

QFT II/QFT

homework III

- submission via the U Tokyo LMS. We request that the file name includes the problem number(s), such as II-1***.pdf or ***-IV-2-IX-1.jpeg. The LMS will show who had submitted the file (student ID and name), so the file name will not have to contain your name or ID number.
- course credit guaranteed by working on $1\#[A] + 1.5\#[B] + 2\#[C] + 4\#[D] + 8\#[E] \geq 9$ problems.
- A sample solution is available as a pdf file for problems with the mark \star .

1. Final State Phase Space [B]

- (a) **cross section 1:** Let us consider a process of a pair of relativistic (approximately massless) particles collide at the center of mass energy \sqrt{s} , and scatter into a pair of non-identical particles. Suppose that this pair of final state particles have the same mass, m (imagine $e^+ + e^- \rightarrow \mu^+ + \mu^-$, for example). Verify that the cross section is

$$d\sigma = \frac{d\varphi d(\cos\theta)}{2\pi} \frac{1}{32\pi s} \beta |\mathcal{M}|^2, \quad (1)$$

where (φ, θ) are the azimuthal angle and the scattering angle at the center of mass frame. [i.e., the initial state particles come along the z -axis, and the final state particles are moving out to $\vec{p} = |\vec{p}|(\sin\theta \cos\varphi, \sin\theta \sin\varphi, \cos\theta)$] β is the velocity $|\vec{p}|/E$ of the final state particles.

(The primary lesson to take out from this problem is that the cross section has an overall dependence β/s due to the kinematics (final state phase space). To see how powerful this understanding is, visit the web page

https://pdg.lbl.gov/2020/reviews/contents_sports.html

and click on the last entry, “Plots of cross sections and related quantities”, and look at Figure 52.2.)

- (b) **cross section 2:** Suppose now that the final state particles are (approximately) massless, for simplicity. Verify, then, that the expression above can also be written as

$$d\sigma \simeq \frac{d\varphi dt}{2\pi} \frac{1}{16\pi s^2} |\mathcal{M}|^2, \quad (2)$$

where t is the Mandelstam variable of the 2-body \rightarrow 2-body scattering. (it is an option to skip this problem)

- (c) **decay rate 1** Let us think of a particle at rest with mass M decaying to a pair of non-identical (approximately) massless particles. Verify, then, that

$$d\Gamma \simeq \frac{d^2\Omega}{4\pi} \frac{1}{16\pi M} |\mathcal{M}|^2. \quad (3)$$

- (d) **decay rate 2** Suppose that a particle at rest with mass M decays to a pair of a particle with mass $M' = M - \Delta m$ and an approximately massless particle. Verify, when $0 < \Delta m \ll M$, that

$$d\Gamma \simeq \frac{d^2\Omega}{4\pi} \frac{\beta}{8\pi M} |\mathcal{M}|^2, \quad (4)$$

where β is the velocity of the massive particle in the final state. [A lesson: when a scattering/decay process is barely allowed kinematically, the measure over the final state phase space yields a positive power of $\beta \ll 1$, and hence the cross section/decay rate is suppressed.]

- (e) **decay rate 3** Let us now think of a particle at rest with mass M decaying to three non-identical particles (e.g., $\mu^- \rightarrow e^- + \bar{\nu}_e + \nu_\mu$). The final state phase space is of 5-dimensions, after exploiting the energy-momentum conservation. When the decaying particle is a scalar, or with a spin but without a polarization, however, there is $\text{SO}(3)$ symmetry of space (\mathbb{R}^3) rotation acting on the final state phase, and the integral over the 5-dimensional phase space is reduced to one on a 2-dimensional space. We can take, for example, (E_1, E_2) —the energy of two particles (e.g., ν_μ and $\bar{\nu}_e$) as the coordinates of the 2-dimensional space, and the integral is in the form of

