TWO INSTANCES OF FAKE MINIMAL FANO 3-FOLDS II.

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Abstract. In section 1 we show that differential operators L_n[1] of type $D3$ annihilate periods for two pencils of hypersurfaces $1 - tf_n = 0$ in 3-torus given by levels of Laurent polynomials f_n, in particular L_n are of Picard–Fuchs type. However these operators are not associated to any Fano threefolds of first series. in section 2 we prove that they annihilate regularized G-series (generating functions for 1-point Gromov–Witten invariants) of two Fano threefolds of degrees 28 and 30 with second Betti numbers 2 and 3, so they indeed come from symplectic geometry of Fano threefolds. In section 3 we demonstate the mirror symmetry between two deformation classes of smooth Fano threefolds Y_{28},Y_{30} from section 2 and two Laurent polynomials f_{14}, f_{15} from section 1.

Let t be a coordinate on 1-dimensional torus $T = \text{Spec } \mathbb{C}[t, t^{-1}]$ and $D = t\frac{d}{dt}$.

Definition 0.1 ([11])**.** Operators of type $nD3$ is a family of linear differential operators

$$L_{(b_1,b_2,b_3,b_4,b_5)} = D^3 - t \cdot b_1 D(D+1)(2D+1) - t^2 \cdot (D+1)(b_2 D(D+2) + 4b_3) - \\ - t^3 \cdot b_4(D+1)(D+2)(2D+3) - t^4 \cdot b_5(D+1)(D+2)(D+3) \tag{0.2}$$

with parameters b_i[1].

Operator L_b of type $nD3$ has a unique power series solution F_b with initial condition $F_b(0) = 1$:

$$F_b = 1 + b_3\frac{t^2}{2} + (5b_1b_3 + 2b_4)\frac{t^3}{9} + \dots \tag{0.3}$$

We consider two particular operators L_n of type $nD3$ (from [13, 1]):

$$L_{14} = L_{(1,59,16,68,80)}, \tag{0.4}$$

$$L_{15} = L_{(1,43,12,78,216)}. \tag{0.5}$$

with solutions

$$F_{14}(t) = 1 + 8t^2 + 24t^3 + 240t^4 + 1440t^5 + 11960t^6 + 89040t^7 + \dots \tag{0.6}$$

$$F_{15}(t) = 1 + 6t^2 + 24t^3 + 162t^4 + 1080t^5 + 7620t^6 + 55440t^7 + \dots \tag{0.7}$$

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[1]Original definition has the other basis $a_{01}, a_{02}, a_{03}, a_{11}, a_{12}$ for parameter space $\mathbb{A}^5$. Bases a and b are equivalent over $\mathbb{Z}$: $b_1 = a_{11}$, $b_2 = a_{12} + 2a_{01} - a_{11}^2$, $b_3 = a_{01}$, $b_4 = a_{02} - a_{01}a_{11}$, $b_5 = a_{03} - a_{01}^2$; $a_{01} = b_3$, $a_{02} = b_4 + b_1b_3$, $a_{03} = b_5 + b_3^2$, $a_{11} = b_1$, $a_{12} = b_2 - 2b_3 + b_1^2$.

1. Laurent polynomials

Consider the Laurent polynomials f_n:

$$f_{14} = x + y + z + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + xy + \frac{1}{xy} + xz + \frac{1}{yz} \tag{1.1}$$

$$f_{15} = x + y + z + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{x}{y} + \frac{y}{z} + \frac{z}{x} \tag{1.2}$$

Proposition 1.3 (*Dutch trickcor Utrecht trick*, see e.g. [15])**.** *Let* $\omega = \prod_{k=1}^{d} \frac{dx_k}{2\pi i \cdot x_k}$ *be a logarithmic volume form on the torus* $X = (\mathbb{C}^*)^d = Spec\ \mathbb{C}[x_1, x_1^{-1}, \cdots, x_d, x_d^{-1}]$. *For Laurent polynomial* $w(x) \in \mathbb{C}[X]$ *define regular* constant term series *as* $G_w^r(t) = \int_{|x_i|=1} \frac{\omega}{1-t\cdot w} = \sum_{k\geqslant 0} < w^k > t^k$, *where* $< u >$ *is a constant term of Laurent polynomial* u. *Let* $\omega_t = Res_{1-t\cdot w=0} \frac{\omega}{1-t\cdot w}$ *be a fiberwise residue of* $\frac{\omega}{1-tw}$. *Then* $G_w^r(t)$ *is a solution to Picard–Fuchs equation for the pencil of hypersurfaces* $1 - tw(x) = 0$ *equipped with volume form form* ω_t.

