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G-minimal varieties are quantum minimal

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Abstract. We show that G-minimal Fano varieties are quantum minimal.

The aim of this note is to provide a conceptual explanation (theorem 15, remarks 23.24) for all discovered (so far) phenomena of minimality in quantum cohomology. In particular we show that 8 known differential operators of type $D3$ (see example 39) that do not correspond to any minimal Fano threefolds, do actually come from geometry of G-Fano threefolds.

Let X be a smooth $(n-1)$-dimensional variety embedded by linear system $\mathcal{L}^{\otimes k}$. Assume that X admits action of some finite (possibly trivial) group G. Consider cohomology algebra $H = H^\bullet(X,\mathbb{Q})$, its subalgebra H_a of algebraic cycles, its G-invariant subalgebra H_a^G, and subring H_L generated by $L = c_1(\mathcal{L})$: $H_L = \mathbb{Q}[L]/L^n \subset H$.

Inclusions $H_L \subset H_a^G \subset H_a \subset H$ imply inequalities

$$n = \dim H_L \leqslant \dim H_a^G \leqslant \dim H_a \leqslant \dim H. \tag{1}$$

Definition 2. Variety X is called *homologically minimal* (or just *minimal*) if its cohomology H is n-dimensional, and *algebraically G-minimal* if $\dim H_a^G = n$.

Remark 3. Variety X is called *G-Fano* if $H^2(X,\mathbb{Q})^G$ is 1-dimensional, i.e. $H^2(X,\mathbb{Q})^G = \mathbb{Q}c_1(X)$ and anticanonical line bundle is ample. Obviously, algebraically G-minimal Fano varieties are G-Fano, and G-Fano varieties of dimension $\leqslant 3$ are algebraically G-minimal.

For classes $\gamma_i \in H^\bullet(X)$ and $\beta \in H_2(X,\mathbb{Z})$ let $\langle\gamma_1,\gamma_2,\gamma_3\rangle_\beta$ be 3-pointed genus-0 Gromov–Witten invariant (naively equal to number of rational curves of homology class β passing through three cycles Poincare-dual to classes γ_i if β is effective class and 0 otherwise).

Definition 4. *Very small* (algebraic) *quantum cohomology* of X is a trivial $\mathbb{Q}[q]-module$ $qH_a(X) = H_a(X)[q]$ with multiplication $\star$ defined by

$$\int_{[X]} (\gamma_1 \star \gamma_2) \cup \gamma_3 = \sum_{\beta\in H_2(X,\mathbb{Z})} \langle\gamma_1,\gamma_2,\gamma_3\rangle_\beta \, q^{c_1(X)\cdot\beta}. \tag{5}$$

Ring $qH(X)$ is a commutative associative unital ring, and it is homogeneous with degree of q equal to 1 and degree of $\gamma\otimes 1$ equal to half of usual degree of $\gamma \in H^\bullet(X)$.

Definition 6. *Minimal quantum cohomology* $qH_m(X)$ is a subring of very small quantum cohomology generated by $c_1(X)\otimes 1$ and q.

Definition 7. *Quantum rank* $N(X)$ of X is the rank of $qH_m(X)$ as $\mathbb{Q}[q]$-module.
Fano variety X is called *quantum maximal* if $N(X) = \dim H_a$ i.e. $qH_m(X) = qH_a(X)$.
Fano variety X is called *quantum minimal* if $N(X) = n$.

Some reconstruction theory for quantum minimal varieties is developed in [10].

Remark 8. Upper bound for quantum rank is obvious and congruence $c_1(X)^{\star k} = c_1(X)^{\cup k} \mod q$ implies that $N(X)$ cannot be less than n.

Example 9. Variety X is algebraically minimal $\iff$ it is both quantum minimal and maximal.
Grassmannian $G(2,4)$ is quantum minimal, but not maximal.
Grassmannian $G(2,5)$ is quantum maximal, but not minimal.
Grassmannian $G(2,6)$ is neither quantum minimal, nor quantum maximal.

