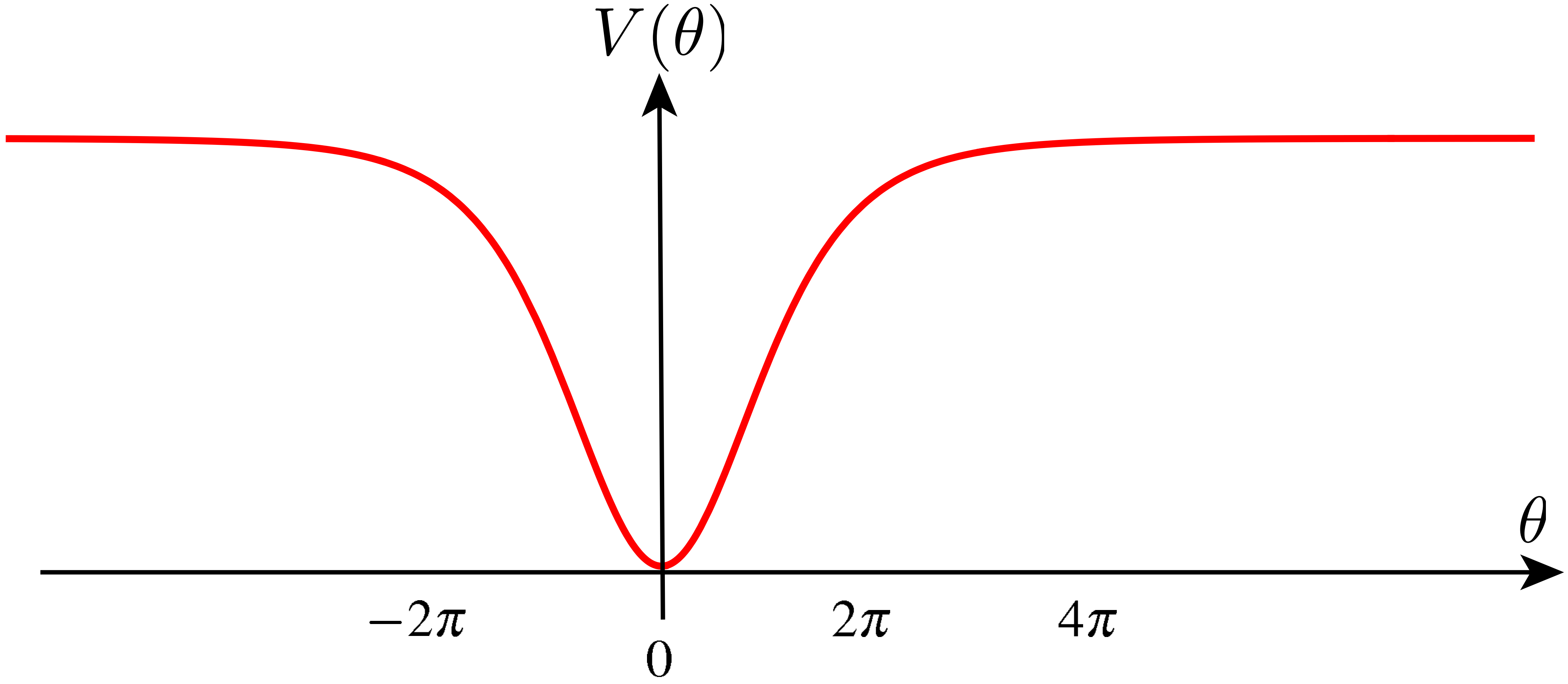
multiple branches

Witten ‘80; cf. Dashen ‘71

Finite N?

Expectation for large but finite N

Based on several papers by MY and collaborators

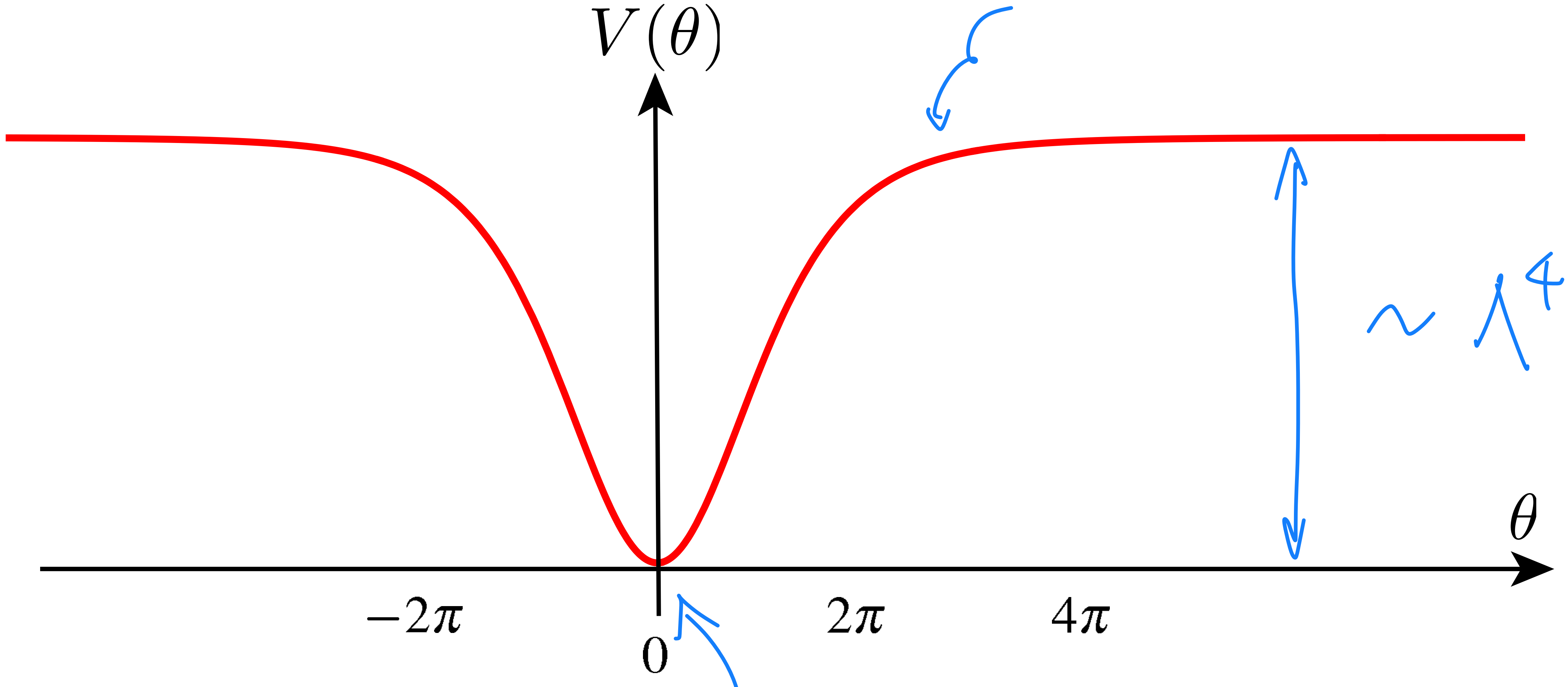


Expectation for large but finite N

Based on several papers by MY and collaborators

$\theta \to -\theta$

cf. holographic analysis
Dubovsky-Lawrence-Robert's '11

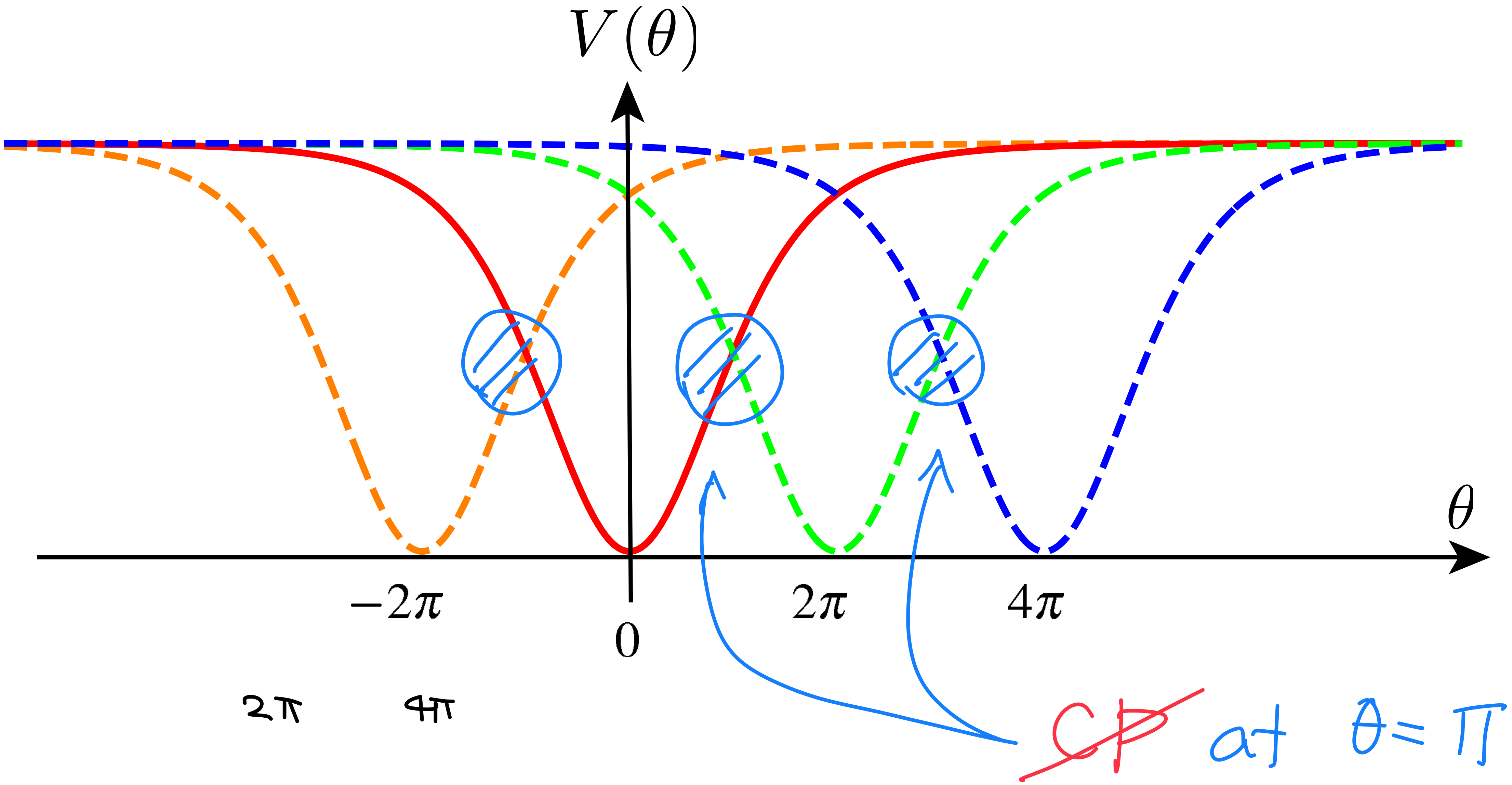


$V \sim \theta^2$ near $\theta = 0$

cf. Vafa-Witten '84

Expectation for large but finite N

Based on several papers by MY and collaborators



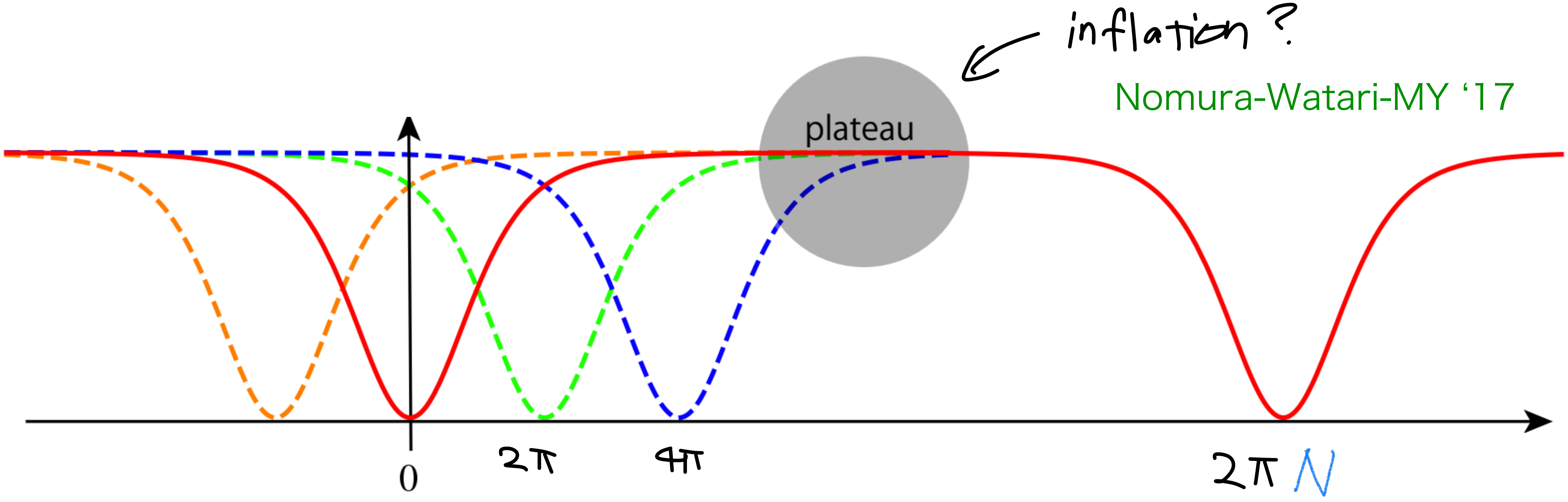
$\cancel{CP}$ at $\theta = \pi$

claim: N branches

Yonekura-MY '17, cf. Witten '80

Expectation for large but finite N

Based on several papers by MY and collaborators



claim: N branches

Yonekura-MY '17, cf. Witten '80

4d vs. 2d

MY + Yonekura, MY ‘17

* 4d $SU(N)$ pure YM ($\mathbb{Z}_N$ center sym.)