Theorem 1.4. *For* $n = 14, 15$ *we have* $G_{f_n} = F_n$. *In particular, differential operators* L_n *are Picard–Fuchs differential operators for level families* $1 - tf_n = 0$ *of Laurent polynomials* f_n.

The proof is straightforward once one knows the formulation: functions G_{f_n} satisfy *some* differential equation, and each coefficient of G_f gives a linear relation for coefficients of the differential equation.

2. Fano varieties

2.1. **Introduction.** An important *symplectic* invariant of Fano variety Y [2] is its *regularized G-series* $RG_Y \in \mathbb{Q}[[t]]$ (see 2.4). If Y is a Fano threefold of first series then RG_Y is the unique analytic solution of the unique linear differential operator L_Y from 5-parametric family $nD3$ 0.2 with the initial condition $RG_Y(0) = 1$. In [11] Golyshev specified 17 points in the parameter space of $D3$'s that correspond to Fano threefolds of first series. He pointed out to me two other interesting differential operators 0.4 and 0.5 in family $nD3$. These equations are also listed in [1]. In [13] Lian and Yau realized F_{14} and F_{15} as periods for two families of $K3$ surfaces. In the first chapter we find Laurent polynomials $f_{14}(x_1, x_2, x_3)$ and $f_{15}(x_1, x_2, x_3)$ such that $F_n = \frac{1}{(2\pi i)^3} \int_{|x_i|=1} \frac{1}{1-tf_n} \frac{dx_1 \wedge dx_2 \wedge dx_3}{x_1 x_2 x_3}$ i.e. we realized F_n as periods of hypersurfaces $1 - tf_n = 0$ in 3-dimensional torus Spec $\mathbb{C}[x_1, x_1^{-1}, x_2, x_2^{-1}, x_3, x_3^{-1}]$. Laurent polynomials f_n are associated with the singular toric Fano 3-folds X_{2n} that admit smoothing Y_{2n} in the unique deformation class. In this article we prove that F_n coincides with $RG_{Y_{2n}}$:

Theorem 2.1. *Series* $RG_{Y_{2n}}$ *is annihilated by differential operator* L_n: $RG_{Y_{2n}} = F_n$.

We note that Fano threefolds Y_{28} and Y_{30} are not of first series ($b_2(Y_{28}) = 2$ and $b_2(Y_{30}) = 3$), but nevertheless they turn out to be *quantum minimal*: the minimal degree in D of differential operators annihilating their RG-series is 3, and not naively expected 4 and 5.

[2] *Fano variety* is a smooth variety Y with an ample anticanonical line bundle $\det T_Y$. Fano threefold Y is said to be *of first series* if its second Betti number is equal to one: $b_2(Y) = \dim H^2(Y, \mathbb{Q}) = 1$. Iskovskikh shown that there are exactly 17 deformation classes of such varieties, and his classification was redone with different methods by Mukai and Ciliberto-Miranda-Lopez. We refer the reader to the textbook[12] for the details on algebraic geometry of Fano threefolds.

2.2. **G-series.** Take $D = t\frac{d}{dt}$ where t is a coordinate on one-dimensional torus $T = \mathrm{Spec}\,\mathbb{C}[t, t^{-1}]$.

Let $\star$ be quantum multiplication on cohomologies of Fano variety Y.

Define *quantum differential equation* as a trivial vector bundle over T with fibre $H^\bullet(Y)$ and connection

$$D\Phi = c_1(Y) \star \Phi \tag{2.2}$$

where $\Phi(t) \in H^\bullet(Y)$.

Let $\mathcal{G}_Y(t) = [pt] + \sum_{n\geqslant 1} \mathcal{G}^{(n)} t^n$ be the unique power series solution of 2.2 with initial condition $\mathcal{G}_Y(0) = [pt]$, where $[pt] \in H^{2\dim Y}(Y, \mathbb{Z})$ is the cohomology class Poincare-dual to the the class of the point. Define series

$$G_Y(t) = \int_Y \mathcal{G}_Y(t) = 1 + \sum_{n\geqslant 1} g_n(Y) t^n \tag{2.3}$$

and

$$RG_Y(t) = 1 + \sum_{n\geqslant 1} g_n(Y) n! t^n \tag{2.4}$$

Remark 2.5. String equation implies $g_1(Y) = 0$.