$$d\Gamma = \frac{1}{(2\pi)^3 8M} |\mathcal{M}|^2 dE_1 dE_2. \quad (5)$$

Now, assume that all the three particles in the final states are approximately massless, and work out the region in the (E_1, E_2) space that is kinematically possible; the area should be $M^2/8$. So, when the matrix element $\mathcal{M}(E_1, E_2)$ does not have a particular structure (such as singularity), the total decay rate is something like

$$\Gamma \sim \frac{M}{(8\pi)^3} \langle |\mathcal{M}|^2 \rangle. \quad (6)$$

[The absence of structure in $|\mathcal{M}|^2$ is an appropriate assumption for $\mu \rightarrow e\nu\bar{\nu}$, but not quite for $t \rightarrow W + b + g$.]

(f) (not meant as a homework problem) Suppose that the matrix element \mathcal{M} for a 2-body decay in (3) is approximately M times the matrix element \mathcal{M} for a 3-body decay in (6). The 3-body decay rate is smaller than the 2-body decay rate by a factor $1/32\pi^2$ then.

(g) (not meant as a homework problem) If you are interested in learning more on kinematics (such as Dalits plot), you might think of visiting the web page referred to earlier, and download a review article “Kinematics (rev.)” from that page. Derivation of (5) is also found there.

2. Fermi Surface, Hole Excitation, Friedel Oscillation [B or C] ★

Consider a Lagrangian (and corresponding Hamiltonian) of non-relativistic electron.

$$\mathcal{L} = \psi^\dagger \left[i\partial_t + \frac{1}{2m} \vec{\partial} \cdot \vec{\partial} \right] \psi, \quad (7)$$

$$H = \int d^3x \psi^\dagger \left[-\frac{1}{2m} \vec{\partial} \cdot \vec{\partial} \right] \psi. \quad (8)$$

ψ is a 2-component spinor field (i.e., in the 2-dimensional representation of the space rotation SO(3) symmetry group). With the creation and annihilation operators of states with a given momentum, the field operators are written as

$$\psi(\vec{x}, t) = \int \frac{d^3p}{(2\pi)^3} \sum_r \xi_r a_{\vec{p},r} e^{-iE_{\vec{p}}t + i\vec{p}\cdot\vec{x}}, \quad \{a_{\vec{p},r}, a_{\vec{q},s}^\dagger\} = \delta_{r,s} (2\pi)^3 \delta^3(\vec{p} - \vec{q}). \quad (9)$$

Here, $E_{\vec{p}} = |\vec{p}|^2/(2m)$, and ξ_r is a dimensionless 2-component spinor; one can take $\xi_{r=\uparrow} = (1, 0)^T$ and $\xi_{r=\downarrow} = (0, 1)^T$, for example.

[OK to skip (a-c) if trivial for you. [B] until (e), [C] to go beyond.]

(a) Verify that

$$H' \equiv H - \epsilon_F \int d^3x \psi^\dagger \psi = \int \frac{d^3p}{(2\pi)^3} \sum_r (E_{\vec{p}} - \epsilon_F) a_{\vec{p},r}^\dagger a_{\vec{p},r} + \text{const.} \quad (10)$$

(b) Verify that $\psi^\dagger \psi$ is the $\mu = 0$ component of the Noether current J^μ corresponding to the electron-number symmetry (phase rotation of ψ) in (7). [remark: Thus, $H' = H - \epsilon_F N_e$, where N_e is the electron number. ϵ_F is regarded as the chemical potential (Fermi energy).]

(c) (not a homework problem) Note that this system can be described by a Lagrangian

$$\mathcal{L}' = \psi^\dagger \left[i\partial_t + \frac{1}{2m} \vec{\partial} \cdot \vec{\partial} + \epsilon_F \right] \psi. \quad (11)$$

(d) Let us define $b_{\vec{p},r} \equiv a_{-\vec{p},r}^\dagger$ for $|\vec{p}|$ below the Fermi momentum p_F ; $\epsilon_F = E_{p_F}$. Rewrite H' in terms of $a_{\vec{p},r}$ with $|\vec{p}| \geq p_F$ and $b_{\vec{p},r}$ with $|\vec{p}| < p_F$, and show that the state with all the levels below the Fermi surface filled is the ground state of this Hamiltonian H' . It will be easy to see that $b_{\vec{p}}$ and $b_{\vec{p}}^\dagger$ are the annihilation and creation operators of a hole.