$$\Big\downarrow T^2$$

cf. Atiyah ‘84

Looijenga ’77, ‘80

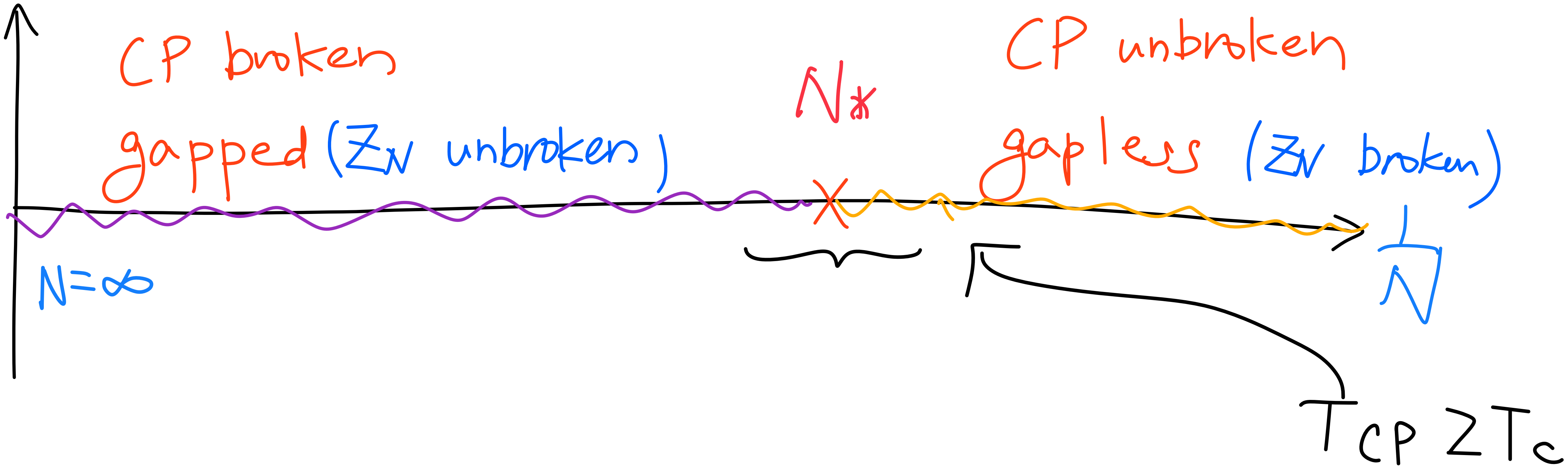
$$\mathcal{M}^{SU(N)}_{flat}(T^2) \simeq \mathbb{CP}^{N-1}$$

+(singularities)

* 2d "$\mathbb{CP}^{N-1}$ - model" ($\mathbb{Z}_N$ flavor sym.)

Small N ? SU(2)??

4d SU(N) YM$_{(\theta=\pi)}$



4d vs. 2d

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+(singularities)

* 2d "$\mathbb{CP}^{N-1}$ - model" ($\mathbb{Z}_N$ flavor sym.)

$\Big\downarrow S^1$

(+ $\mathbb{Z}_N$ twist)

fractional instanton

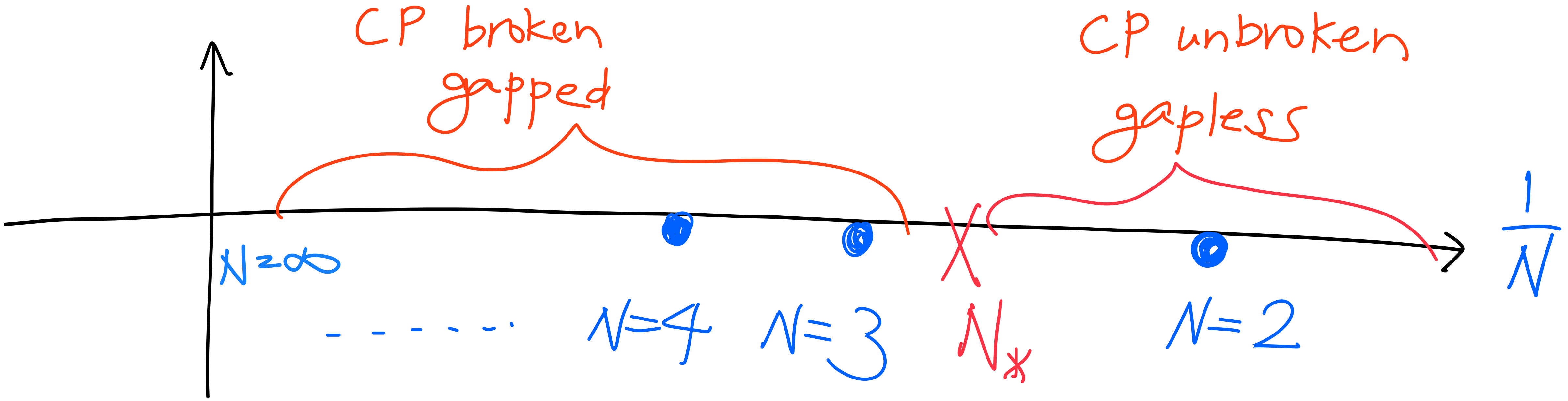
≒ renormalon

* 1d quantum mechanics

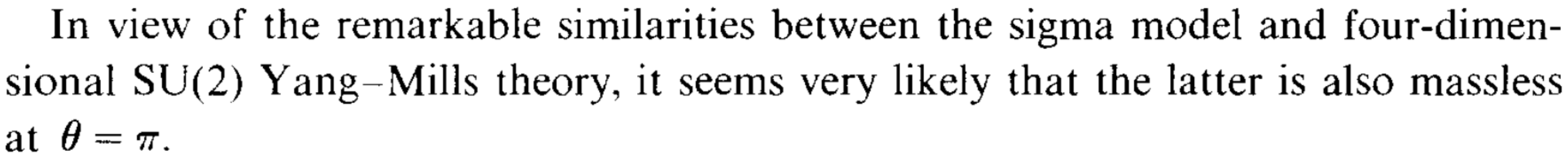
cf. Dunne-Unsal ’12 for CP^N model

Small N ? SU(2)??

2d CP^{N-1} model $(\theta=\pi)$

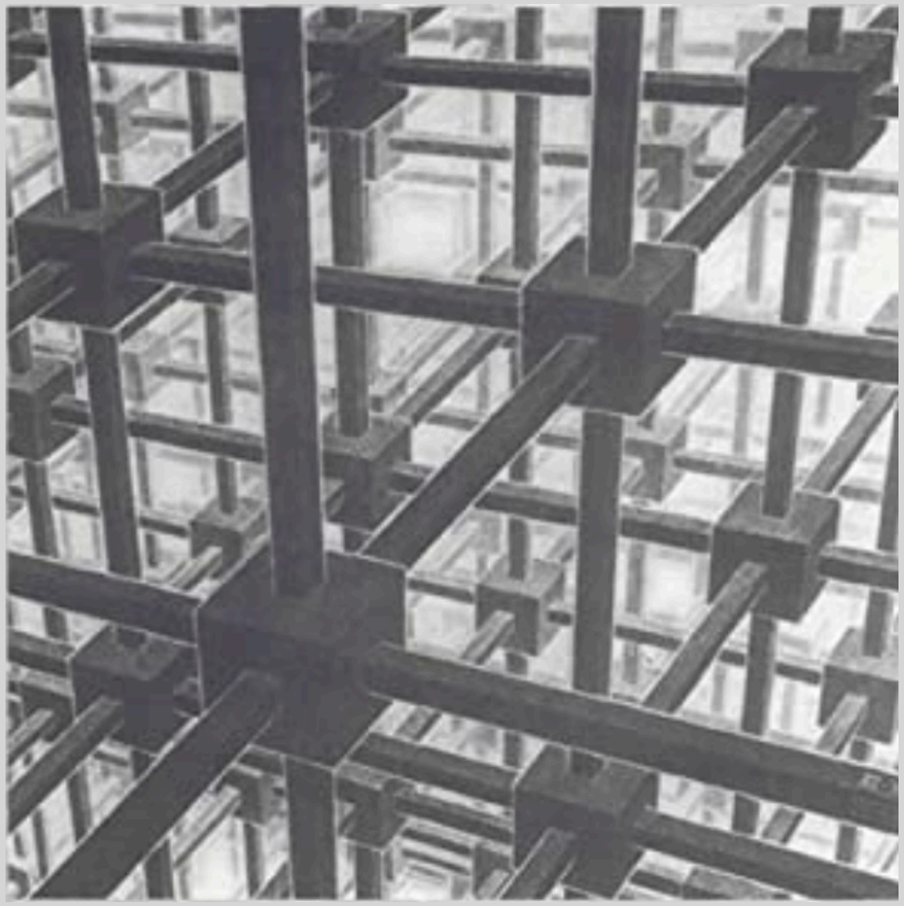


Haldane '83, Affleck-Haldane '87, Shankar-Read '90,⋯



In view of the remarkable similarities between the sigma model and four-dimensional SU(2) Yang–Mills theory, it seems very likely that the latter is also massless at $\theta = \pi$.

Computer Simulations

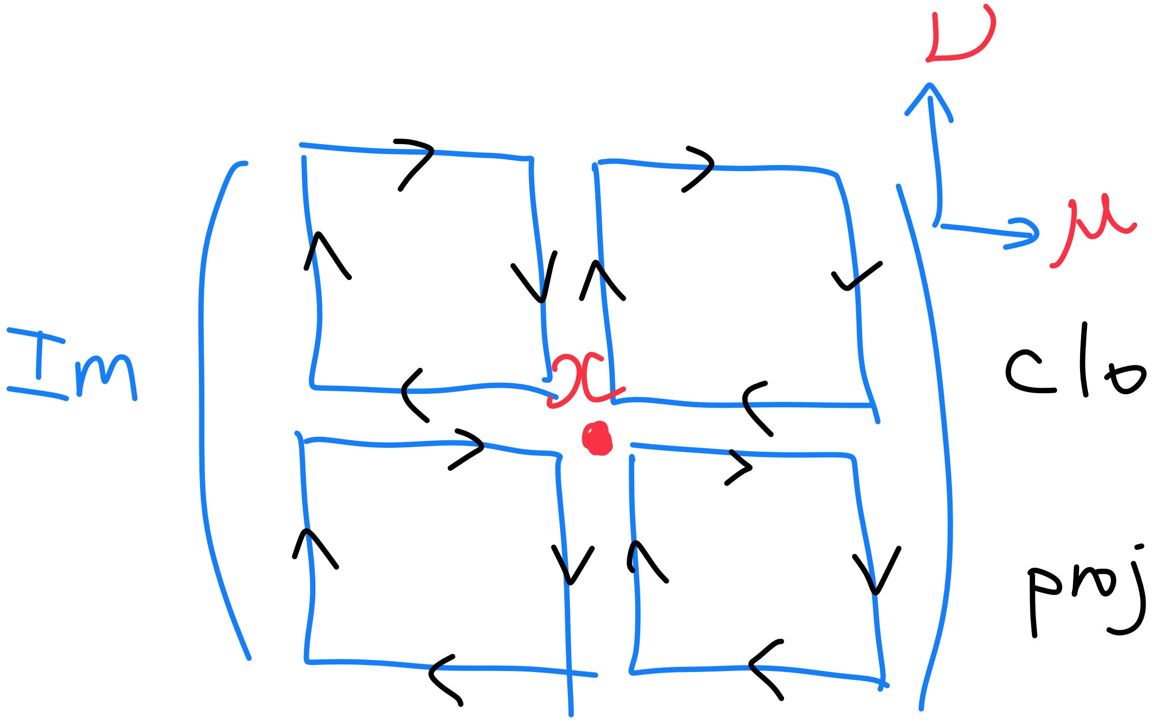


… requires computational resources
(and several years of my research time!)

