On Apery constants of homogeneous varieties.

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Abstract. We do numerical computations of Apery constants for homogeneous varieties G/P for maximal parabolic groups P in Lie groups of type A_n, $n \leqslant 10$, B_n, C_n, D_n, $n \leqslant 7$, E_6, E_7 , E_8, F_4 and G_2. These numbers are identified to be polynomials in the values of Riemann zeta-function $\zeta(k)$ for natural arguments $k \geqslant 2$.

1. Introduction

The article is devoted to the computations of Apery numbers for the quantum differential equation of homogeneous varieties, so first we introduce these 3 notions.

Let X be a Fano variety of index r: $-K_X = rH$, and q be a coordinate on the anticanonical torus $\mathbb{Z} - K_X \otimes \mathbb{C}^* = G_m \in \mathrm{Pic}(X) \otimes \mathbb{C}^*$, and $D = q\frac{d}{dq}$ be an invariant vector field. Cohomologies $H^\bullet(X)$ are endowed with the structure of quantum multiplication $\star$, and associativity of $\star$ implies that first Dubrovin's connection given by

$$D\phi = H \star \phi \tag{1.1}$$

is flat.

If we replace in equation 1.1 quantum multiplication with the ordinary cup-product, then it's solutions are constant Lefschetz coprimitive (with respect to H) classes in $H^\bullet(X)$. Dimension μ of the space of homolorphic solutions of 1.1 is the same and equal to the number of admissible initial conditions (of the recursion on coefficients) modulo q, i.e. the rank of the kernel of cup-multiplication by H in $H^\bullet(X)$, that is the dimension of coprimitive Lefschetz cohomologies.

Solving equation 1.1 by Newton's method one obtains a matrix-valued few-step recursion reconstructing all the holomorphic solutions from these initial conditions.

Givental's theorem states that the solution $A = 1 + \sum_{n\geqslant 1} a^{(n)}q^n$ associated with the primitive class $1 \in H^0(X)$ is the I-series of the variety X (the generating function counting some rational curves of X). Choose a basis of other solutions $A_1, \ldots, A_{\mu-1}$ associated with homogeneous primitive classes of nondecreasing codimension [1]

Put $A = \sum_{n\geqslant 0} a^{(n)}t^n$ and $A_i = \sum_{n\geqslant 0} a_i^{(n)}t^n$. We call the number

$$\lim_{n\to\infty} \frac{a_i^{(n)}}{a^{(n)}}$$

i-th Apery constant after the renown work [2], where $\zeta(3)$ and $\zeta(2)$ were shown to be of that kind for some differential equations and such a presentation was used for proving the irrationality of these two numbers. If there is no choosen basis, for any coprimitive class γ one still may consider

[1]One could also consider other bases, e.g. it is often exists a base with ith element B_i determined by the condition $B_i = t^i(mod t^\mu)$. But the answer in this base looks worse. Finally one may reject to choose any basis and express everything invariantly in the dual space of primitive classes.

the solution $A_\gamma = \sum_{n\geqslant 1} a_\gamma^{(n)} q^n = Pr_0(\gamma + \sum_{n\geqslant 1} A_\gamma^{(n)} q^n)$ and the limit

$$Apery(\gamma) = \lim_{n\to\infty} \frac{a_\gamma^{(n)}}{a^{(n)}} \tag{1.2}$$

Defined in that way, *Apery* is a linear map from coprimitive cohomologies to $\mathbb{C}$. A linear map on coprimitive cohomologies is dual [2] to some (nonhomogeneous) primitive cohomology class with coefficients in $\mathbb{C}$. We name it *Apery characteristic class* $A(X) \in H^{\leqslant \dim X}(X, \mathbb{C})$.

Consider the homogeneous ring $R = \mathbb{Q}[c_1, c_2, c_3, \dots], \deg c_i = i$ and a map $ev: R \to \mathbb{C}$ sending c_1 to Euler constant C [3], and c_i to $\zeta(i)$.

The main conjecture we verify is the following

Conjecture 1.3. *Let X be any Fano variety and $\gamma \in H^\bullet(X)$ be some coprimitive with respect to $-K_X$ homogeneous cohomology class of codimension n. Consider two solutions of quantum D-module: A_0 associated with 1 and A_γ associated with γ. Then Apery number for A_γ (i.e. $\lim_{k\to\infty} \frac{a_\gamma^{(k)}}{a_0^{(k)}}$) is equal to $ev(f_\gamma)$ for some homogeneous polynomial $f_\gamma \in R^{(n)}$ of degree n.*

Actually, in our case there is no Euler constant contributions, and the conjecture seems too strong to be true - it would imply that some of differential equations studied in [1] has non-geometric origin (at least come not from quantum cohomology), because their Apery numbers does not seem to be of the kind described in the conjecture (e.g. Catalan's constant, π^3, $\pi^3\sqrt{3}$).

From the other point of view, for toric varieties X the solutions of QDE are known to be pullbacks of hypergeometric functions, coefficients of hypergeometric functions are rational functions of Γ-values, and the Taylor expansion

$$\log\Gamma(1+x) = Cx + \sum_{k\geqslant 2} \frac{\zeta(k)}{k} x^k \tag{1.4}$$

suggests all Apery constants would probably be rational functions in C and $\zeta(k)$. So whether one believes in toric degenerations or hypergeometric pullback conjecture, he would find natural to believe in 1.3. Also Apery limits like $\frac{91}{432}\zeta(3) - \frac{1}{216}\pi^3\sqrt{3}$ may appear as "square roots" or factors (convolutions with quadratic character or something) of geometric ones like $\frac{91^2}{432^2}\zeta(3)^2 - \frac{3}{216^2}\pi^6$.

This is not even the second paper (the computations of this paper were described by Golyshev 2-3 years ago) discussing the natural appearance of ζ-values in monodromies of QDEs. In case of fourfolds X the expression of monodromies in terms of $\zeta(3), \zeta(2k)$ and characteristic numbers of anticanonical section of X was given by van Straten [14], Γ-class for toric varieties appears in Iritani's work [9], and in general context in [10].