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More generally Grassmannians $G(k,p)$ for prime values of p (e.g. $G(2,5)$) are quantum maximal (see corollary 14) and algebraically G-minimal varieties are quantum minimal (see theorem 15).

Definition 10 (see also [5])**.** *Quantum characteristic polynomial* of variety X is the characteristic polynomial of the operator $\star c_1(X)$ acting on the vector space $qH_a(X)\otimes\mathbb{Q}(q)$. (Anticanonical) *spectrum* of variety X is the set of roots of its quantum characteristic polynomial.

Lemma 11. *Quantum rank $N(X)$ is not less than number of distinct non-zero elements in the spectrum of X.*

Proof. By definition $N(X)$ is the degree of the minimal polynomial for operator $\star c_1(X)$ acting on $qH_a(X)\otimes\mathbb{Q}(q)$. □

Corollary 12. *If variety X is quantum minimal then number of distinct non-zero elements in its spectrum is not more than $n=\dim X+1$.*

Corollary 13. *If the spectrum of variety X is simple i.e. all eigenvalues of $\star c_1(X)$ are distinct and non-zero, then variety X is quantum maximal.*

Corollary 14. *Grassmannians $G(k,p)$ are quantum maximal for prime values of p.*

Proof. By [4] roots of quantum characteristic polynomial of $G(k,p)$ are proportional to sums of the elements in all different k-tuples of distinct p-th roots of unity. For prime values of p all these numbers are distinct, so previous corollary 13 implies $G(k,p)$ is quantum maximal. □

Theorem 15. *If Fano variety X is algebraically G-minimal then X is quantum minimal.*

This theorem holds because quantum multiplication respects the group action:

Lemma 16. *For any $g\in G$, $\gamma_1,\gamma_2\in qH_a(X)$ we have $g^*\gamma_1\star g^*\gamma_2=g^*(\gamma_1\star\gamma_2)$, where action of G on $H_a[q]$ is defined by the base change i.e. $g^*q=q$.*

Corollary 17. *Vector space $H_a^G[q]$ is closed with respect to $\star$-multiplication.*

Proof of the corollary 17. If $g^*\gamma_1=\gamma_1$ and $g^*\gamma_2=\gamma_2$ then $g^*(\gamma_1\star\gamma_2)=g^*\gamma_1\star g^*\gamma_2=\gamma_1\star\gamma_2$. □

Corollary 18. *Minimal quantum cohomology $qH_m(X)$ is a subring of $H_a^G[q]$.*

Proof. Anticanonical class is algebraic and G-invariant, so $c_1(X)\in H_a^G[q]$ and by corollary 17 the whole subring $qH_m(X)$ generated by $c_1(X)$ lies in $H_a^G[q]$. □

Proof of the lemma 16. Since Gromov–Witten invariants are well defined and are indeed invariant with respect to the isomorphims for all classes $\beta\in H_2(X)$ and $\gamma_i\in H^\bullet(X)$. one has

$$\langle g^*\gamma_1,...,g^*\gamma_n\rangle_\beta=\langle\gamma_1,\dots,\gamma_n\rangle_{g_*\beta}\tag{19}$$

Anticanonical class $c_1(X)$ is G-invariant, so any automorphism preserves anticanonical degrees:

$$g^*c_1(X)=c_1(X)\tag{20}$$

$$c_1(X)\cdot\beta=(g^{-1})^*c_1(X)\cdot\beta=c_1(X)\cdot g_*\beta\tag{21}$$

For fixed γ_1,γ_2 and arbitrary γ_3 using equalities 19, 20, 21 we derive:

$$\begin{aligned}\int_{[X]}(g^*\gamma_1\star g^*\gamma_2)\cup\gamma_3=\sum_\beta\langle g^*\gamma_1,g^*\gamma_2,\gamma_3\rangle_\beta\, q^{c_1(X)\cdot\beta}=\sum_\beta\left\langle\gamma_1,\gamma_2,(g^{-1})^*\gamma_3\right\rangle_{g_*\beta}q^{c_1(X)\cdot g_*\beta}=\\=\int_{[X]}(\gamma_1\star\gamma_2)\cup g^{-1})^*\gamma_3=\int_{[X]}(g^*(\gamma_1\star\gamma_2))\cup\gamma_3\end{aligned}\tag{22}$$