(e) Use the expression (9) and the observation in (d) to verify that

$$\begin{aligned} G(\vec{x}, t; \vec{y}, t') &:= \langle 0|T \left\{ \psi(\vec{x}, t) \psi^\dagger(\vec{y}, t') \right\} |0\rangle \\ &= \mathbf{1}_{2 \times 2} \int \frac{d^3 p}{(2\pi)^3} \left(\Theta(t - t') \Theta(|p| - p_F) e^{-i(E_p - \epsilon_F)(t - t') + i\vec{p} \cdot (\vec{x} - \vec{y})} \right. \\ &\quad \left. - \Theta(t' - t) \Theta(p_F - |p|) e^{-i(E_p - \epsilon_F)(t - t') - i\vec{p} \cdot (\vec{x} - \vec{y})} \right). \end{aligned} \quad (12)$$

On the other hand, the path integral formulation allows one to conclude right away from the action (11) that

$$G(\vec{x}, t; \vec{y}, t') = \mathbf{1}_{2 \times 2} \int d\omega \frac{d^3 p}{(2\pi)^3} \frac{i e^{-i\omega(t - t') + i\vec{p} \cdot (\vec{x} - \vec{y})}}{\omega - E_p + \epsilon_F + i \text{sgn}(E_p - \epsilon_F) \epsilon}. \quad (13)$$

Verify that the former expression (in the operator formalism) is reproduced from the latter (in the path integral formalism) by integrating over ω .

(f) Verify that

$$\begin{aligned} &\int_{-\infty}^{+\infty} dt' \langle 0|T \left\{ \psi^\dagger \Gamma_x \psi(\vec{x}, t) \psi^\dagger \Gamma_y \psi(\vec{y}, t') \right\} |0\rangle \\ &= \int_{-\infty}^{+\infty} dt'' \langle 0|T \left\{ \psi^\dagger \Gamma_x \psi(\vec{r}, t'') \psi^\dagger \Gamma_y \psi(\vec{0}, 0) \right\} |0\rangle, \end{aligned} \quad (14)$$

where $\vec{r} := \vec{x} - \vec{y}$ and $t'' = t - t'$, is equal to

$$(14) = -i \text{tr}_{2 \times 2} [\Gamma_x \Gamma_y] \int \frac{d^3 q}{(2\pi)^3} e^{i\vec{q} \cdot \vec{r}} f(q), \quad (15)$$

with

$$f(q) = 2 \int \frac{d^3k}{(2\pi)^3} \frac{\Theta(|\vec{q} - \vec{k}| - p_F) \Theta(p_F - |\vec{k}|)}{(\omega_{\vec{q}-\vec{k}} - \omega_{\vec{k}})i}, \quad (16)$$

$$= \int \frac{d^3k}{(2\pi)^3} \frac{\Theta(p_F - |\vec{k}|) - \Theta(p_F - |\vec{q} - \vec{k}|)}{(2m)^{-1}[q^2 - 2\vec{k} \cdot \vec{q}]}, \quad (17)$$

$$= \int \frac{d^3k}{(2\pi)^3} \frac{2\Theta(p_F - |\vec{k}|)}{(2m)^{-1}[q^2 - 2\vec{k} \cdot \vec{q}]}, \quad (18)$$

$$= \dots = \frac{mk_F}{(2\pi)^2} \left\{ 1 + \frac{4k_F^2 - q^2}{4k_F q} \ln \left| \frac{2k_F + q}{2k_F - q} \right| \right\}. \quad (19)$$

Here, Γ_x and Γ_y are 2x2 matrices yet to be specified at this moment. This $f(q)$ is known as Lindhard function of a free electron gas (or of conduction band electrons). [Once you verify the equivalence between (14) and (15, 16), the rest—in between (16) and (19)—is not so much about QFT, but just a math exercise. It is an option to just accept that (16)=(19), and move on.]