Proposition 2.6 ([11])**.** *If Y is Fano threefold of first series then RG_Y is annihilated by unique normalized differential operator of type $D3$.*

Let $M = \overline{M}_{0,1}(Y, \beta)$ be the moduli space of stable maps $\phi : (C, p) \to Y$ from rational curves C with marked point p to Y of degree $\phi_*[C] = \beta \in H_2(Y, \mathbb{Z})$, ψ_1 be the first Chern class of the tautological line bundle on M, and $ev_1 : M \to Y$ be the evaluation map $ev_1(\phi, C, p) = \phi(p) \in Y$,

Consider Givental's J-function

$$J_Y(t_i; z) = \prod_i t_i^{a_i \frac{D_i}{z}} \sum_{\beta \in H_2(Y)} \frac{\prod_i t_i^{\beta \cdot D_i}}{z^{c_1(Y)\cdot\beta}} \cdot (ev_1)_* \frac{z}{z - \psi_1}, \tag{2.7}$$

where D_i is a base of $H^2(Y, \mathbb{Z})$ and $c_1(Y) = \sum a_i D_i$.

Next proposition is a corollary of Givental's theorem that relates solutions of quantum differential equation and J-function.

Proposition 2.8. *G-series equals to the fundamental term of Givental's J-series:*

$$G_Y = \int_{[Y]} J_Y(t_i = t^{a_i}, z = 1) \cup [pt].$$

2.3. **The threefolds.** Define degree of Fano variety Y to be its anticanonical degree

$$deg(Y) = (-K_Y)^{\dim Y} = \int_{[Y]} c_1(Y)^{\dim Y},$$

Euler number as topological Euler characteristic $\chi(Y) = \int_{[Y]} c_{\dim Y}$ and Picard number $\rho(Y)$ as $\mathrm{rk}\, H^2(Y, \mathbb{Z})$.

2.3.1. *Degree* 30.

Definition 2.9. Fano threefolds Y_{30} are blowups of a curve of bidegree $(2,2)$ on W, where W be a hyperplane section of bidegree $(1,1)$ of $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$. This deformation class of varieties has number 13 in table 3 of [14], it has degree 30, $\chi = 8$ and $\rho = 3$.

Proposition 2.10. *Y_{30} is a complete intersection of three numerically effective divisors of tridegrees $(1,1,0)$, $(1,0,1)$ and $(0,1,1)$ on $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$.*

Description 2.10 shows that Y_{30} is a complete intersection of sections of numerically effective line bundles in a smooth toric Fano variety. This allows us to compute G-series $G_{Y_{30}}$ using Givental's theorem [10].

So $G_{Y_{30}}$ is given by the pulback of hypergeometric series from three-dimensional torus

$$G_{Y_{30}} = e^{-3t} \sum_{a,b,c\geqslant 0} \frac{(a+b)!(a+c)!(b+c)!}{a!^3 b!^3 c!^3} t^{a+b+c} = 1 + 3t^2 + 4t^3 + \frac{27}{4}t^4 + 9t^5 + \dots \tag{2.11}$$

2.3.2. *Degree* 28.

Definition 2.12. Fano threefolds Y_{28} are blowups of a twisted quartic on a 3-dimensional quadric Q. This deformation class of varieties has number 21 in table 2 of [14], it has degree 28, $\chi = 6$ and $\rho = 2$.

Realize Q as a linear section of 4-dimensional quadric $Gr(2,4) \subset \mathbb{P}^5$ parametrizing lines in $\mathbb{P}^3$. Let U be the universal vector bundle of rank 2 on $Gr(2,4)$, and $\mathcal{O}_{Gr}(1) = \det U^*$ be the ample generator of Picard group on this Grassmannian.