Let G be a (semi)simple Lie group, W be it's Weyl group, P be a (maximal) parabolic subgroup associated with the subset (or just one) of the simple roots of Dynkin diagram, and denote factor G/P by X. X is a homogeneous Fano variety with $\operatorname{rk}\operatorname{Pic} X$ equal to the number of chosen roots. In case when G is simple and P is maximal we have $\operatorname{Pic} X = \mathbb{Z}H$, where H is an ample generator, $K_X = -rH$.

For homogeneous varieties with small number of roots in Dynkin diagram (being more precise, with not too big total dimension of cohomologies) by the virtue of Peterson's version of Quantum

[2] One may choose between Poincare and Lefschetz dualities. We prefer the first one.

[3] $C = \lim_{n\to\infty}(\sum_{k=1}^n \frac{1}{k}) - \ln n$

Chevalley formula [4][Theorem 10.1] we explicitly compute the operator $H\star$ [4], and hence find 1.1 with all it's holomorphic solutions. Then we do a numerical computation of the ratios $\frac{a_\gamma^{(k)}}{a_0^{(k)}}$ for big k (e.g. $k = 20$ or 40 or 100), and guess the values of the corresponding Apery constants, then state some conjectures (refining 1.3) on what these numbers should be.

2. Grassmanian Gr(2,N)

Let V be the tautological bundle on Grassmanian $Gr(2, N)$, consider $H = c_1(V)$ and $c_2 = c_2(V)$. Cohomologies $H^\bullet(Gr(2, N), \mathbb{C})$ is a ring generated by H and c_2 with relations of degree $\geqslant N - 1$. So there is at least 1 primitive (with respect to H) Lefschetz cohomology class p_{2k} in every even codimension $2k$, $0 \leqslant k \leqslant \frac{N-2}{2}$. Since

$$\dim H^\bullet(Gr(2, N), \mathbb{C}) = \binom{N}{2} = \sum_{k=0}^{\frac{N-2}{2}} (2N - 3 - 4k)$$

they exhaust all the primitive classes.

$$p_0 = 1$$
$$p_2 = c_2 - \frac{c_2 \cdot c_1^{2N-6}}{c_1^{2N-4}} c_1^2$$
$$\cdots$$

The associated conjectural Apery numbers are listed in the following table, Apery numbers associated with the primitive cohomology classes of codimension $2k$ are rational multiples of $\zeta(2k) \simeq_{\mathbb{Q}^*} \pi^{2k}$.

X	μ	p_2	p_4	p_6	p_8
$Gr(2,4)$	2	0			
$Gr(2,5)$	2	$\zeta(2)$			
$Gr(2,6)$	3	$2\zeta(2)$	0		
$Gr(2,7)$	3	$3\zeta(2)$	$\frac{27}{4}\zeta(4)$		
$Gr(2,8)$	4	$4\zeta(2)$	$16\zeta(4)$	0	
$Gr(2,9)$	4	$5\zeta(2)$	$\frac{111}{4}\zeta(4)$	$\frac{675}{16}\zeta(6)$	
$Gr(2,10)$	5	$6\zeta(2)$	$42\zeta(4)$	$108\zeta(6)$	0
$Gr(2,11)$	5	$7\zeta(2)$	$\frac{235}{4}\zeta(4)$	$\frac{3229}{16}\zeta(6)$	$\frac{18375}{64}\zeta(8)$
$Gr(2,12)$	6	$8\zeta(2)$	$78\zeta(4)$	$328\zeta(6)$	$768\zeta(8)$,
$Gr(2,13)$	6	$9\zeta(2)$	$\frac{399}{4}\zeta(4)$	$\frac{7855}{16}\zeta(6)$	$\frac{96111}{64}\zeta(8)$,
$Gr(2,14)$	7	$10\zeta(2)$	$124\zeta(4)$	$695\zeta(6)$	$\frac{7664}{3}\zeta(8)$,
$Gr(2,15)$	7	$11\zeta(2)$	$\frac{603}{4}\zeta(4)$	$\frac{15113}{16}\zeta(6)$	$\frac{768085}{192}\zeta(8)$,

Remark 2.1. $Gr(2, 5)$ case is essentially Apery's recursion for $\zeta(2)$ (see remark 7.1).

[4]We used computer algebra software LiE [11] for the computations in Weyl groups. The script is avaiable at `http://www.mi.ras.ru/~galkin/work/qch.lie`, and the answer is available in [6]. We used PARI/GP computer algebra software [12] for solving the recursion and finding the linear dependencies between the answers and zeta-polynomials. Script for this routine is available at `http://www.mi.ras.ru/~galkin/work/apery.gp`.

Remark 2.2. Constants for p_2 depend linearly on N, constants for p_4 depend quadratically on N, constants for p_6 looks like they grow cubically in N. So we conjecture constants for p_{2k} is $\zeta(2k)$ times polynomial of degree k of N.

The proof for the computation of p_2 (in slightly another $\mathbb{Q}$-basis) was given recently in [7]. Let us describe a transparent generalization of this method for the all primitive p_{2k} of $Gr(2,N)$. Quantum D-module for $Gr(r,N)$ is the r'th wedge power of quantum D-module for $\mathbb{P}^{N-1}$ (solutions of QDE for $Gr(r,N)$ are $r \times r$ wronskians of the fundamental matrix of solutions for $\mathbb{P}^{N-1}$). Let N be either $2n$ or $2n+1$. Consider the deformation of quantum differential equation for $\mathbb{P}^{N-1}$:

$$(D-u_1)(D+u_1)(D-u_2)(D+u_2)\cdot\cdots\cdot(D-u_n)(D+u_n)\cdot D^{N-2n}-q \tag{2.3}$$