Oakforest-PACS in Kashiwa

Cygnus in Tsukuba

“Just do it” on the lattice?



clover average

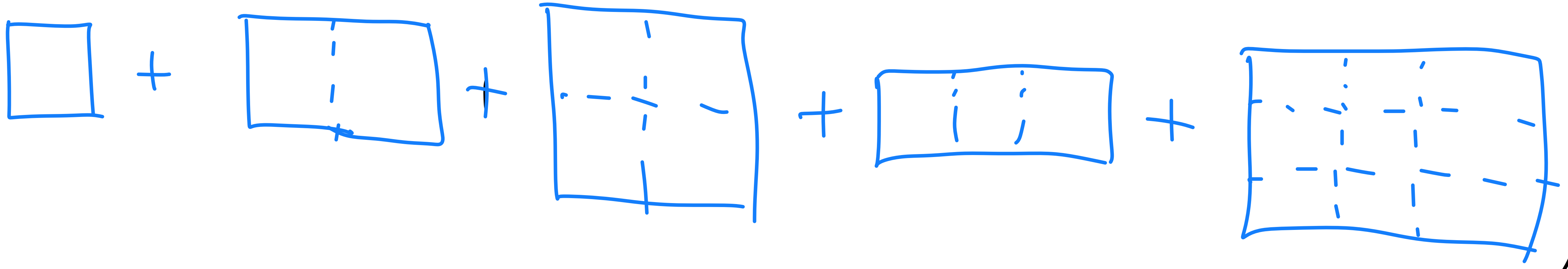
project.

local $F_{\mu\nu}(x)$ $\rightsquigarrow$ Q

improvement

de Forcrand-Garcia Perez-Stamatescu ‘97

no $\mathcal{O}(a^2), \mathcal{O}(a^4)$

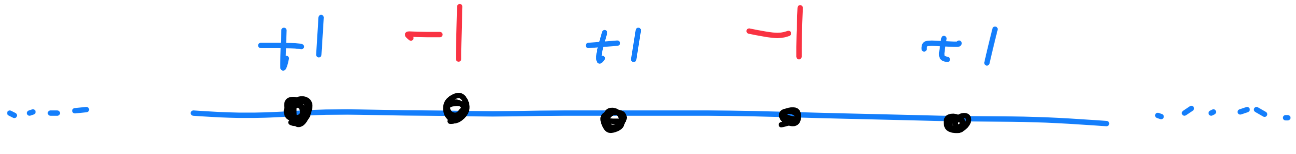


However…

⚠ Q is not quantized on the lattice

(short-distance fluctuations)

e.g.



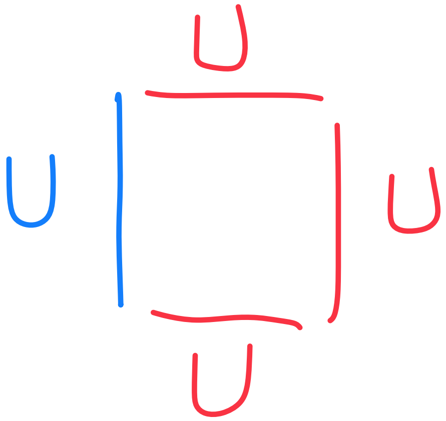
Luscher '82

(cf. definition of Q in terms of overlap fermion does not have this problem but computationally very expensive)

Practical solution: smearing

APE smearing

Albanese+ '87



$$U_\mu^{(\mathrm{new})} = \mathrm{Proj}\left[(1-\rho)\mathrm{U}_\mu^{(\mathrm{old})}(\mathrm{x}) + \rho\,\mathrm{X}_\mu(\mathrm{x})\right] \ ,$$

$$X_\mu(x) = \sum_{\nu\neq\mu}\Big[U_\nu^{(\mathrm{old})}(x)U_\mu^{(\mathrm{old})}(x+\hat\nu)U_\nu^{(\mathrm{old})\dagger}(x+\hat\mu) + U_\nu^{(\mathrm{old})\dagger}(x-\hat\nu)U_\mu^{(\mathrm{old})}(x-\hat\nu)U_\nu^{(\mathrm{old})}(x-\hat\nu+\hat\mu)\Big] \ ,$$

gradient flow

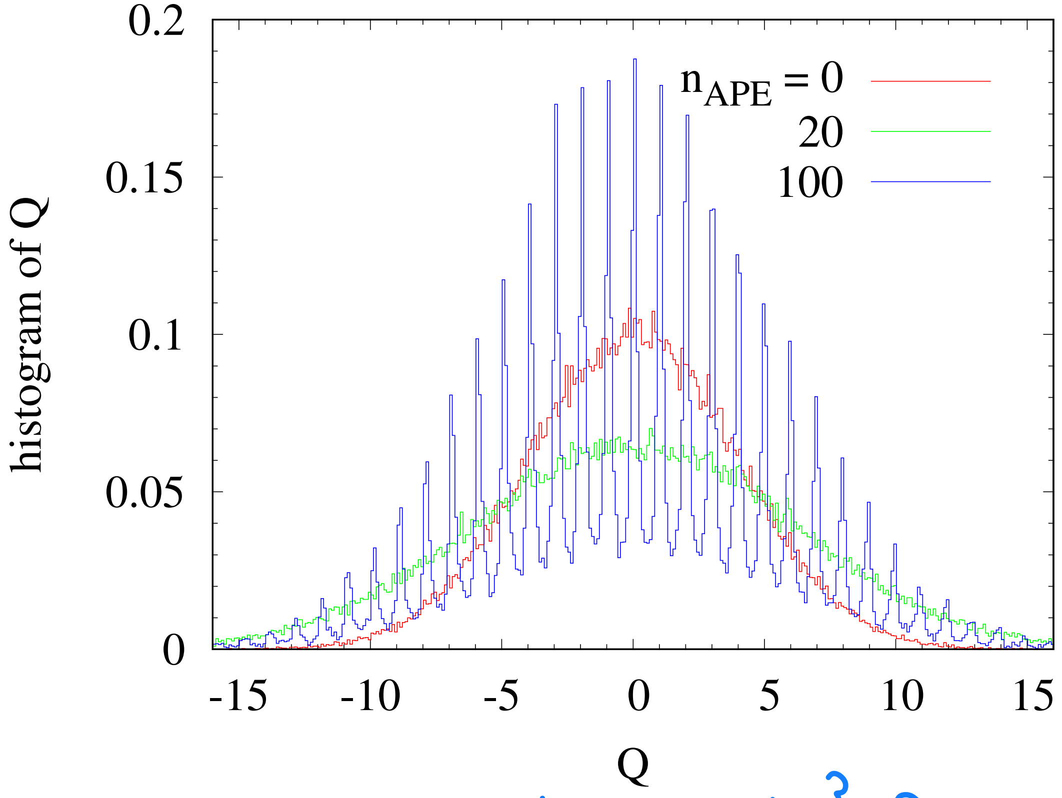
Luscher '10

$$\frac{\partial A_\mu(s)}{\partial s} = \frac{\delta \mathcal{L}}{\delta A_\mu(s)}$$