This proves the lemma. □

Proof of the theorem 15. By corollary 18 minimal quantum cohomology $qH_m(X)$ is contained inside $H_a^G[q]$. That implies $N(X)\leqslant\dim H_a^G$, but by definition of algebraic G-minimality $\dim H_a^G=n$, so $N(X)\leqslant n$ i.e. X is quantum minimal. □

Remark 23. Actually theorem 15 can be considered as a particular manifestation of monodromy action for family $[X/G]\to[pt/G]$ over stack $[pt/G]$ with fibers X. Geometrically deformation invariance of Gromov–Witten is invariance of correlators with respect to Gauss-Manin connection.

Remark 24. There are two frameworks for quantum cohomology — symplectic and algebraic. One may notice neither of these definitions were used in the proof. Geometrical part is hidden behind the equalities 19, 20, 21 and the fact that correlators are invariant with respect to algebraic or symplectic isomorphisms.

Moreover, one can even apply the theorem in the case of non-geometric action of the Galois group (or mixed geometric and Galois action) on variety X and it's cohomologies (e.g. $H_{et}(X, \mathbb{Q}_l)$) if X is defined over $\mathbb{Q}$ (or over some number field). This is true since in algebraic framework is motivic and everything is defined over the base field of X: $M_{g,n}(X, \beta)$, evaluation map $ev : M_{g,n}(X, \beta) \to X^n$, ψ-classes and the virtual fundamental class.

Remark 25. Since quantum cohomology is deformation-invariant we deduce from corollary 18 that minimal quantum cohomology $qH_m(X)$ is contained in the intersection of all G-invariant algebraic cycles for all possible complex structures and actions of groups on X.

We'd like to note that since quantum rank $N(X)$ doesn't depend on any action of group G, corollary 18 says that knowledge of quantum cohomology provides us with a lower bound for dimension of G-invariant cohomology for any G. In small dimensions this gives bound for G-invariant Picard number $\rho^G = \dim H^2(X, \mathbb{Q})^G$. In particular

Lemma 26. *For any action $G : X$ we have lower bound $\dim H_a(X)^G \geqslant N(X)$.*
For any action $G : S$ of group on del Pezzo surface we have lower bound $\rho^G(S) \geqslant N(X) - 2$
For any action $G : V$ of group on Fano threefold we have lower bound $\rho^G(V) \geqslant \frac{N(X)}{2} - 1$.

In practice bound of lemma 26 is sharp in a sense that there exists some deformation and action of some group $G : X'$ with inequality becoming equality. Manin argues [1] that quantum cohomology should be promoted to quantum motive and this promotion turns *every* algebraic variety into a sort of homogeneous space (over operad of motives of moduli stacks of curves). Motivated by this insight we propose the following conjecture (which should be a "consequence of quantum Torelli"):

Conjecture 27. *For any deformation class of Fano threefolds the bounds 26 are sharp i.e. there exists some family of threefolds in this class over some base such that monodromy-invariant part of cohomology coincides with minimal quantum cohomology (after change of scalars).*

In collaboration [1] we compute minimal quantum cohomology of Fano threefolds and, in particular, verify the conjecture 27.

Quantum differential equation and its solution.

Let t be a coordinate on $\mathbf{G}_m = \operatorname{Spec} \mathbb{C}[t, t^{-1}]$ and $D = t\frac{d}{dt}$.