This equation has (at least) $2n$ formal solutions:

$$R_a = \sum_{k-a\in\mathbb{Z}_+} \frac{1}{\Gamma(k-u_1)\Gamma(k+u_1)\cdot\cdots\cdot\Gamma(k-u_n)\Gamma(k+u_n)\cdot\Gamma(k)^{N-2n}} q^k$$

for $a = u_1, -u_1, \ldots, u_n, -u_n$. Let $S_i = R'_{u_i}R_{-u_i} - R'_{-u_i}R_{u_i}$ be the wronskians. Then $S_i = \sum_{k\geqslant 0} s_i^{(k)} q^k$ for $i = 1,\ldots,n$ are n holomorphic solutions of the wedge square of the deformed equation 2.3. Using his explicit calculation for the monodromy of hypergeometric equation 2.3 and Dubrovin's theory, Golyshev computes the monodromy of $\wedge^2(2.3)$ and demonstrates *the formula of sinuses*:

$$\lim_{k\to\infty} \frac{s_i^{(k)}}{s_j^{(k)}} = \frac{\sin(2\pi u_i)}{\sin(2\pi u_j)} \tag{2.4}$$

So in the base of $S_1,\ldots,S_n$ Apery numbers are $\frac{\sin(2\pi u_i)}{\sin(2\pi u_1)}$. One then reconstructs the required Apery numbers by applying the inverse fundamental solutions matrix to this vector of sinuses, and limiting all u_i to 0.

3. Other grassmannians of type A

Let V be the tautological bundle on Grassmanian $Gr(3,N)$, consider $H = c_1(V)$, $c_2 = c_2(V)$ and $c_3 = c_3(V)$.

Cohomologies $H^\bullet(Gr(3,N),\mathbb{C})$ are generated by H, c_2 and c_3 with relations of degree $\geqslant N-2$. In particular, if $N > 7$, then 1, c_2, c_3, c_2^2 and c_2c_3 generate $H^{\leqslant 10}(X,\mathbb{Q}) = H^\bullet(X)/H^{>10}(X)$ as $\mathbb{Q}[c_1]$-module. So there is 1 primitive class in codimensions 0,2,3,4 and 5.

X	μ	p_2	p_3	p_4	p_5	$p_{\geqslant 6}$
$Gr(3,6)$	3	0	$-6\zeta(3)$			
$Gr(3,7)$	4	$\zeta(2)$	$-7\zeta(3)$	$-\frac{17}{4}\zeta(4)$		$-\frac{49}{2}\zeta(3)^2 - \frac{945}{16}\zeta(6)$
$Gr(3,8)$	5	$2\zeta(2)$	$-8\zeta(3)$	0	$-8\zeta(2)\zeta(3) - 4\zeta(5)$	$-32\zeta(3)^2 - 62\zeta(6)$
$Gr(3,9)$	8	$3\zeta(2)$	$-9\zeta(3)$	$\frac{27}{4}\zeta(4)$	$-\frac{27}{2}\zeta(2)\zeta(3) - \frac{9}{2}\zeta(5)$	$\pm(\frac{81}{2}\zeta(3)^2 + \frac{871}{16}\zeta(6)), \ldots$
$Gr(3,10)$	10	$4\zeta(2)$	$-10\zeta(3)$	$16\zeta(4)$	$-20\zeta(2)\zeta(3) - 5\zeta(5)$	$\pm(50\zeta(3)^2 + 32\zeta(6)), \ldots$
$Gr(3,11)$	13	$5\zeta(2)$	$-11\zeta(3)$	$\frac{111}{4}\zeta(4)$	$-\frac{55}{2}\zeta(2)\zeta(3) - \frac{11}{2}\zeta(5)$	$(-\frac{121}{2}\zeta(3)^2 + \frac{110}{16}\zeta(6)) \pm \frac{45}{16}\zeta(6), \ldots$

Remark 3.1. One may notice that the Apery constants of p_2 and p_4 for $Gr(3,N)$ are equal to the Apery constants of p_2, p_4 for $Gr(2,N-2)$. Why? Is it possible to make an analogous statement for p_6 (obviously one should choose another basis of two elements in $H^{12}(Gr(3,N))$ to vanish appearing $\zeta(3)^2$ terms)?

Remark 3.2. p_2 is linear of N, p_4 is quadratic of N, p_3 is linear of N, p_5 is quadratic of N.

Remark 3.3. p_5 is quadratic polynomial of N times $\zeta(2)\zeta(3)$ plus linear polynomial of N times $\zeta(5)$. Actually it is $-\frac{p_2p_3-N\zeta(5)}{2}$. This gives a suggestion on a method of separating e.g. $\zeta(4)$ and $\zeta(2)^2$ in p_4 — $\zeta(4)$ term should be only linear and $\zeta(2)^2$ is quadratic in N. Similarly the coefficient at $\zeta(3)^2$ is quadratic in N (and in the choosen basis p_6'th $\zeta(3)^2$-part is $\frac{p_3^2}{2}$).

For $Gr(4,N)$ we still do have a unique primitive class of codimension 5.

X	μ	p_2	p_3	p_4	p'_4	p_5	$p_{\geqslant 6}$
$Gr(4,8)$	8	0	$-8\zeta(3)$	$-6\zeta(4)$	0	none	$32\zeta(3)^2+50\zeta(6)$ twice and 0_8
$Gr(4,9)$	12	$\zeta(2)$	$-9\zeta(3)$	$\frac{21}{4}\zeta(4)$	$\zeta(4)$	$-\frac{9}{2}(\zeta(2)\zeta(3)+\zeta(5))$	$(\frac{81}{2}\zeta(3)^2+\frac{117}{4}\zeta(6))\pm\frac{159}{16}\zeta(6)$, ...
$Gr(4,10)$	18	$2\zeta(2)$	$-10\zeta(3)$	$-2\zeta(4)$	$2\zeta(4)$	$-10\zeta(2)\zeta(3)-5\zeta(5)$	$50\zeta(3)^2+31\zeta(6)$, $50\zeta(3)^2$, 0_6, ...
$Gr(4,11)$	24	$3\zeta(2)$	$-11\zeta(3)$	$\frac{15}{4}\zeta(4)$	$3\zeta(4)$	$-\frac{33}{2}\zeta(2)\zeta(3)-\frac{11}{2}\zeta(5)$	$(\frac{121}{2}\zeta(3)^2+\frac{35}{2}\zeta(6))\pm\frac{197}{16}\zeta(6)$, $\frac{27}{16}\zeta(6)$,...