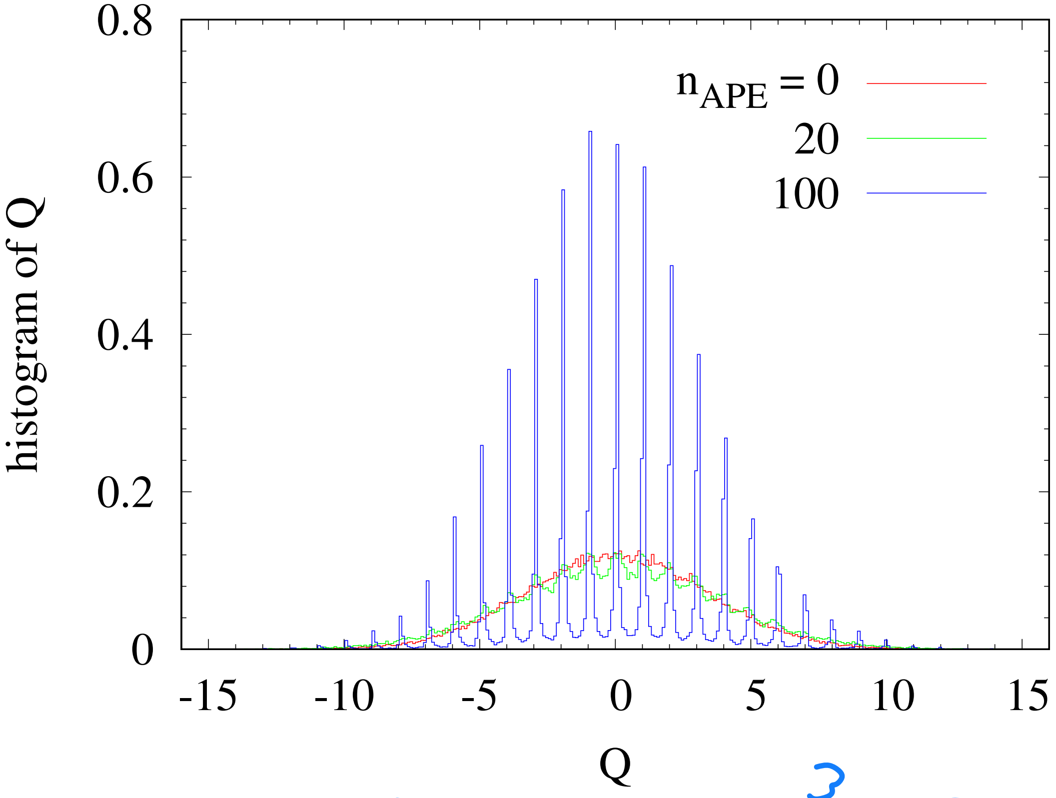
different methods consistent

cf. Alexandrou et al., '17

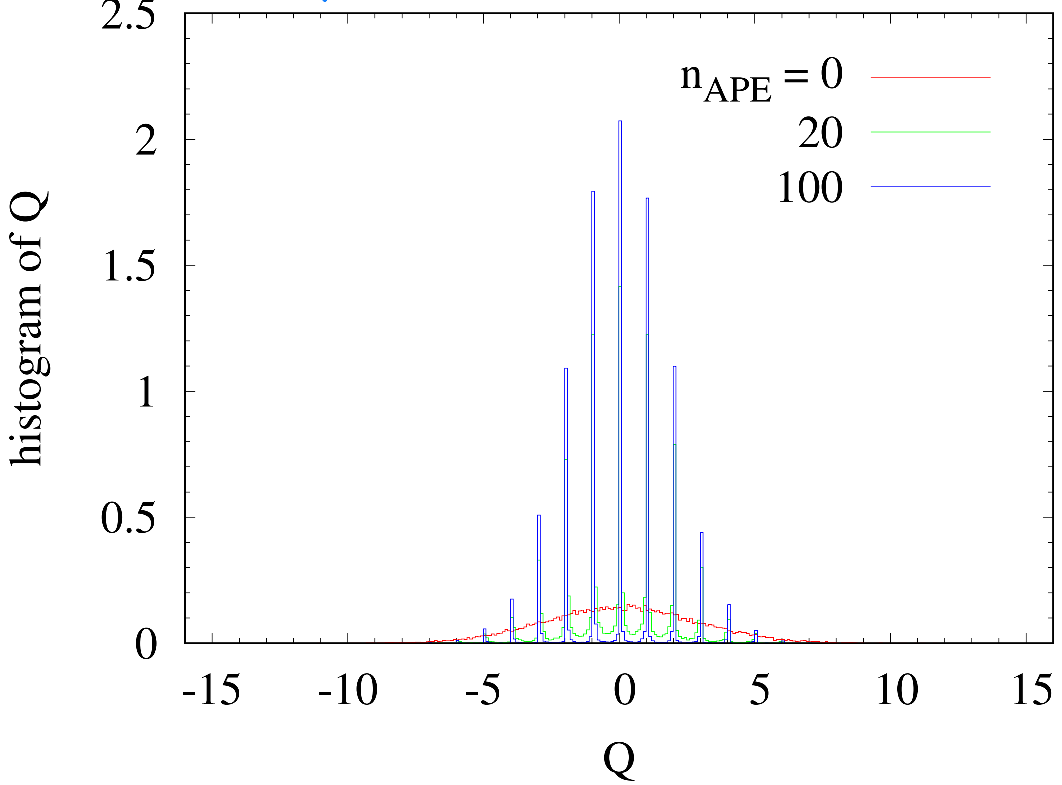
$\beta = 1,75$ $16^3 \times 32$



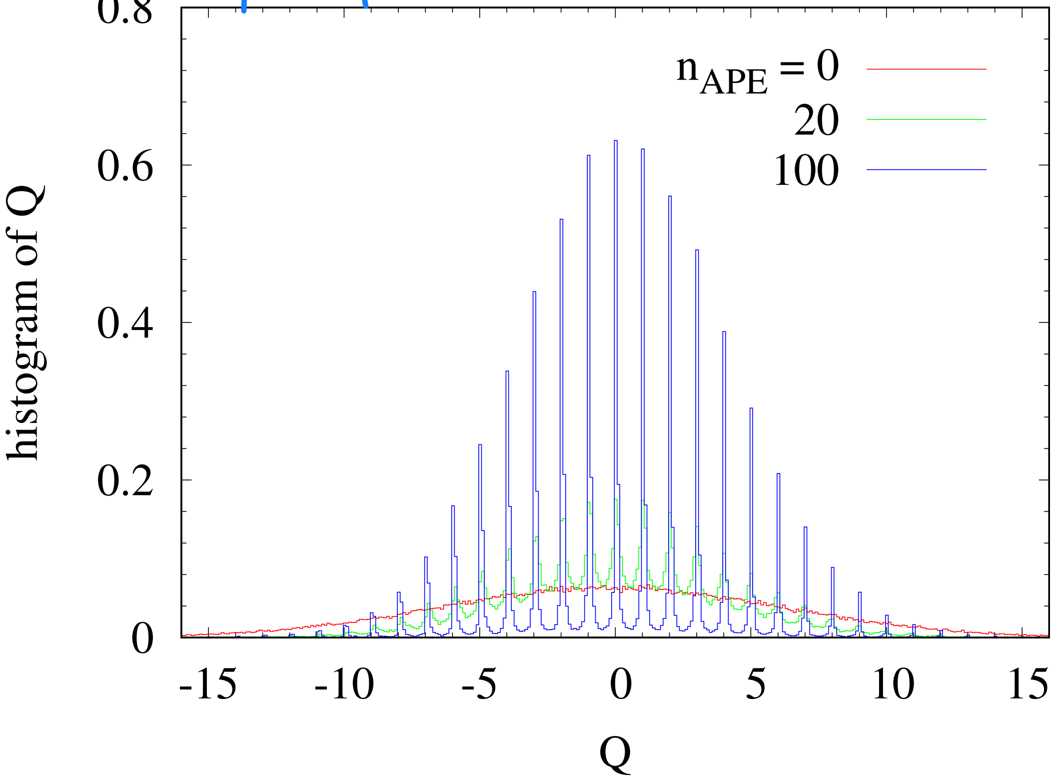
$\beta = 1,85$ $16^3 \times 32$

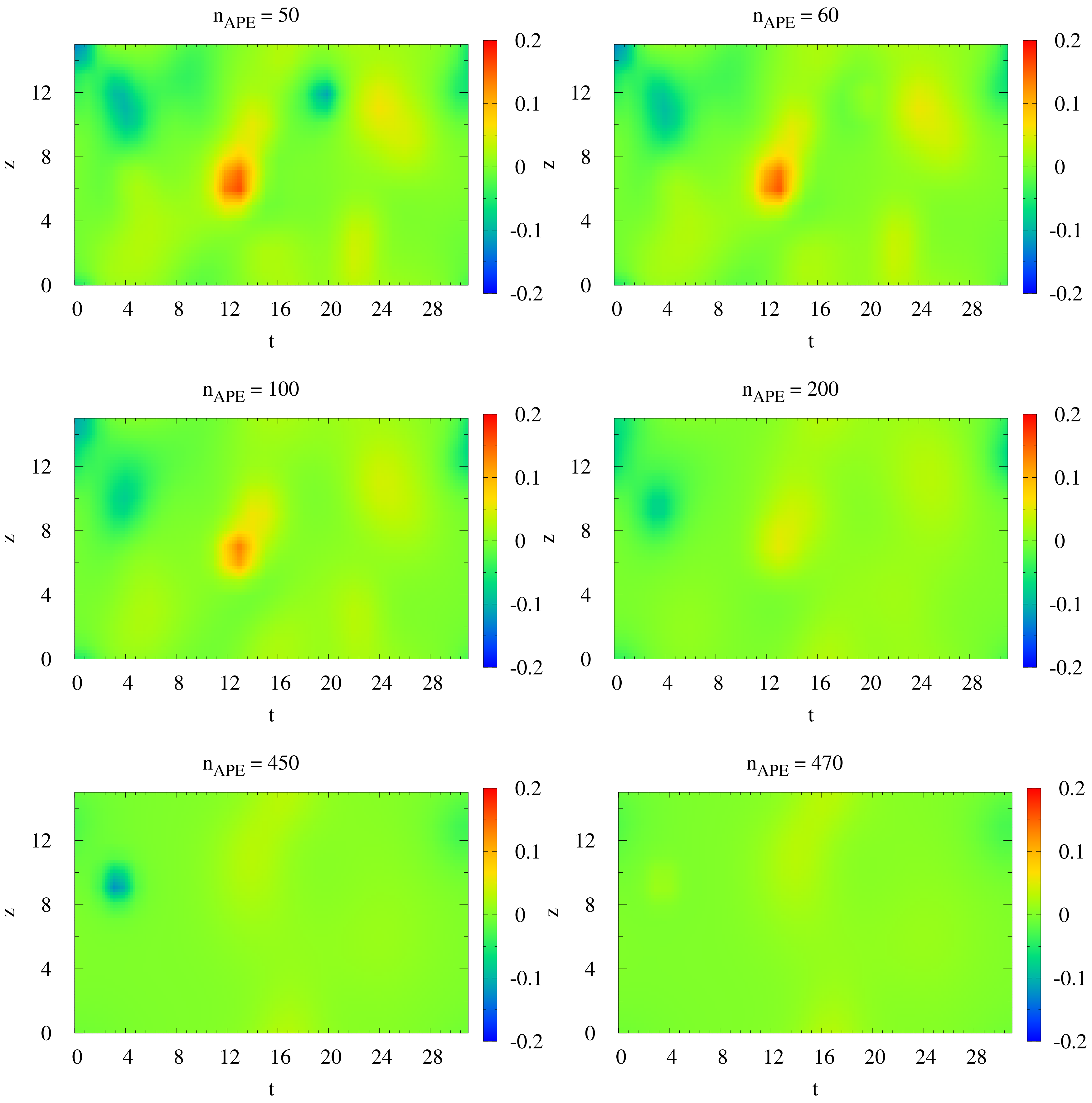


$\beta = 1,975$ $16^3 \times 32$



$\beta = 1,975$ $24^3 \times 48$





cf.[Bilson-Thompson, Leinweber, Williams, Dunne ’03]

“Just do it” on the lattice? However…

sign problem

$$e^{-S_g + \underbrace{i\theta Q}_{\text{widely fluctuating}}}$$

@ $\theta \sim \pi$

expansion around $\theta = 0$

Kitano-Yamada-MY ‘20

sub-volume method

Kitano-Yamada-Matsudo-MY ’21

cf. analytic continuation in θ^2 ← Masazumi’s talk

expansion around $\theta = 0$

generate gauge conf. at $\theta=0$ ← no sign problem

↓

measure top. charge Q

↓

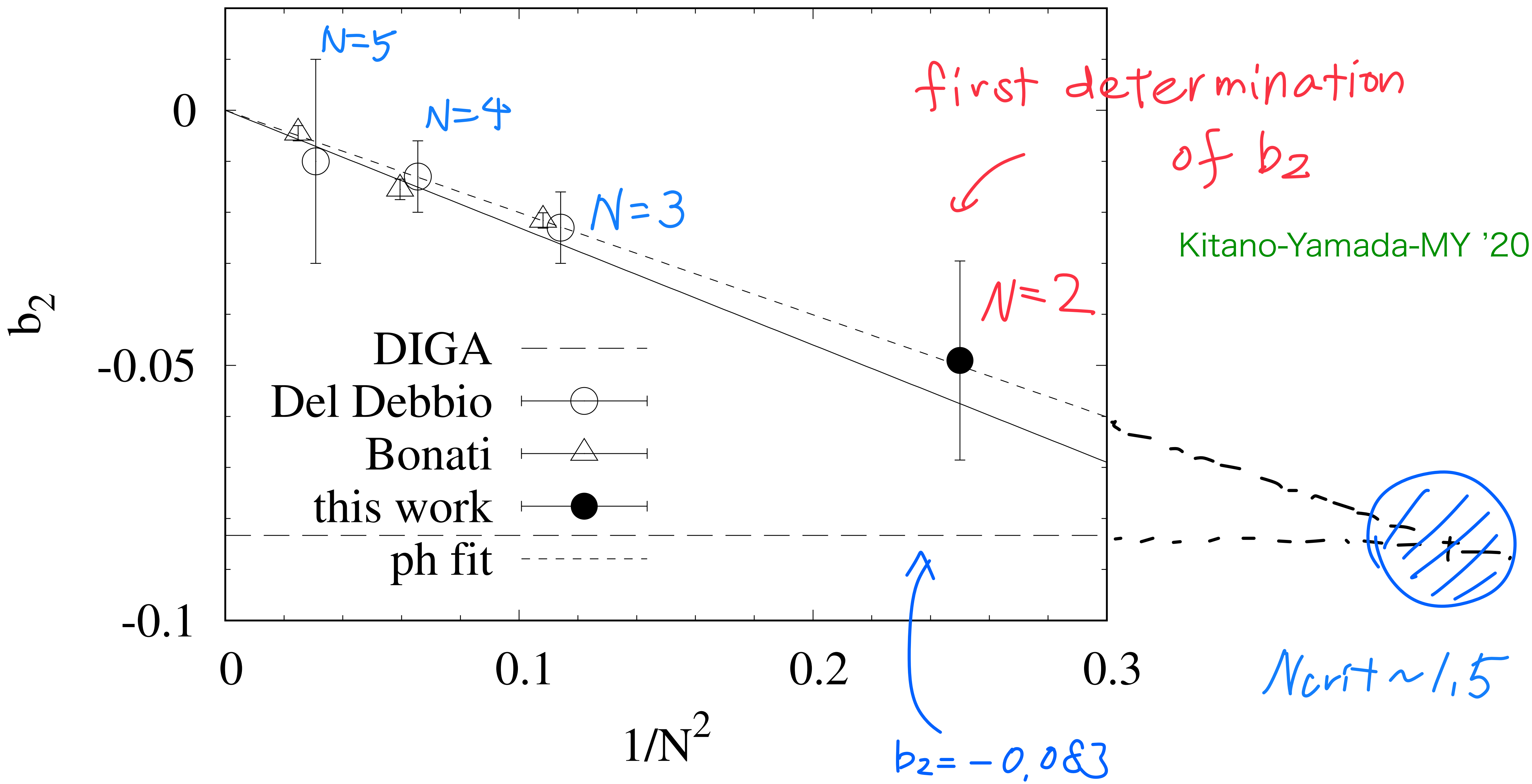
$$\chi = \frac{\langle Q^2\rangle_{\theta=0}}{V} \ ,$$

$$b_2 = -\frac{\langle Q^4\rangle_{\theta=0} - 3\,\langle Q^2\rangle_{\theta=0}^2}{12\,\langle Q^2\rangle_{\theta=0}} \ ,$$

$$b_4 = \frac{\langle Q^6\rangle_{\theta=0} - 15\,\langle Q^2\rangle_{\theta=0}\,\langle Q^4\rangle_{\theta=0} + 30\,\langle Q^2\rangle_{\theta=0}^3}{360\,\langle Q^2\rangle_{\theta=0}} \ ,$$

$$F(\theta) = \frac{1}{2}\chi\,\theta^2\left(1 + b_2\,\theta^2 + b_4\,\theta^4 + \cdots\cdots\right)$$

$$F(\theta) = \frac{1}{2}\chi\theta^2\left(1 + b_2\theta^2 + b_4\theta^4 + \cdots\right)$$