Proposition 2.13. *Consider a generic section of rank 5 vector bundle $(U^* \otimes \mathcal{O}_{\mathbb{P}^4}(1))^{\oplus 2} \oplus \mathcal{O}_{Gr}(1)$ on the 8-fold $Gr(2,4) \otimes \mathbb{P}^4$. Then it is a smooth Fano threefold from deformation class of Y_{28}.*

Consider *abelianization* for Y_{28}: a complete intersection A_{28} in $\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^4$ of line bundles $\mathcal{O}(1,0,1)$ twice, $\mathcal{O}(0,1,1)$ twice and $\mathcal{O}(1,1,0)$.

J-series of Y_{28} (and hence G-series $G_{Y_{28}}$) can be computed using abelian/non-abelian correspondence [3, 4] of Bertrand, Ciocan-Fontanine, Kim and Sabbah:

Proposition 2.14. *$G_{Y_{28}}$ equals e^{-t} times coefficient at H_2 of*

$$\sum_{a_1,a_2,b\geqslant 0} t^{a_1+a_2+b} \cdot \frac{g(a_1+b,H_1)^2 g(a_2+b,H_2)^2 g(a_1+a_2,H_1+H_2)}{g(a_1,H_1)^4 g(a_2,H_2)^4 b!^5} (a_2-a_1+(H_2-H_1))(-1)^{a_2-a_1}$$

where $g(d,D) = d! \cdot (1+h_d D)$ is approximation of Γ-function, $h_d = \sum_{i=1}^{d} \frac{1}{i}$ is the harmonic number, and H_1, H_2 are positive generators of $H^2(\mathbb{P}^3,\mathbb{Z})$ pulled back to $H^2(\mathbb{P}^3 \times \mathbb{P}^3,\mathbb{Z})$ by two projections.

So $G_{Y_{28}} = 1 + 4t^2 + 4t^3 + 10t^4 + 12t^5 + \frac{299}{18}t^6 + \dots$.

We'll present a detailed proof of these computations in [6].

Denote the G-series of Y_{2n} as $G_n = G_{Y_{2n}}$.

3. Toric degenerations and mirror symmetry

Definition 3.1 ([15])**.** Let $G_Y(t) = 1 + g_2(Y)t^2 + \dots$ be the G-series of Y. Laurent polynomial w is said to be *mirror* for Y if general element of pencil $1 - tw(x) = 0$ is birational to Calabi–Yau variety and $G_Y(t) = G_w(t)$.

Remark 3.2. In particular constant term of w is zero.

Definition 3.3. For toric Fano variety X let $\Delta(X)$ be its fan polytope. For lattice polytope Δ let $\mathbb{P}_\Delta$ be the toric Fano variety with fan polytope Δ. Define *vertex Laurent polynomial* w_Δ as sum of all monomials corresponding to the vertices of Δ:

$$w_\Delta = \sum_{v \in Vertices(\Delta)} x^v \tag{3.4}$$

For Laurent polynomial w let $\Delta(w)$ be its Newton polytope and define $\mathbb{P}(w) = \mathbb{P}_{\Delta(w)}$

Theorem 3.5 (Mirror construction for smooth toric Fano varieties [10])**.** *If variety X is a smooth toric Fano variety then $w_{\Delta(X)}$ is a mirror for X.*

Definition 3.6 (Small toric degeneration [2])**.** Assume X is a toric Fano variety that admits only terminal Gorenstein singularities. We say that X is a small toric degeneration of smooth Fano variety Y if there exists a flat projective morphism to irreducible curve $\pi : \mathcal{X} \to C$, such that X and Y are isomorphic to some fibers of π, and the restriction map $Pic(\mathcal{X}) \to Pic(\mathcal{X}_t)$ is an isomorphism for all $t \in C$.

Conjecture 3.7 (Small toric degeneration hypothesis [2])**.** *If X is a small toric degeneration of smooth Fano variety Y, then $w_{\Delta(X)}$ is a mirror for Y.*

Theorem 3.8 ([8])**.** *For $n = 14, 15$ toric Fano variety $\mathbb{P}(f_n)$ have terminal Gorenstein singularities and is small toric degeneration of smooth Fano threefold Y_{2n} in the uniquely determined deformation class (number* 2.21 *and* 3.13 *in* [14]*).*

Combination of theorems 1.4 and 2.1 implies

Theorem 3.9. *For $n = 14, 15$ Laurent polynomial f_n is a mirror for family of smooth Fano threefolds Y_{2n}.*

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