Remark 3.4. Apery of p_3 for $Gr(3,N)$ and $Gr(4,N)$ coincide. Apery of p_2 for $Gr(4,N)$ is equal to Apery of p_2 for $Gr(3,N-2)$ and Apery of p_2 for $Gr(2,N-4)$.

For $Gr(5,10)$ we have 20 Lefschetz blocks, they correspond to 20 solutions, and hence 19 Apery constants. Some of them vanish, while some other coincide (because solutions differ only by some character).

X	μ	p_2	p_3	p_4	p'_4	p_5	p'_5
$Gr(5,10)$	20	0	$-10\zeta(3)$	$-6\zeta(4)$	0	$10\zeta(5)$	$-10\zeta(5)$
$Gr(5,11)$	32	$\zeta(2)$	$-11\zeta(3)$	$-\frac{21}{4}\zeta(4)$	$\zeta(4)$	$11(\zeta(5)-\zeta(2)\zeta(3))$	$-11\zeta(5)$

4. B,C,D cases

The picture for other 3 series of classical groups is similar.

For $1\leqslant k\leqslant n$ let $D(n,k)$ denote homogeneous space of isotropic (with respect to nondegenerate quadratic form) k-dimensional linear spaces in $2n$-dimensional vector space. $D(n,k)=OGr(k,2n)=G/P$ where G is $Spin(2n)$, and maximal parabolic subgroup $P\subset G$ corresponds to k'th simple root counting from left to right. Similarly define $B(n,k)=OGr(k,2n+1)$ and $C(n,k)=SGr(k,2n)$.

X	μ	Apery numbers
$B(3,2)$	2	$-2\zeta(2)$.
$B(4,2)$	3	$\zeta(2)$, $-\frac{41}{2}\zeta(4)$.
$B(4,3)$	3	$-4\zeta(2)$, $-4\zeta(3)$.
$B(4,4)$	2	$2\zeta(3)$.
$B(5,2)$	4	$3\zeta(2)$, $\frac{3}{2}\zeta(4)-\frac{1191}{8}\zeta(6)$.
$B(5,3)$	8	0_2, $-8\zeta(3)$, $-24\zeta(4)$, $20\zeta(5)$, $\frac{64}{3}\zeta(3)^2+\frac{80}{3}\zeta(6)$, $32\zeta(3)\zeta(4)+\frac{232}{3}\zeta(7)$, $\frac{256}{21}\zeta(3)^3+\frac{320}{7}\zeta(3)\zeta(6)-\frac{480}{7}\zeta(4)\zeta(5)-\frac{1000}{21}\zeta(9)$.
$B(5,4)$	8	$-6\zeta(2)$, $-6\zeta(3)$, $-45\zeta(4)$, $9\zeta(2)\zeta(3)+21\zeta(5)$, $15\zeta(3)^2+\frac{1141}{24}\zeta(6)$, $56\zeta(2)\zeta(5)+30\zeta(3)\zeta(4)+52\zeta(7)$, $\frac{266}{5}\zeta(3)^3-\frac{171}{5}\zeta(2)\zeta(7)-\frac{222}{5}\zeta(3)\zeta(6)-\frac{263}{5}\zeta(4)\zeta(5)+\frac{136}{5}\zeta(9)$.

X	μ	Apery numbers
$B(5,5)$	3	$4\zeta(3)$, $20\zeta(5)$.
$B(6,2)$	5	$5\zeta(2)$, $\frac{87}{4}\zeta(4)$, $-\frac{485}{8}\zeta(6)$, $-\frac{35073}{32}\zeta(8)$.
$B(6,3)$	12	$2\zeta(2)$, $-6\zeta(3)$, $-12\zeta(4)$, $-12\zeta(2)\zeta(3)+18\zeta(5)$, $-36\zeta(3)^2-146\zeta(6)$, $36\zeta(3)^2+2\zeta(6)$, $24\zeta(2)\zeta(5)+24\zeta(3)\zeta(4)+76\zeta(7)$, $\frac{360\zeta(3)^2\zeta(2)-1080\zeta(3)\zeta(5)+1176\zeta(8)}{11}$, $803\zeta(3)^3-528\zeta(2)\zeta(7)+318\zeta(3)\zeta(6)-244\zeta(4)\zeta(5)-35\zeta(9)$, $75\zeta(3)^3-336\zeta(2)\zeta(7)-395\zeta(3)\zeta(6)-22\zeta(4)\zeta(5)-70\zeta(9)$,...
$B(6,4)$	18	$-1\zeta(2)$, $-10\zeta(3)$, $-\frac{17}{4}\zeta(4)$, $-14\zeta(4)$, $5\zeta(2)\zeta(3)+19\zeta(5)$, $50\zeta(3)^2+317\zeta(6)$, $-50\zeta(3)^2-\frac{4135}{8}\zeta(6)$,
$B(6,5)$	14	$-8\zeta(2)$, $-8\zeta(3)$, $-84\zeta(4)$, $64\zeta(2)\zeta(3)+16\zeta(5)$, $-64\zeta(2)\zeta(3)$, $\frac{80}{3}\zeta(3)^2+24\zeta(6)$, $110\zeta(2)\zeta(5)+\frac{49}{2}\zeta(3)\zeta(4)+\frac{101}{2}\zeta(7)$,
$B(6,6)$	5	$6\zeta(3)$, $18\zeta(5)$, $-18\zeta(3)^2-60\zeta(6)$, $36\zeta(3)^3+360\zeta(3)\zeta(6)+332\zeta(9)$
$B(7,2)$	6	$7\zeta(2)$, $\frac{211}{4}\zeta(4)$, $\frac{1733}{8}\zeta(6)$, $-\frac{76699}{96}\zeta(8)$, $-\frac{5368203}{640}\zeta(10)$.
$B(7,7)$	8	$8\zeta(3)$, $16\zeta(5)$, $-30\zeta(3)^2-60\zeta(6)$, $-112\zeta(7)$, $\frac{256}{3}\zeta(3)^3+480\zeta(3)\zeta(6)+\frac{992}{3}\zeta(9)$, ...