first determination of b_2

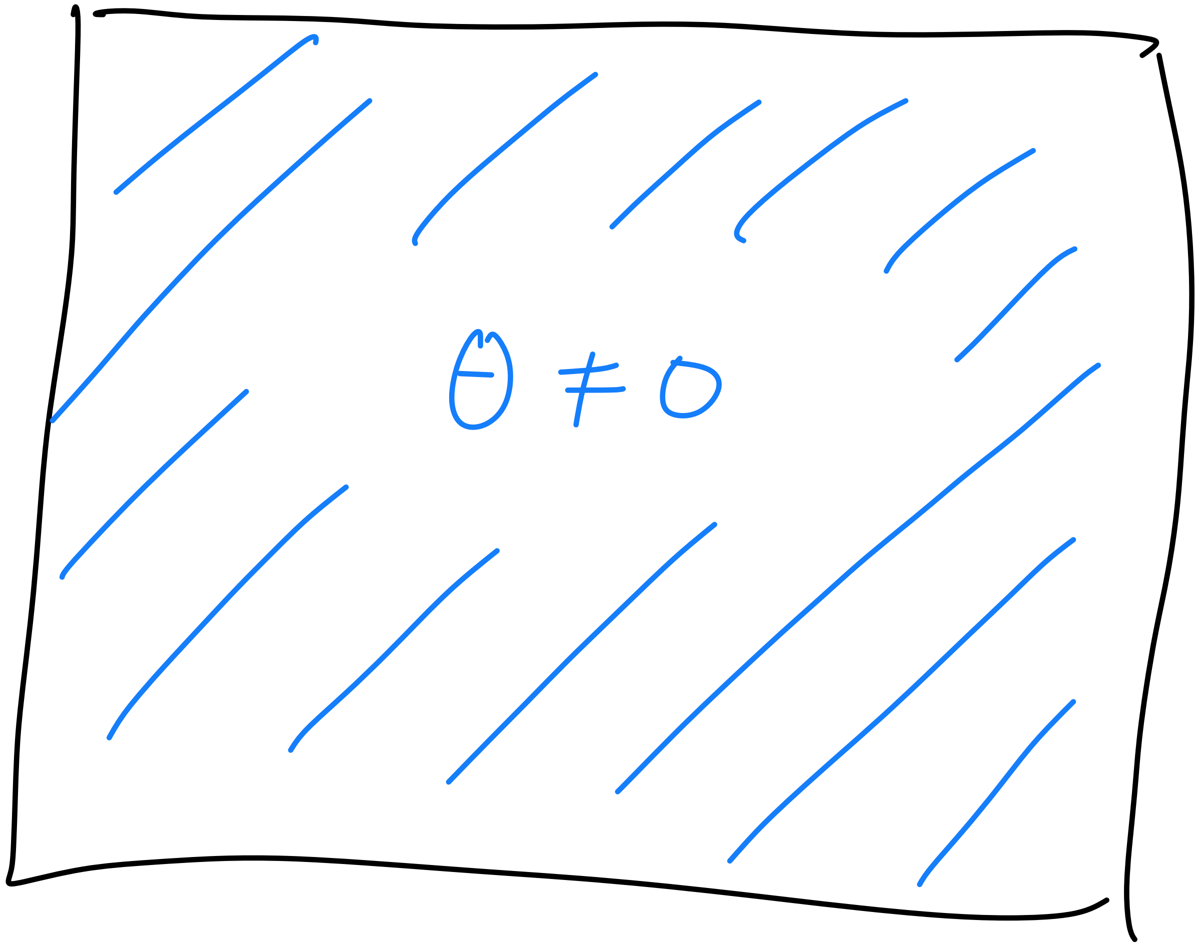
Kitano-Yamada-MY '20

$N_{crit} \sim 1.5$

$b_2 = -0.083$ (instanton)

Subvolume method

Kitano-Yamada-Matsudo-MY '21



$$Q \in \cancel{\mathbb{Z}}$$

$$\left(e^{iQ\pi} = (-1)^{Q} = \pm 1 \right)$$

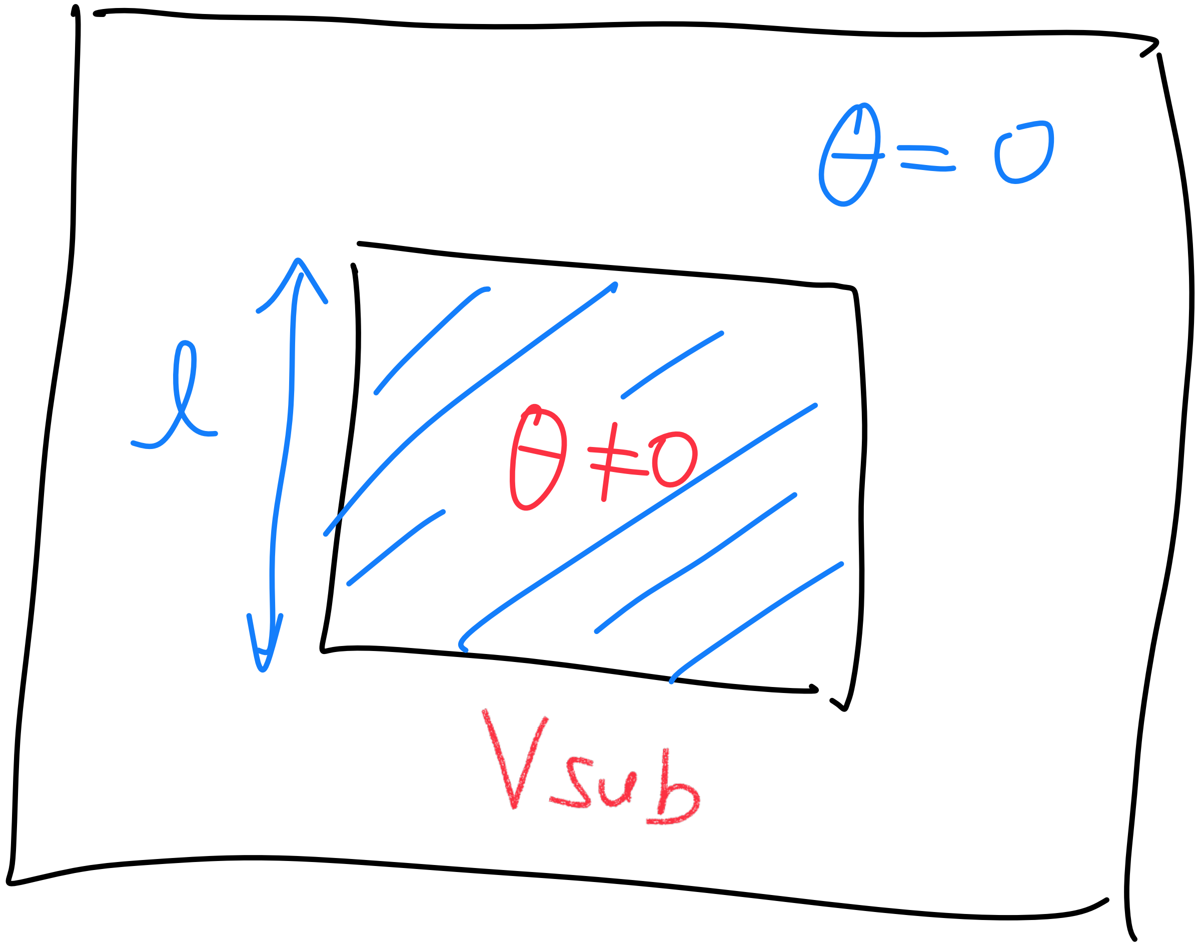
Subvolume method

Kitano-Yamada-Matsudo-MY '21

Idea

(measure

Q_{sub} inside V_{sub})



$Q_{sub} \notin \mathbb{Z}$

better w/ sign problem!

Subvolume method

Kitano-Yamada-Matsudo-MY '21

$$e^{-V_{sub}F_{sub}(\theta)} = \frac{1}{Z}\int \mathcal{D}U\, e^{-S_g + i\theta Q_{sub}}$$

$$= \left\langle e^{i\theta Q_{sub}} \right\rangle$$

$\uparrow$ "reweighting"

Subvolume method

Kitano-Yamada-Matsudo-MY '21

$$e^{-V_{sub} F_{sub}(\theta)} = \frac{1}{Z} \int \mathcal{D}U \, e^{-S_g + i\theta Q_{sub}}$$

$$= \left\langle e^{i\theta Q_{sub}} \right\rangle$$

Fit

$$F_{sub}(\theta) \sim F(\theta) + \frac{S(\theta)}{\ell} + \mathcal{O}\left(\frac{1}{\ell^2}\right)$$