Definition 28. *Quantum differential equation* (QDE) is a trivial vector bundle over $\mathbf{G}_m$ with fibre H and connection

$$D\Phi = L \star \Phi \tag{29}$$

where $\Phi \in H[[t]]$.

Definition 30. Let $\mathcal{G}_X(t) = [pt] + \sum_{n \geqslant 1} \mathcal{G}_n(X) t^n$ be the unique analytic solution of 29 with initial condition $\mathcal{G}_X(0) = [pt]$. Define G-series as

$$G_X(t) = \int_{[X]} \mathcal{G}_X(t) = 1 + \sum_{n \geqslant 1} g_n(X) t^n \tag{31}$$

Define *regularized G-series* as Fourier-Laplace transform of G-series:

$$\hat{G}_X(t) = 1 + \sum_{n \geqslant 1} n! \cdot g_n(X) t^n \tag{32}$$

Definition 33. *Scalar QDE* is a differential operator of minimal degree (in D) annihilating $G_X(t)$, and *scalar RQDE* (scalar regularized QDE) is a differential operator of minimal degree annihilating $\hat{G}_X$).

Proposition 34. *Degree of scalar QDE(X) equals to quantum rank $N(X)$.*

Examples.

Let S_d be a del Pezzo surface of degree $d = 1, \ldots, 9$ which is a blowup of projective plane in $(9 - d)$ points in generic position, and $Q = \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ — a smooth quadric surface. Anticanonical linear system maps S_d to $\mathbb{P}^d$ and for $d \geqslant 3$ this map is embedding. (-1)-curves are exceptional curves of blowups and strict transforms of lines passing through 2 points of blowup, conics passing through 5 points of blowup, etc

[1] in his lecture at IHES dedicated to Grothendieck's anniversary

Example 35. Surface $S_8 = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$ has unique (-1)-curve C. Uniqueness implies C is G-invariant for any action $G : S_8$, hence space $H^2(S_8, \mathbb{Q})^G$ is two-dimensional since it contains two linearly independent elements $c_1(S_8)$ and C.

Example 36. Similarly, intersection graph for (-1)-curves on S_7 is A_3 and so the middle (-1)-curve C (one that intersects other two (-1)-curves) is G-invariant for any action $G : S_7$, and space $H^2(S_7, \mathbb{Q})^G$ contains two-dimensional space generated by K_{S_7} and C. Action of $\mathbb{Z}/2\mathbb{Z}$ on S_7 induced from line-interchanging involution on $\mathbb{P}^1 \times \mathbb{P}^1$ swaps two (-1)-curves different from C, so $\dim H^{\mathbb{Z}/2\mathbb{Z}}(S_7) = 4$.

Theorem 37 (see [2])**.** *Del Pezzo surface S_d of degree d is G-Fano $\iff$ S is $\mathbb{P}^2$, Q or $d \leqslant 6$.*

Corollary 38. *Del Pezzo surfaces S_1, S_2, S_3, S_4, S_5, S_6, $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$ are quantum minimal.*

Example 39. There are 8 deformation classes of G-Fano threefolds Y with $\dim H^2(Y, \mathbb{Q}) > 1$, 6 has index one: Y_{20} of degree 20, Y_{24} of degree 24, Y_{28} of degree 28, Y_{30} of degree 30, U_{12} and V_{12} of degree 12. and two has index equal to two: $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and $W \subset \mathbb{P}^2 \times \mathbb{P}^2$ of degree 48.

They produce 8 equations of type $D3$ [6].

Let t be a coordinate on $G_m = \operatorname{Spec} \mathbb{C}[t, t^{-1}]$ and $D = t\frac{d}{dt}$.
Scalar RQDE(Y_{20}) has the form

$$\mathcal{D}_{Y_{20}} = D^3 - t \cdot 2D(D+1)(2D+1) - t^2 \cdot 112(D+1)^3 - \\ - t^3 \cdot 184(D+1)(D+2)(2D+3) - t^4 \cdot 336(D+1)(D+2)(D+3) \tag{40}$$

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