Remark 4.1. $B(4,4)$ case is essentially Apery's recursion for $\zeta(3)$.

X	μ	Apery numbers
$C(3,2)$	2	$2\zeta(2)$.
$C(3,3)$	2	$\frac{7}{2}\zeta(3)$.
$C(4,2)$	3	$4\zeta(2)$, $16\zeta(4)$.
$C(4,3)$	4	$\zeta(2)$, $-9\zeta(3)$, $-\frac{9}{2}(\zeta(2)\zeta(3)+\zeta(5))$.
$C(4,4)$	2	$4\zeta(3)$.
$C(5,2)$	4	$6\zeta(2)$, $42\zeta(4)$, $108\zeta(6)$.
$C(5,3)$	8	$3\zeta(2)$, $-11\zeta(3)$, $\frac{27}{4}\zeta(4)$, $-\frac{33}{2}\zeta(2)\zeta(3)-\frac{11}{2}\zeta(5)$, $\frac{242}{3}\zeta(3)^2+\frac{2383}{48}\zeta(6)$, $-11\zeta(2)\zeta(5)-\frac{99}{4}\zeta(3)\zeta(4)-\frac{11}{3}\zeta(7)$, $108\zeta(3)^3-38\zeta(2)\zeta(7)+\frac{309}{4}\zeta(3)\zeta(6)-\frac{41}{4}\zeta(4)\zeta(5)+36\zeta(9)$
$C(5,4)$	8	0_2, $-10\zeta(3)$, $30\zeta(4)$, $-5\zeta(5)$, $\frac{250}{3}\zeta(3)^2+\frac{175}{3}\zeta(6)$, $-\frac{100}{3}\zeta(3)\zeta(4)-\frac{10}{9}\zeta(7)$, $\frac{2500}{21}\zeta(3)^3+250\zeta(3)\zeta(6)-\frac{150}{7}\zeta(4)\zeta(5)-\frac{10}{21}\zeta(9)$.
$C(5,5)$	3	$\frac{9}{2}\zeta(3)$, $-\frac{21}{2}\zeta(5)$.
$C(6,2)$	5	$8\zeta(2)$, $78\zeta(4)$, $328\zeta(6)$, $768\zeta(8)$.
$C(6,3)$	12	$5\zeta(2)$, $-13\zeta(3)$, $\frac{111}{4}\zeta(4)$, $-\frac{65}{2}\zeta(2)\zeta(3)-\frac{13}{2}\zeta(5)$, $-\frac{169}{2}\zeta(3)^2+\frac{155}{16}\zeta(6)$, $\frac{169}{2}\zeta(3)^2+\frac{65}{2}\zeta(6)$,
$C(6,6)$	4	$\zeta(3)$, $-11\zeta(5)$, $-25\zeta(3)^2-\frac{15}{2}\zeta(6)$, $\frac{500}{3}\zeta(3)^3+150\zeta(3)\zeta(6)-\frac{131}{3}\zeta(9)$.
$C(7,2)$	6	$10\zeta(2)$, $124\zeta(4)$, $695\zeta(6)$, $\frac{7664}{3}\zeta(8)$, $5760\zeta(10)$.
$C(7,7)$	8	$\frac{11}{2}\zeta(3)$, $-\frac{23}{2}\zeta(5)$, $-\frac{121}{4}\zeta(3)^2-\frac{15}{2}\zeta(6)$, $\frac{71}{2}\zeta(7)$, $\frac{1331}{6}\zeta(3)^3+165\zeta(3)\zeta(6)-\frac{263}{6}\zeta(9)$, $\frac{781}{12}\zeta(3)\zeta(7)-\frac{529}{12}\zeta(5)^2-\frac{63}{2}\zeta(10)$,...

Remark 4.2. One may notice that Apery numbers for $C(2,n)=SGr(2,2n)$ coincide with Apery numbers of $Gr(2,2n)$ except the last 0. The reason for this coincidence is that $SGr(2,2n)$ is a quadratic hyperplane section of $Gr(2,2n)$, so by quantum Lefschetz (7.1) has almost the same Apery numbers.

Remark 4.3. For general k spaces $OGr(k,N)$ and $SGr(k,N)$ are sections of ample vector bundles over $Gr(k,N)$ (symmetric and wedge square of tautological bundle). Is it possible to formulate a generalization of quantum Lefshetz principle explaining the relations between Apery numbers of $OGr(k,N)$, $SGr(k,N)$ and $Gr(k,N)$?