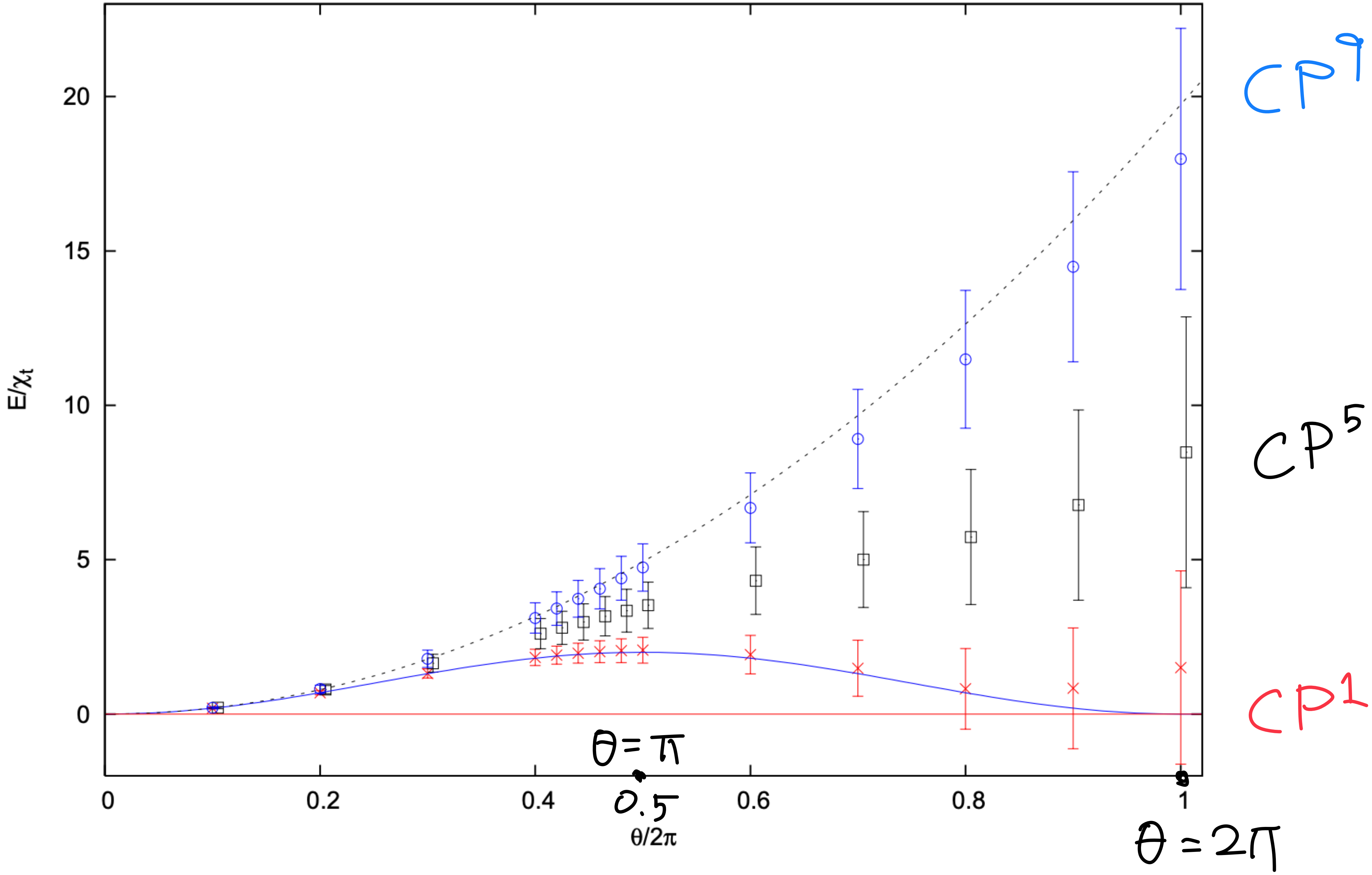
subvol. size

surface tension

$$(aT_c)^{-4} \ll V_{sub} \ll V_{full}$$

Works for CP^N model

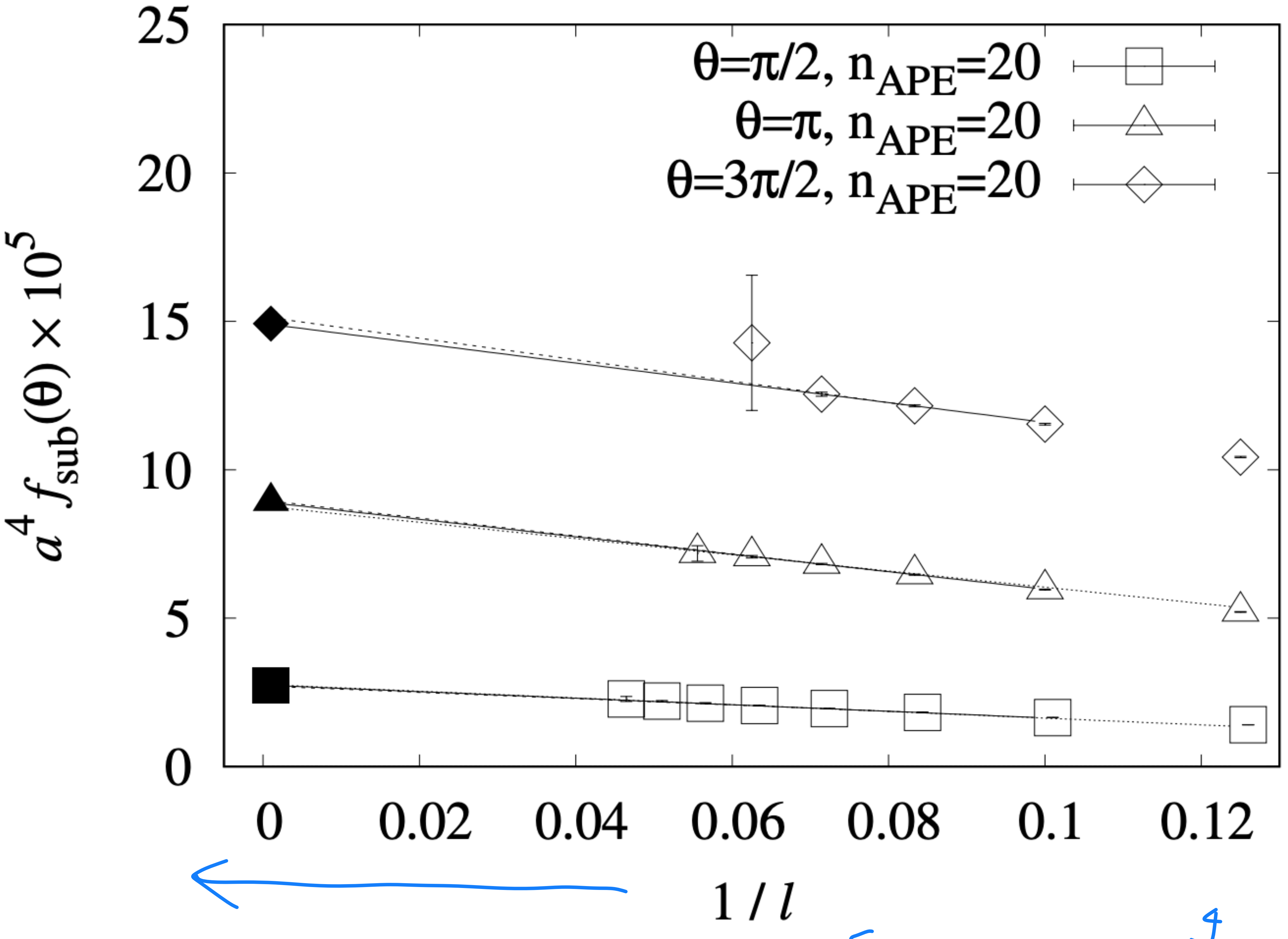
Keith-Hynes & Thacker '08



What about 4d YM ? Let's start with T=0

Linear extrapolation works for subvolume!

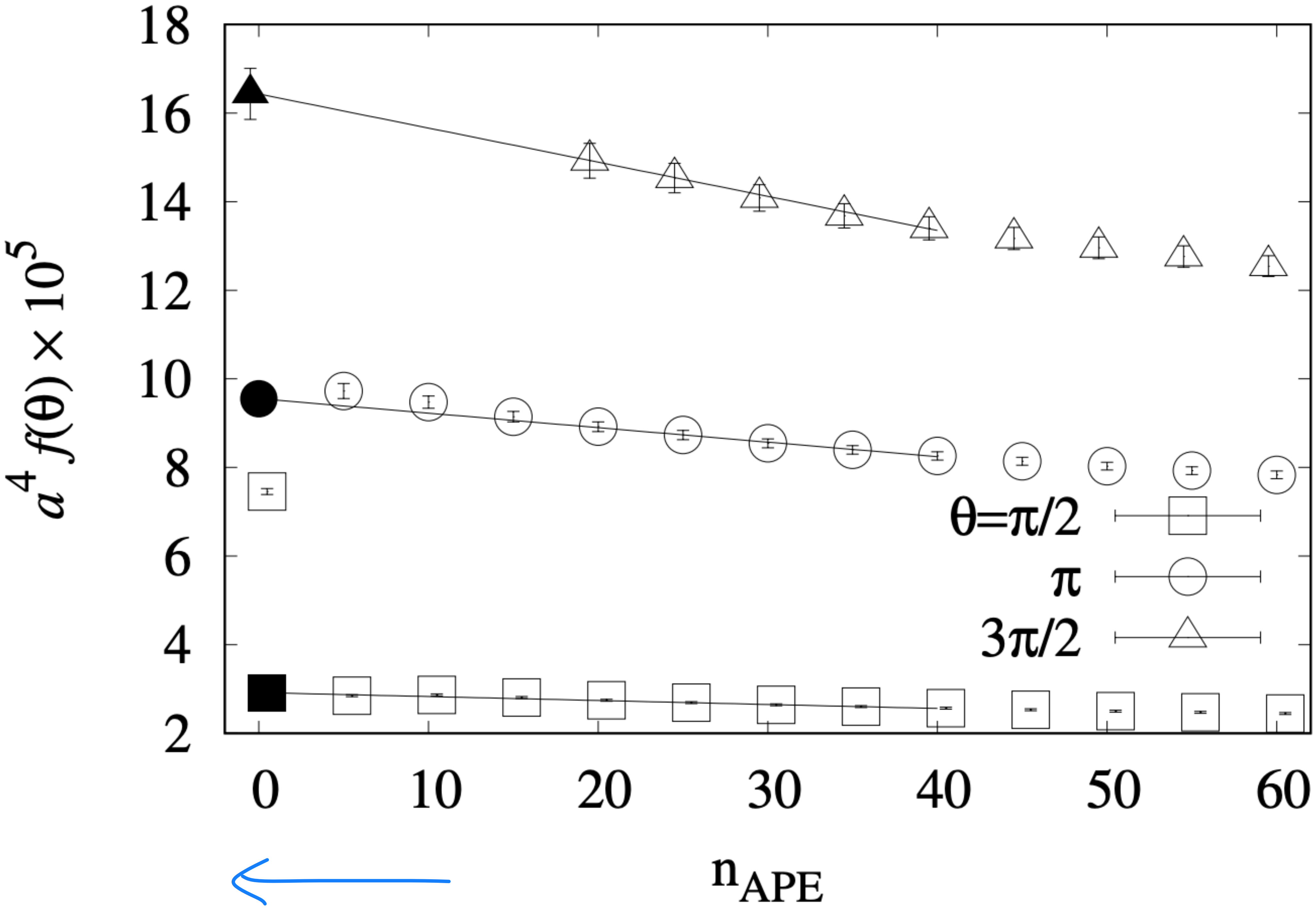
Kitano-Yamada-Matsudo-MY '21



$[10, 12, \cdot \cdot \; 24]^4 \subset 24^3 \times 48$

Let's also not forget to extrapolate in the smearing steps

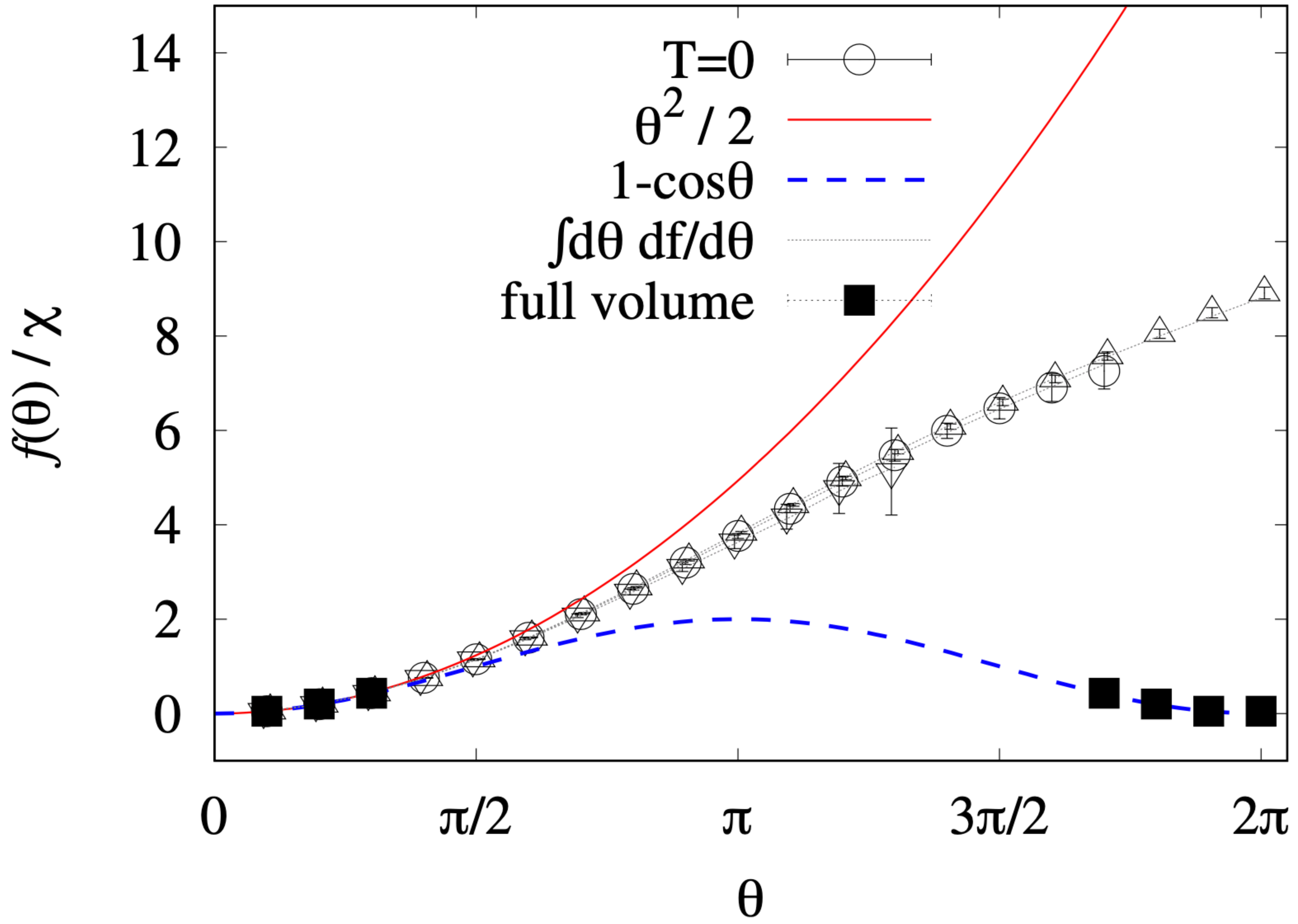
Kitano-Yamada-Matsudo-MY '21



Kitano-Yamada-Matsudo-MY '21

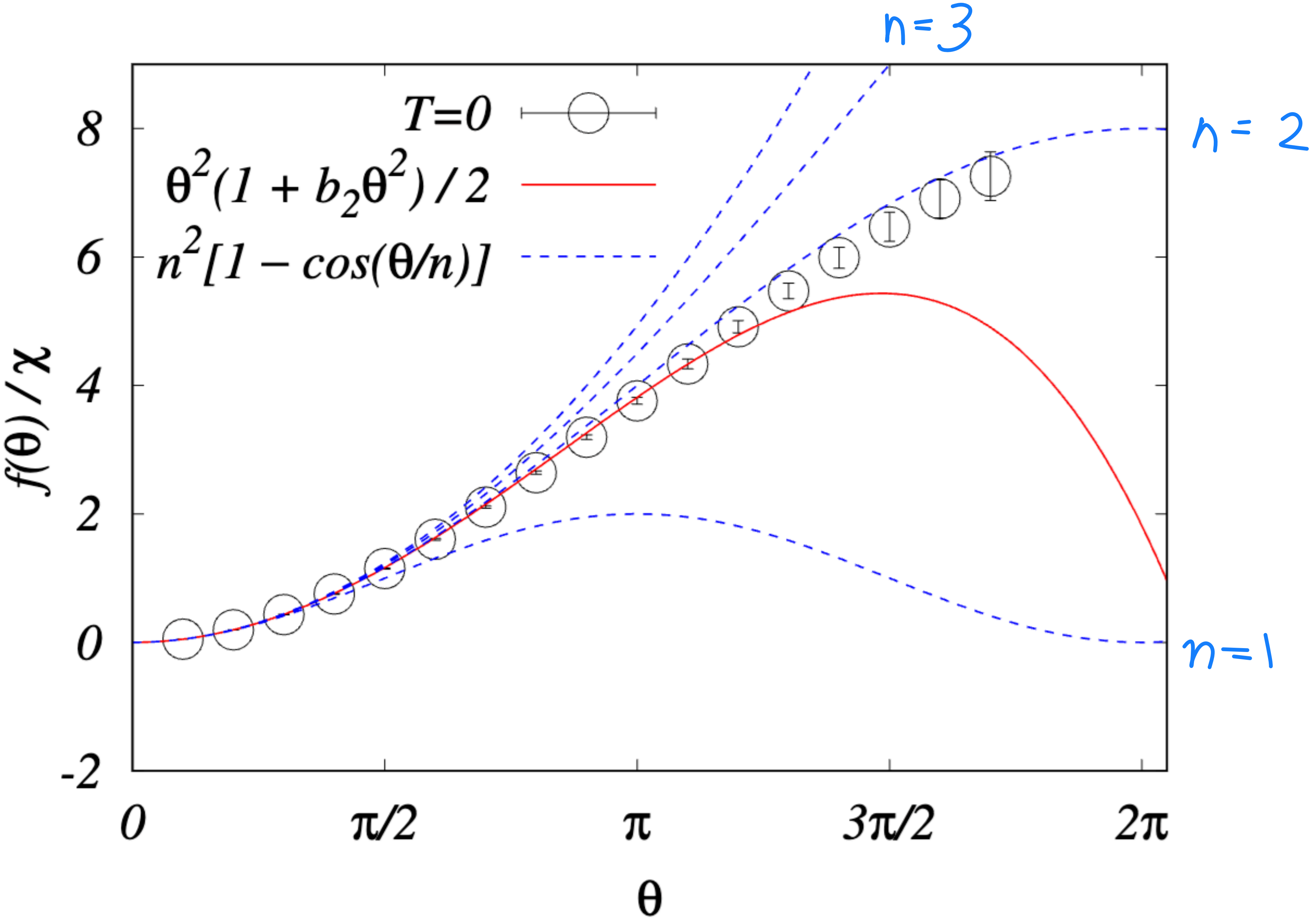
$F(\theta)$ @ $T=0$

clearly NOT 2π-periodic !!