- (g) Suppose that the conduction band electrons (approximated by a free electron gas system) have interactions with other quantum mechanical degrees of freedom

$$\Delta H = \sum_i g_i (\psi^\dagger \Gamma_i \psi)(\vec{r}_i, t) s_i, \quad (20)$$

where s_i is an operator acting on a quantum mechanical degree of freedom localized at \vec{r}_i , Γ_i a constant dimensionless valued 2×2 matrix, and g_i is a coupling constant. The combination $g_i s_i$ has dimension of [energy \times volume].

By setting

$$\langle 0|T \left\{ \exp \left[-i \int dt \Delta H \right] \right\} |0\rangle_{\psi, \psi^\dagger \text{ system}} =: T \exp \left[-i \int dt \Delta H_{\text{eff}} \right], \quad (21)$$

verify that the effective Hamiltonian contains

$$\Delta H_{\text{eff}} \supset - \sum_{i < j} J_{ij} s_i s_j; \quad J_{ij} := \text{tr}_{2 \times 2} [\Gamma_i \Gamma_j] \int \frac{d^3q}{(2\pi)^3} f(q) e^{i\vec{q} \cdot \vec{r}_{ij}}. \quad (22)$$

[This is known as RKKY interaction. To learn more, look up references with a keyword “Friedel oscillation” and “RKKY interaction”.]

3. Time-ordered Exponential for a System with Explicit Time Dependence [B]

In the interaction picture, the difference between the time evolution due to the full Hamiltonian H and the evolution due to the free-part Hamiltonian H_0 is characterized by an operator

$$U(t_1, t_2; t_*) \equiv e^{iH_0(t_1-t_*)} e^{-iH(t_1-t_2)} e^{-iH_0(t_2-t_*)}. \quad (23)$$

This expression is appropriate, however, only when H and H_0 do not have explicit time dependence. But, one may be interested in studying a quantum field theory system under time-varying external field (expanding universe also works as a time-varying external field).

- (a) Find an expression for $U(t_1, t_2; t_*)$ that is appropriate even when H and H_0 have explicit time dependence.
- (b) Show that such $U(t_1, t_2; t_*)$ is still given by a time-ordered exponential of $V_I(t)$. (The definition of $V_I(t)$ should also be stated clearly.)

4. In/out State Normalization and Källén–Lehmann Spectral Representation [B]

A coefficient Z appearing in the Källén–Lehmann spectral representation,¹

$$\int d^4(y-x) \langle \Omega | T \{ \phi(y) \phi(x) \} | \Omega \rangle e^{ik \cdot (y-x)} = \frac{i Z}{k^2 - m^2 + i\epsilon} + (\text{other singularities}), \quad (24)$$

also sets the normalization of in/out-states,

$$|\vec{p}\rangle^{\text{in}} = \frac{1}{\sqrt{Z} \langle \Omega | 0 \rangle_{T_-}} e^{-iH(t_*-T_-)} e^{iH_0(t_*-T_-)} a_{\vec{p}}^\dagger | 0 \rangle \sqrt{2E_{\vec{p}}}, \quad (25)$$

$$|\vec{p}\rangle^{\text{out}} = \frac{1}{\sqrt{Z} \langle \Omega | 0 \rangle_{T_+}} e^{-iH(t_*-T_+)} e^{iH_0(t_*-T_+)} a_{\vec{p}}^\dagger | 0 \rangle \sqrt{2E_{\vec{p}}}. \quad (26)$$

In the following, let us see why this is the case.