X	μ	Apery numbers
$D(4,2)$	4	$0, 0, -24\zeta(4)$.
$D(5,2)$	5	$2\zeta(2)$, 0, $-12\zeta(4)$, $-144\zeta(6)$.
$D(5,3)$	9	$-\zeta(2)$, $-\zeta(2)$, $-6\zeta(3)$, 0_4, $-\frac{45}{2}\zeta(4)$, $3\zeta(2)\zeta(3)+21\zeta(5)$, 0_5, $12\zeta(3)^2+\frac{275}{24}\zeta(6)$.
$D(5,4)$	2	$2\zeta(3)$.
$D(6,2)$	6	$4\zeta(2)$, $10\zeta(4)$, $10\zeta(4)$, $-124\zeta(6)$, $-960\zeta(8)$.
$D(6,3)$	14	$\zeta(2)$, $-5\zeta(3)$, $-5\zeta(3)$, $-\frac{41}{2}\zeta(4)$, 0, $-5\zeta(2)\zeta(3)+19\zeta(5)$, $\frac{25}{2}\zeta(3)^2+\frac{953}{16}\zeta(6)$, $\frac{25}{2}\zeta(3)^2-\frac{937}{16}\zeta(6)$, 0,
$D(6,5)$	3	$4\zeta(3)$, $20\zeta(5)$.
$D(7,2)$	7	$6\zeta(2)$, $36\zeta(4)$, 0, $50\zeta(6)$, $-1072\zeta(8)$, $-6912\zeta(10)$.
$D(7,6)$	5	$6\zeta(3)$, $18\zeta(5)$, $-18\zeta(3)^2-60\zeta(6)$, $36\zeta(3)^3+360\zeta(3)\zeta(6)+332\zeta(9)$.

Remark 4.4. $D(N,N-1)$ is isomorphic to $B(N-1,N-1)$, so in the case $D(6,5)$ we again have Apery's recurrence for $\zeta(3)$ here.

5. Exceptional cases - E, F, G

We provide computations of Apery constants only for a few of 23 exceptional homogeneous varieties, those with not too big spaces of cohomologies.

X	μ	Apery numbers
$E(6,6)$	3	$6\zeta(4)$, 0_8.
$E(6,2)$	6	0_3, $18\zeta(4)$, $90\zeta(6)$, 0_7, $-3456\zeta(10)$.
$E(7,7)$	3	$-24\zeta(5)$, $168\zeta(9)$.
$E(8,8)$	11	$120\zeta(6)$, $-1512\zeta(10)$, ... (of degrees 12, 16, 18, 22, 28).
$F(4,1)$	2	$21\zeta(4)$.
$F(4,3)$	8	$-4\zeta(2)$, 0_3, $-2\zeta(4)$, $-24\zeta(5)$, $-246\zeta(6)$, $32\zeta(2)\zeta(5)+60\zeta(7)$, $2160\zeta(2)\zeta(7)-144\zeta(4)\zeta(5)$.
$F(4,4)$	2	$6\zeta(4)$.

Remark 5.1. There are two roots in the root system of G_2, taking factor by the parabolic subgroup associated with the smaller one we get a projective space, so later by G_2/P we denote the 5-dimensional factor by another maximal parabolic subgroup. There is no literal Apery constants for G_2/P since this variety is minimal, so the only primitive cohomology class is 1, altough one may seek for almost solutions of quantum differential equation (strictly speaking Apery himself also considered such solutions). In [7] Golyshev considers this problem for Fano threefold V_{18} (i.e. a section of G_2/P by two hyperplanes) and using Beukers argument [3] and modularity of the quantum D-module for V_{18} shows that Apery number is equal to $L_{\sqrt{-3}}(3)$

6. Varieties with greater rank of Picard group, non-Calabi-Yau and Euler constant

One may consider the same question for varieties X with higher Picard group. Canonicaly we should put $H=-K_X$, but if we like, we could choose any $H \in \mathrm{Pic}(X)$.

Even for such simple spaces as products of projective spaces one immidiately calculates some non-trivial Apery constants.

X	μ	Apery numbers

$\mathbb{P}^2 \times \mathbb{P}^2$	3	0_1, $6\zeta(2)$.
$\mathbb{P}^2 \times \mathbb{P}^3$	3	0_1, $\frac{14}{3}\zeta(2)$.

In all these cases Apery numbers corresponding to all primitive divisors vanish. Van Straten's calculation [14] relates monodromies of QDE for Fano fourfold X not to Chern numbers of the Fano, but to Chern numbers of it's anticanonical Calabi-Yau hyperplane section Y. Probably C-factors shouls correspond to c_1-factors in the Chern number, and since for Calaby-Yau $c_1(Y) = 0$ we observe Euler constant is not involved. So one should consider something non-anticanonical.

Let's test the case $H = \mathcal{O}(1,1)$ on $\mathbb{P}^2 \times \mathbb{P}^3$. Being exact, we restrict D-module to subtorus corresponding to H, and consider operator of quantum multiplication by H on it (subtorus associated with H is invariant with respect to vector field associated with H).

(X, H)	μ	Apery numbers
$(\mathbb{P}^2 \times \mathbb{P}^3, \mathcal{O}(1,1))$	3	$-C$, $\frac{C^2+7\zeta(2)}{2}$.

7. Irrationality, special varieties and further speculations

First of all let us note that both differential equations considered by Apery for the proofs of irrationality of $\zeta(2)$ and $\zeta(3)$ are essentially appeared in our computations as quantum differential equations of homogeneous varieties $Gr(2,5)$ and $OGr(5,10) = D(5,4)$ (and isomorphic $OGr(4,9) = B(4,4)$). By essentially we mean the following proposition — Apery constants are invariant with respect to taking hyperplane section if the corresponding primitive classes survive:

Proposition 7.1. *Let X be a subcanonically embedded smooth Fano variety*[5] *of index $r > 1$ i.e. X is embedded to the projective space by a linear system $|H|$, and $-K_X = rH$. Consider a general hyperplane section Y — a subcanonically embedded smooth Fano variety of index $r-1$. There is a restriction map $\gamma \to \gamma \cap H$ from cohomologies of X to cohomologies of Y and by Hard Lefschetz theorem except possible of intermideate codimension all primitive classes of Y are restricted primitive classes of X. Consider a homogeneous primitive class of nonintermediate codimension $\gamma \in H^\bullet(X)$. Then Apery numbers for γ calculated from QDE of X and Y coincide.*