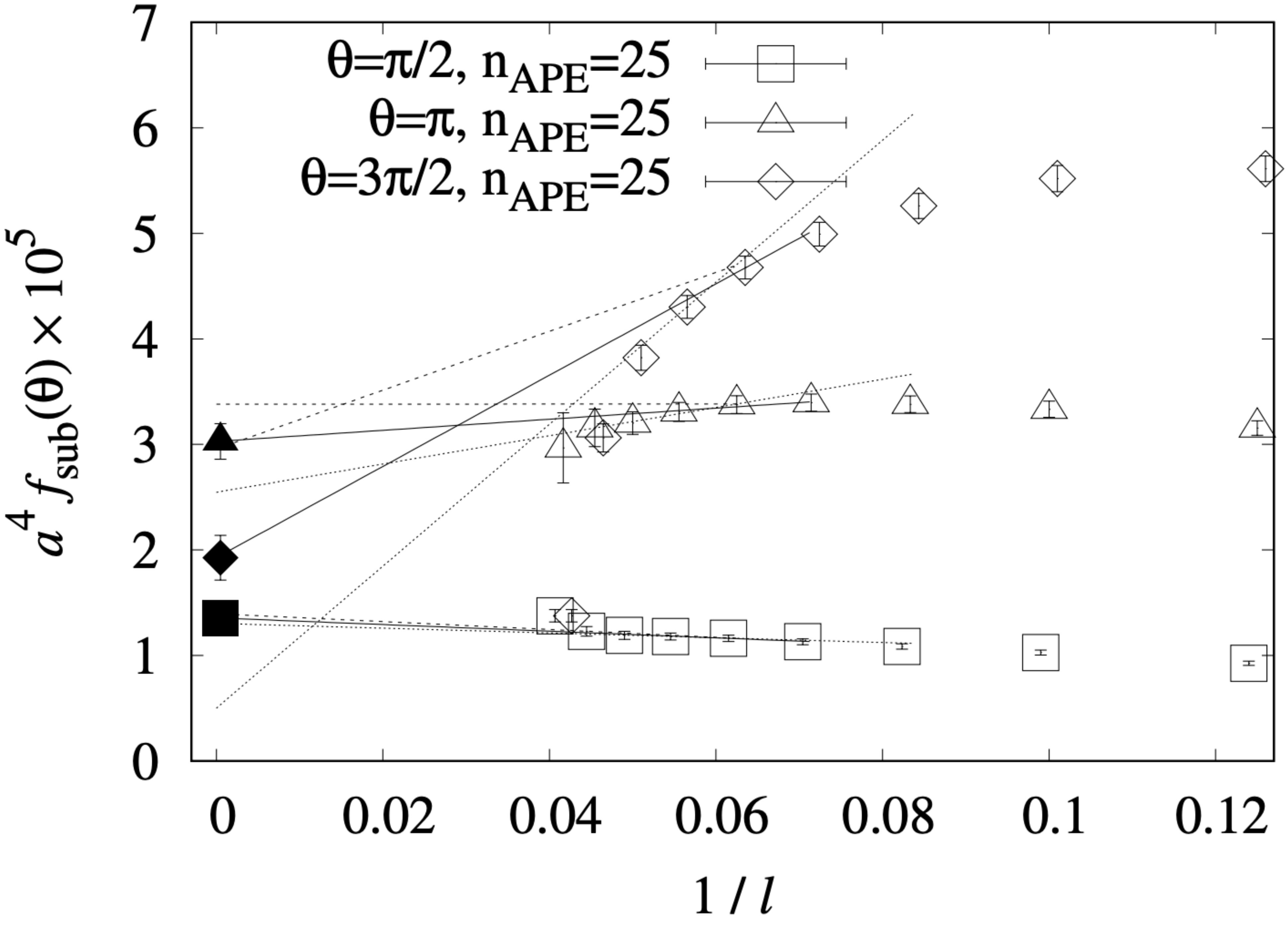
$$f(\theta) \sim 4\chi\left(1-\cos\frac{\theta}{2}\right)$$

4π-periodic, consistent with
Yonekura-MY, ‘17



We can also try T>Tc

(T= 1.2Tc >Tc)

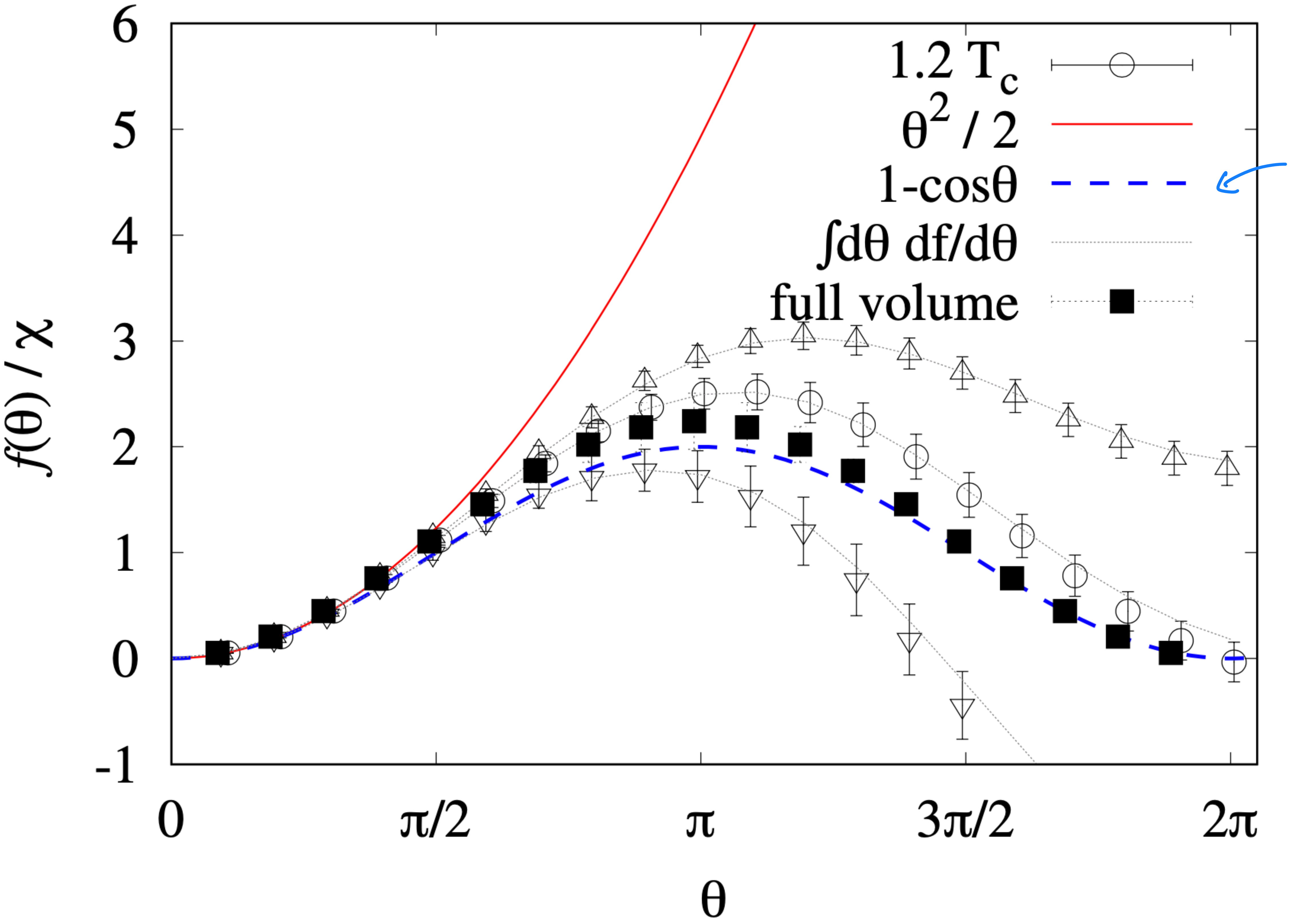


$F(\theta)$ @ $T = 1.2\, T_c > T_c$

Kitano-Yamada-Matsudo-MY '21

Consistent with high T expectation,

Gross-Pisarsky-Yaffe '81



← instanton

We can now explore (T, θ)-plane !

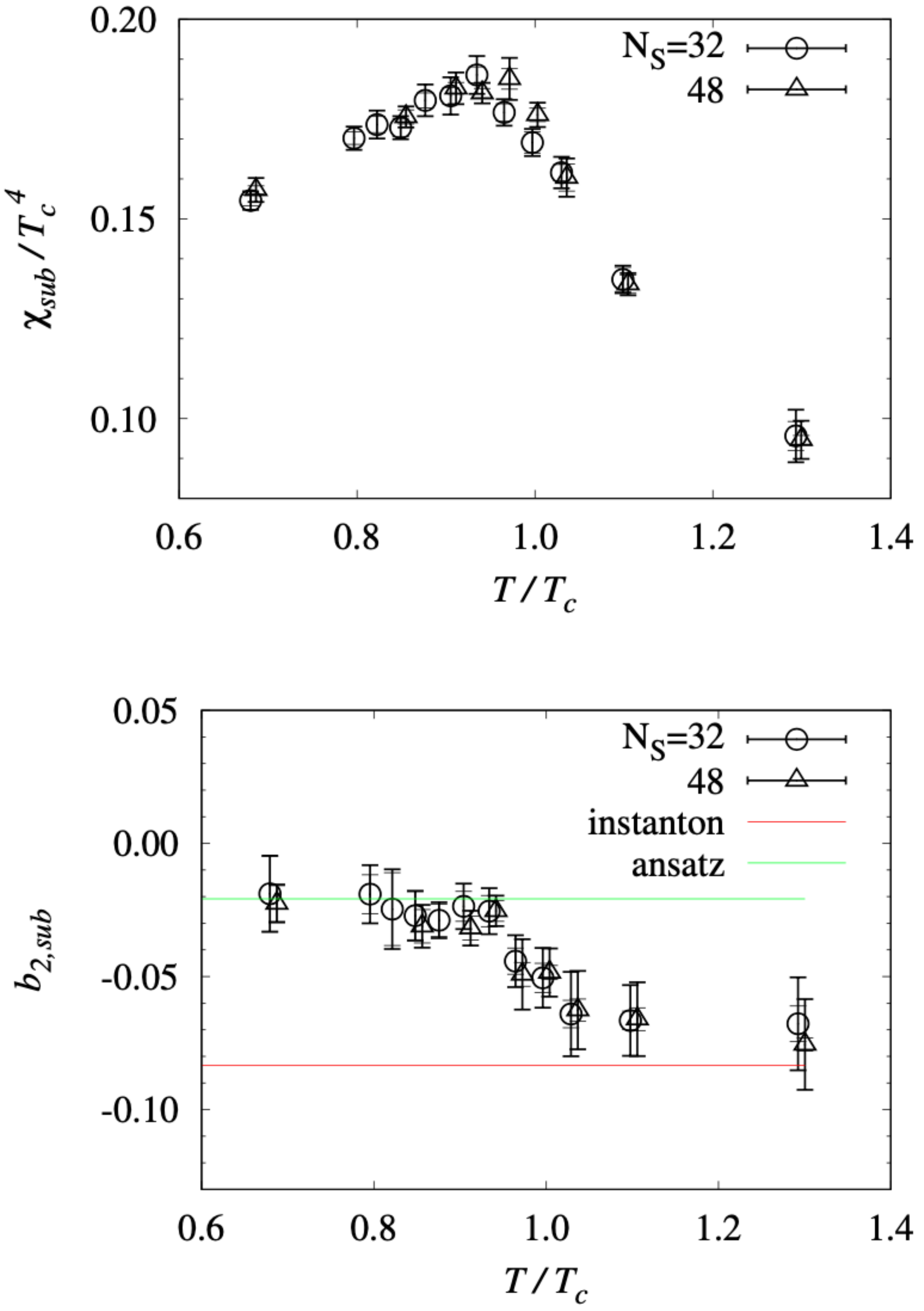
e.g.