- (a) First, show i) that an integral

$$\int_{-\infty}^{T_-} dx^0 \int d^3x \phi_I^*(x) | 0 \rangle e^{-iq \cdot x} \quad (27)$$

¹To learn more, see [PS] section 7.2 or [W-I] Chap. 10.

converges if q^0 is (real positive) $\times (1 + i\epsilon)$, and ii) that a one-particle state in the free theory

$$|\vec{p}\rangle^{\text{free}} = a_{\vec{p}}^\dagger |0\rangle \sqrt{2E_{\vec{p}}} \quad (28)$$

is proportional to the residue of a pole $i/(q^0 - E_{\vec{q}})$ in the complex q^0 plane. Under an understanding that q^0 is always chosen to be real positive (plus a little positive imaginary part), one can also say that this is a residue of a pole at $i/(q^2 - m^2)$.

(b) Second, by using the result above, show that the inner product of

$$\left[\frac{1}{\langle \Omega | 0 \rangle_{T_+}} e^{-iH(t_* - T_+)} e^{iH_0(t_* - T_+)} a_{\vec{p}}^\dagger |0\rangle \sqrt{2E_{\vec{p}}} \right]^\dagger = \left[\frac{\sqrt{2E_{\vec{q}}}}{\langle 0 | \Omega \rangle_{T_+}} \langle 0 | a_{\vec{q}} e^{iH_0(T_+ - t_*)} e^{-iH(T_+ - t_*)} \right]$$

and

$$\left[\frac{1}{\langle \Omega | 0 \rangle_{T_-}} e^{-iH(t_* - T_-)} e^{iH_0(t_* - T_-)} a_{\vec{q}}^\dagger |0\rangle \sqrt{2E_{\vec{q}}} \right]$$

is the same as the residue of

$$\int d^4y \int d^4x e^{ip \cdot y} e^{-iq \cdot x} \langle \Omega | T \{ \phi(y) \phi^*(x) \} | \Omega \rangle \quad (29)$$

at the singularity

$$\frac{i}{(p^2 - m^2)} \frac{i}{(q^2 - m^2)}. \quad (30)$$

As before, assume that p^0 and q^0 are both real and positive with an infinitesimally small positive imaginary part. It is O.K. to assume for now that interactions switch off before T_- and after T_+ .

In order to show this, one can use

$$\langle 0(T_+) | \Omega(T_+) \rangle \langle \Omega(T_-) | 0(T_-) \rangle = \langle 0 | T \left\{ \exp \left(-i \int dt' V_I(t') \right) \right\} | 0 \rangle, \quad (31)$$

which is equivalent to $\langle \Omega | \Omega \rangle = 1$.

(c) Finally, show that the residue of Eq. (29) at (30) is

$$(2\pi)^3 \delta^3(\vec{p} - \vec{q}) (2E_{\vec{p}}) Z, \quad (32)$$

by using Eq. (24). This completes the proof of the normalization (25, 26).

(d) (not a report problem) Normalization of multi-particle in-states / out-states is set as follows. First think of a theory where particles in a group 1 interact within the group, and those in another group 2 also do so within group 2, but not with particles in the other group. Then the in-state with one particle (species n_1) in the group 1 and one particle (species n_2) in the group 2 should be set as follows:

$$|\{n_1, \vec{p}_1; n_2, \vec{p}_2\}\rangle^{\text{in}} = \frac{1}{\sqrt{Z_1}\sqrt{Z_2} [\langle \Omega_1 | \otimes \langle \Omega_2 | 0 \rangle]_{T_-}} e^{-iH(t_*-T_-)} e^{iH_0(t_*-T_-)} |\{n_1, \vec{p}_1; n_2, \vec{p}_2\}\rangle^{\text{free}}. \quad (33)$$

In theories where all the particles interact, therefore,

$$|\{n_i, \vec{p}_i\}_{i \in I}\rangle^{\text{in}} = \frac{1}{\langle \Omega | 0 \rangle_{T_-} \prod_{i \in I} \sqrt{Z_i}} e^{-iH(t_*-T_-)} e^{iH_0(t_*-T_-)} |\{n_i, \vec{p}_i\}_{i \in I}\rangle^{\text{free}}. \quad (34)$$