Proof. By the quantum Lefschetz theorem of Givental-Kim-Gathmann we have a relation between the I-series (solution of 1.1 associated with $1 \in H^\bullet$) of X and Y: e.g. if $r > 2$ and $\mathrm{Pic}(X) = \mathbb{Z}H$ and $H_2(X, \mathbb{Z}) = \mathbb{Z}\beta$ then $d'th$ coefficient of $I - series$ of X should be multiplied by $\prod_{i=0}^{dH\beta}(H+i)$, if $r \leqslant 2$ one should also do a change of coordinate. One may show the similar relation between solutions of 1.1 associated with γ and $\gamma_{|Y}$: either directly repeating the arguments of original proof, or by Frobenius method of solving differential equation. So the limit of the ratio is the same. □
One may rephrase the previous proposition in the following way

Proposition 7.2. *Apery class is functorial with respect to hyperplane sections.*

Proposition 7.2 is slightly stronger then 7.1: indeed, the intermediate primitive classes of X vanish restricted on Y, but also it states that "parasitic" intermediate primitive classes of Y has Apery constant equal to 0. Following notations of [8] let's call all smooth varieties related to each other by hyperplane section or deformation *a strain*, and if Y is a hyperplane section of X let's call X *an unsection* of Y; if Y has no unsections we call it *a progenitor* of the strain. The stability

[5] One may state this proposition in higher generality, but we are going to use it for homogeneous spaces, and as stated it will be enough.

of Apery class is quite of the same nature as the stability of spectra in the strain described in [8]. Propositions 7.1 and 7.2 suggest to consider some kind of stable Apery class on the infinite hyperplane unsection. Such a stable framework of Gromov–Witten invariants was constructed by Przyalkowski for the case of quantum minimal Fano varieties in [13], using only Kontsevich-Manin axioms. The next proposition shows that literaly this construction gives nothing from our perspective

Proposition 7.3. *If a Fano variety X is quantum minimal then all Apery constants vanish i.e. Apery class A is equal to* 1.

Proof. It is a trivial consequence of the definition of quantum minimality — since all primitive classes except 1 are quantum orthogonal to $\mathbb{C}[K_X]$ the operator of quantum multiplication by K_X restricted to nonmaximal Lefschetz blocks coincides with the cup-product, in particular it is nilpotent, so the associated solutions A_γ of QDE are polynomial in q i.e. their coefficients $a_\gamma^{(k)}$ vanish for $k >> 0$, hence the Apery number is 0. □

Conjecture 7.4. *The converse to 7.3 statement is true as well.*

So for our purposes the framework of [13] should be generalized taking into account the structure of Lefschetz decomposition. Another obstacle is geometrical nonliftability of varieties to higher dimensions — one can show both Grassmanian $Gr(2,5)$ (and any other Grassmanian except projective spaces and quadrics) and $OGr(5,10)$ are progenitors of their strains, i.e. cannot be represented as a hyperplane section of any nonsingular variety, this follows e.g. from the fact that these varieties are selfdual, but of course they are hyperplane sections of their cones. We insist that the quantum recursions for the progenitors $Gr(2,5)$ and $OGr(5,10)$ are the most natural in the strain, in particular in both cases we consider two exact solutions of the recursion, and in Apery's case one considers an almost solution with polynomial error term — because for the linear sections of dimension $\leqslant 3$ ($\leqslant 5$) the second Lefschetz block vanishes. One may ask a natural question whether any of the experimentaly or theoreticaly calculated Apery numbers (and their representations as the limits of the ratios of coefficients of two solutions of the recurrence) may be proven to be irrational by Apery's argument. At least we know it works in two cases of $Gr(2,5)$ and $OGr(5,10)$. Remind that for irrationality of $\alpha = \zeta(2)$ or $\alpha = \zeta(3)$ one shows that $(\alpha - \frac{a_\gamma}{q_n})$ is smaller then $\frac{1}{q_n}$, so we are interested in the sign of $\lim \log(|\alpha - \frac{a_\gamma}{q_n}|) - \log(q_n)$ (or equivalently in the sign of

$$\lim \log\log(|\alpha - \frac{a_\gamma}{q_n}|) - \log\log q_n. \tag{7.5}$$

There were many attempts to find any other recurencies with this sign being negative, and most of them failed to the best of our knowledge. The quantum recursions we considered in this article is not an exception (we calculated convergence speed 7.5 numerically for $n \geqslant 20$). For example the convergence speed for $\zeta(2)$ approximation from $Gr(2,N)$ decreases as N grows, and is suitable only in the case of $Gr(2,5)$. So we come to the question: what is so special about $Gr(2,5)$ and $OGr(5,10)$? One immideately reminds the famous theorem of Ein (see e.g. [15])

Theorem 7.6. *Let $X \subset \mathbb{P}^N$ be a smooth nondegenerate irreducible n-dimensional variety, such that X has the same dimension as it's projectively dual X^*. Assume $N \geqslant \frac{3n}{2}$. Then X is either a hypersurface, or one of*

(1) *a Segre variety $\mathbb{P}^1 \times \mathbb{P}^r \subset \mathbb{P}^{2r+1}$*

(2) *the Plucker embedding* $Gr(2,5) \subset \mathbb{P}^9$
(3) $OGr(5,10)$

Three last cases are selfdual: $X \simeq X^*$*.*

Remark 7.7. For 7.6 we have the coincidence of the coherent and topological cohomologies

$$N + 1 = \dim H^0(X, \mathcal{O}(H)) = \dim H^\bullet(X) \tag{7.8}$$

In all 3 cases there are exactly two Lefschetz blocks, the codimensions of the grading of second Lefschetz block are corr. 1, 2 and 3.

Remark 7.9. Apery number for $\mathbb{P}^1 \times \mathbb{P}^r$ should approximate some multiple of C, but for $r = 1, 2, 3$ it is 0. As pointed out in section 6 we haven't got any natural approximations for Euler constant in anticanonical Landau–Ginzburg model. From the other point of view, the variety $\mathbb{P}^1 \times \mathbb{P}^r$ in the statement of the theorem 7.6 is not (sub)anticanonically embedded, but embedded by the linear system $\mathcal{O}(1,1)$. Calculations of 6 are what we expect to be the quantum recursion for X embedded by $\mathcal{O}(1,1)$, they indeed approximate C, but the speed of convergence is too slow. Either our guess is not correct (or not working here) or Landau–Ginzburg corresponding to the linear system $|\mathcal{O}(1,1)|$ is something else.