$$F(\theta, T) = \frac{1}{2}\chi(T)\,\theta^2\left(1 + b_2(T)\,\theta^2 + \cdots\right)$$

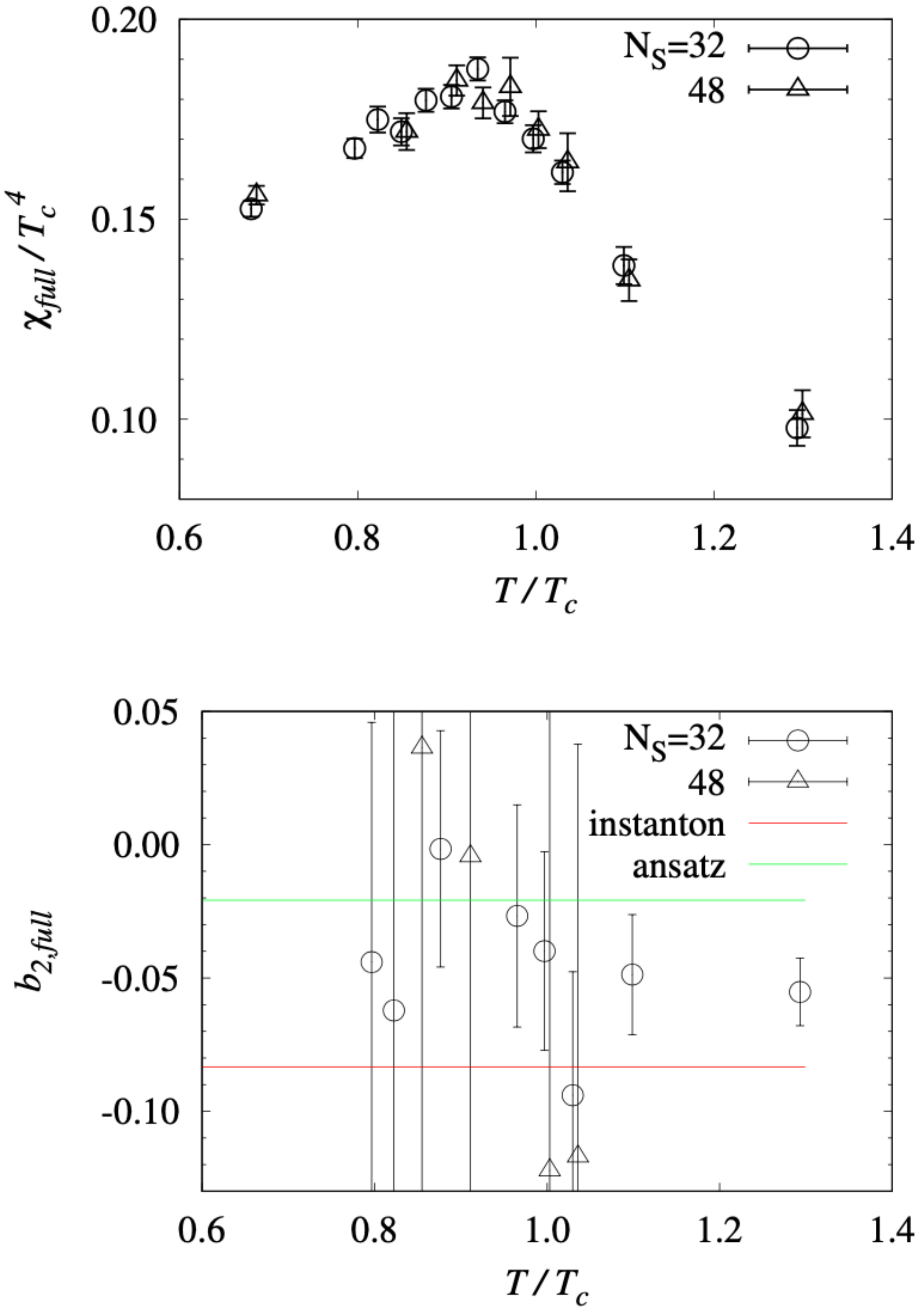
$$T_c(\theta)$$

[Kitano-Yamada-MY, 2403.10767, 2411.00375]

Subvolume method

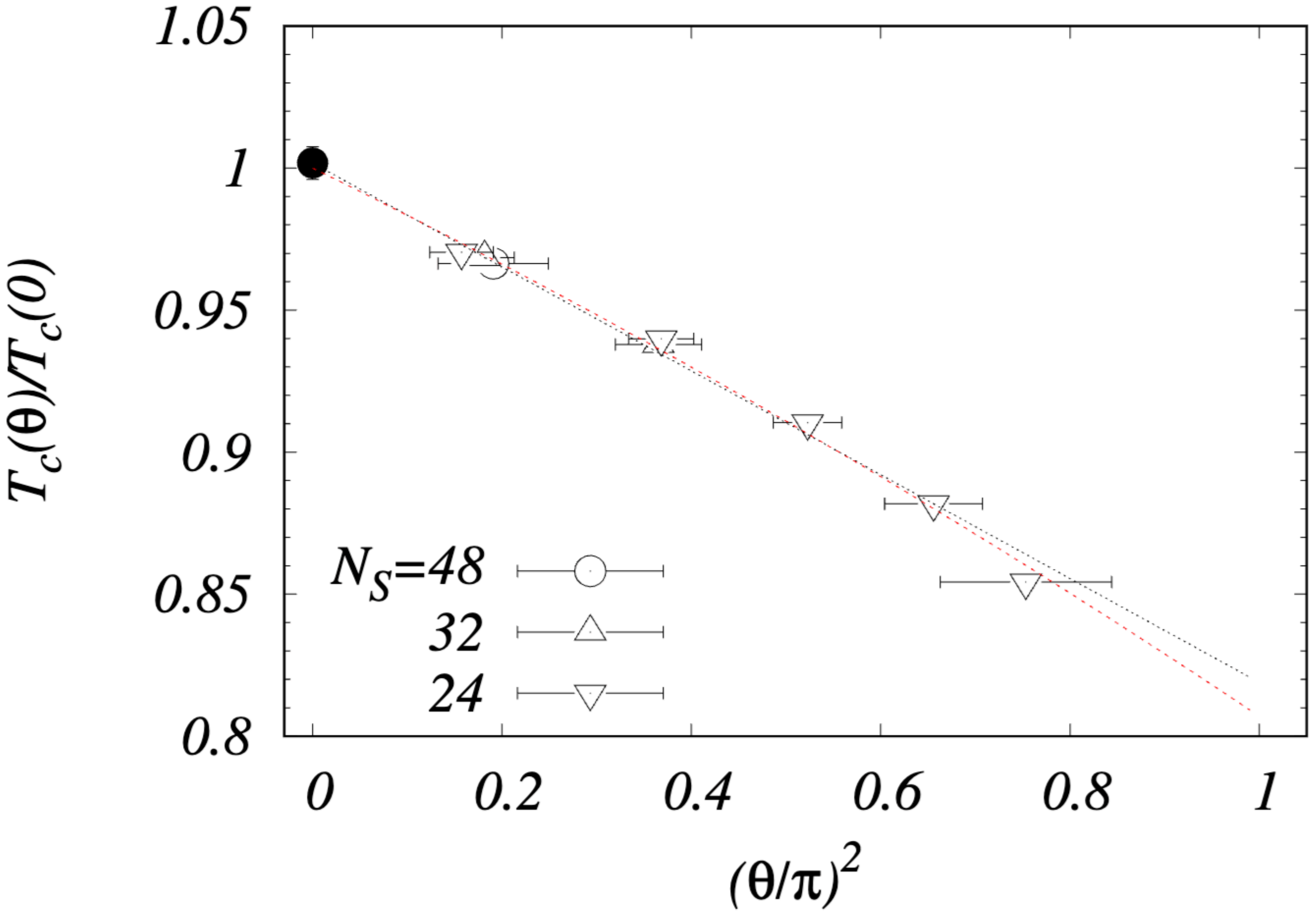


fullvolume method



[Kitano-Yamada-MY, 2403.10767]

$$T_c(\theta)/T_c(0) = 1 - 0.16(2)\,(\theta/\pi)^2 - 0.03(4)\,(\theta/\pi)^4$$



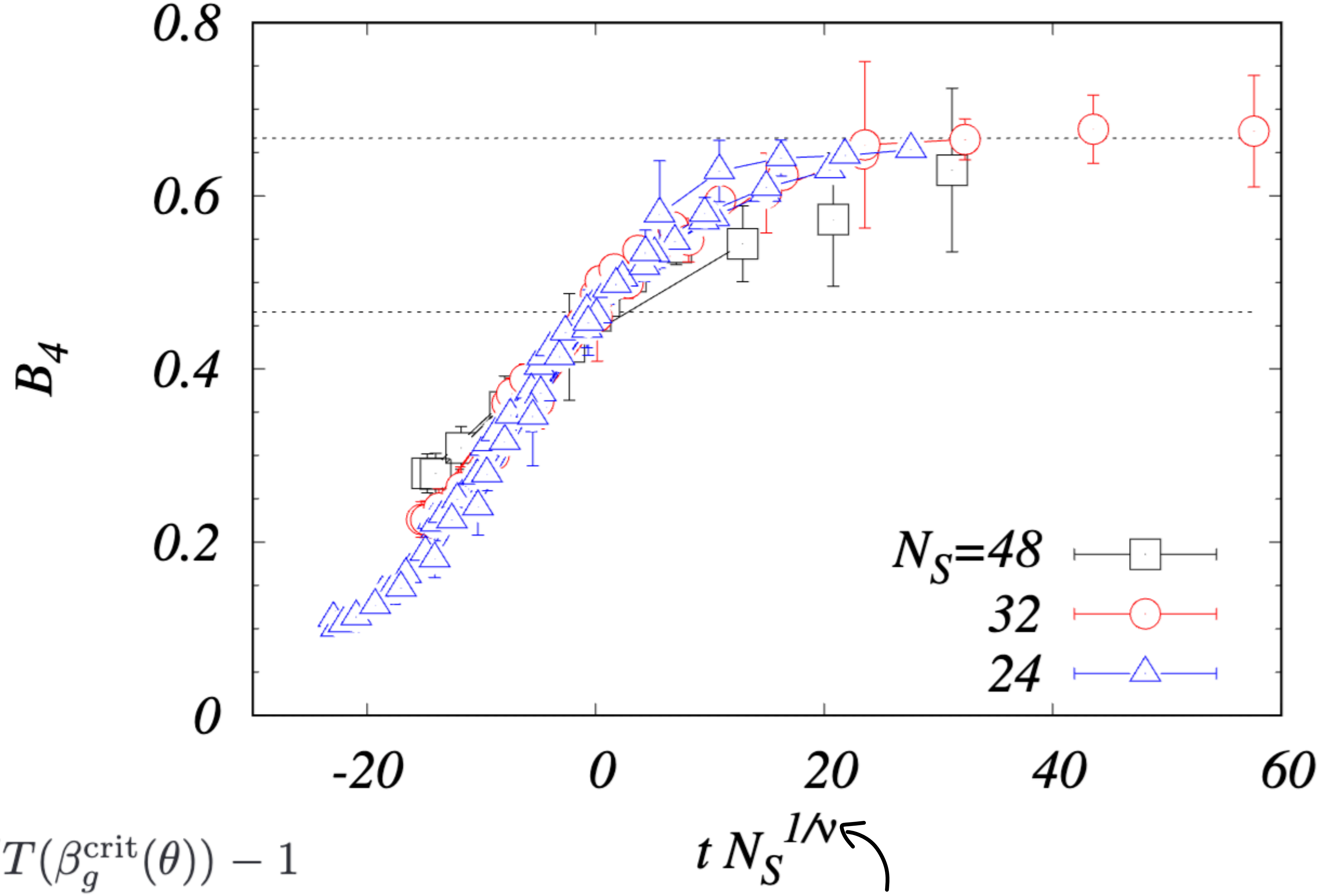
[Kitano-Yamada-MY, 2411.00375]

check of universality class

$$B_4(\beta_g, \theta) = 1 - \frac{\langle \omega^4 \rangle_{\beta_g,\theta}}{3\,\langle \omega^2 \rangle^2_{\beta_g,\theta}}$$

[Kitano-Yamada-MY, 2411.00375]

(ω: Polyakov loop)



$$t = T(\beta_g)/T(\beta_g^{\mathrm{crit}}(\theta)) - 1$$

ν =0.6301 for 3d Ising