So the theorem 7.6 suggests the irrationality of Apery approximations are ruled by either self-duality or extremal defectiveness of the progenitor. Varieties 7.6 are related by the famous construction: let X be one of them, choose any point $p \in X$ (they are homogeneous so all points are equivalent), then take an intersection of X with it's tangent space $Y = X \cap T_pX$. Then Y is a cone over the previous one:

$$T_pGr(2,5) \cap Gr(2,5) = Cone(\mathbb{P}^1 \times \mathbb{P}^2) \tag{7.10}$$

$$T_pOGr(5,10) \cap OGr(5,10) = Cone(Gr(2,5)) \tag{7.11}$$

In that way $OGr(5,10)$ can be "lifted" one step further to Cartan variety $E(6,6) = E(6,1)$:

$$T_pE(6,6) \cap E(6,6) = Cone(OGr(5,10)).$$

$E(6,6)$ is one of the four famous Severi varieties (or more general class of Scorza varieties) classified by Fyodor Zak in [15]:

Theorem 7.12. *Let* $X \subset \mathbb{P}^{N=\frac{3n+4}{2}}$ *be* n*-dimensional Severi variety i.e.* X *can be isomorphically projected to* $\mathbb{P}^{N-1}$*. Then* X *is projectively equivalent to one of*

(1) *the Veronese surface* $v_2(\mathbb{P}^2) \subset \mathbb{P}^5$
(2) *the Segre fourfold* $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$
(3) *the Grassmanian* $Gr(2,6) \subset \mathbb{P}^{14}$
(4) *the Cartan variety* $E(6,6) \subset \mathbb{P}^{26}$

Remark 7.13. Apart from the first case that should be correctly interpreted (e.g. taking symmetric square of D-module for $\mathbb{P}^2$), in the other 3 cases coincidence 7.8 holds (this is general fact for the closures of highest weight orbits of algebraic groups). The Lefschetz decompositions now consist of 3 blocks — first associated with 1, next one, and one block of length 1 in intermediate codimension. The last block has Apery number equal to 0.

Neither of Severi varieties provides us with a fast enough approximation, but the speeds of convergence for them seem to be better then for arbitrary varieties. So it may be possible that these speeds are related with the defect of the variety (it is also supported by the fact that for Grassmanians defect decreases when N grows).

From the other perspective, when there are more then two Lefschetz blocks in the decomposition one may try to use the simultaneous Apery-type approximations of a tuple of zeta-polynomials as in the works of Zudilin.

We would like to note that the recursion 1.1 contains more then one approximation of every Apery number appearing. Clearly speaking, in the definition of Apery numbers we considered the limit of the ratios of fundamental terms i.e. projections of two solutions A_0 and A_γ to $H^0(X)$. It is natural to ask if we get anything from considering the limits of ratios of the other coordinates. Our experiments support the following

Conjecture 7.14. *$A_\gamma^{(k)}$ is approximately equal to $Apery(\gamma) \cdot A_0^{(k)}$ as $k \to \infty$.*

One may divide $A_\gamma^{(k)}$ by $A_0^{(k)}$ in the nilpotent ring of $H^\bullet(X)$ and state the limit of such ratio exists and is equal to $Apery(\gamma) \in H^0(X)$. For homogeneous γ_2 the ratio $\frac{(A_\gamma^{(k)},\gamma_2)}{(A_\gamma^{(k)},1)}$ grows as $k^{\mathrm{codim}\,\gamma_2}$ and the cooordinates in the same Lefshetz block are linearly dependant.

Finally let us provide some speculations explaining why the described behaviour is natural and also why zeta-values should appear. Assume for simplicity that the matrix of quantum multiplication by H has degree 1 in q (it is often the case for homogeneous varieties). Let M_0 be the operator of cup-product by H and M_1 be the degree 1 coefficient of quantum product by H. Then the quantum recursion is one-step:

$$A^{(n)} = \frac{1}{n - M_0} M_1 A^{n-1} = \frac{1}{n}(1 + \frac{M_0}{n} + \frac{M_0^2}{n^2} + \dots) \cdot M_1 A^{n-1} \tag{7.15}$$

Assume M_0 and M_1 commutes (actually, this is never true in our case). Then

$$A^{(l)} = \frac{1}{l!} \prod_{n=1}^{l} (1 + \frac{M_0}{n} + \frac{M_0^2}{n^2} + \dots) \cdot M_1^l A^{(0)}$$

Put

$$N_l = \prod_{n=1}^{l} (1 + \frac{M_0}{n} + \frac{M_0^2}{n^2} + \dots) = \exp(\sum_{n=1}^{l} \sum_{k \geqslant 1} \frac{1}{k} \frac{M_0^k}{n^k}).$$

Up to normalization $\lim N_l$ is $\Gamma(1 + M_0)$. Assume further that largest (by absolute value) eigenvalue α of M_1 has the unique eigenvector β of multiplicity 1. Then $A^{(l)}$ is approximately equal to

$$C(A^{(0)}, \beta) \cdot \frac{1}{l!} \cdot \alpha^l \cdot N_l \beta$$

Since M_0 and M_1 doesn't commute there are additional terms from the commutators of $\Gamma(1+M_0)$ and M_1.

Acknowledges. The database [6] of quantum cohomologies of homogeneous varieties was prepared jointly with V. Golyshev during our interest in the spectra of Fano varieties (inspired by [5]), and the interest to the Apery constant appeared after V. Golyshev's deresonance computation [7] of the first Apery constant for $Gr(2, N)$.

Author thanks Duco van Straten for the fruitful discussions and the invitation to the SFB/TR 45 program at Johannes Gutenberg Universitaet.

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