

QFT II/QFT

homework V

- submission via the U Tokyo LMS. We request that the file name includes the problem number(s), such as II-1***.pdf or ***-IV-2-IX-1.jpeg. The LMS will show who had submitted the file (student ID and name), so the file name will not have to contain your name or ID number.
- A sample solution is available as a pdf file for problems with the mark \star .

1. Quantization of constrained systems [D] ([C]) \star

Read some appropriate materials and learn quantization of constrained systems using Dirac bracket. Weinberg's text book volume 1, Chapter 7.6 (and the appendix to Chap.7) is an example. Prof. Fukushima's lecture note for the QFT I course (summer semester) is another example. (or look up online) ... and how does that work in the following systems in practice?

Because the QFT I lecture covered the case (a) and (b), undergraduate students should skip (a, b) and proceed to (c, d). If a graduate student wants to work on this by starting with (a), (b) and stops there, this problem V-1 is category [C]; by going beyond (a,b), this is category [D].

- (a) Dirac fermion with the Lagrangian $\mathcal{L} = \bar{\Psi}(i\gamma^\mu D_\mu - m)\Psi$. Introductory QFT textbooks often choose to deal with the Hermitian conjugate Ψ^\dagger of the field Ψ as the canonical conjugate field of Ψ , instead of dealing with Ψ^\dagger as a field independent of Ψ (so both Ψ and Ψ^\dagger could have their own canonical conjugate fields (as in the case of a QFT with a complex scalar field)). What if we choose to deal with both Ψ and Ψ^\dagger in the Dirac Lagrangian as independent field, and apply the Dirac bracket procedure for quantization?
- (b) photon $\mathcal{L} = -(1/4)F_{\mu\nu}F^{\mu\nu}$.
- (c) Chern–Simons action $\int d^3x \mathcal{L} = \kappa \int d^3x \epsilon^{\lambda\mu\nu} A_\lambda (\partial_\mu A_\nu)$ in 2+1-dimensional space-time; $\epsilon^{\mu\nu\lambda}$ is the totally anti-symmetric tensor with $\epsilon^{012} = +1$, and κ is a non-zero constant.

[If you have even more time to kill, discuss what the Hilbert space is like when the two-dimensional space is compact and has $2g$ independent homological 1-cycles.]

- (d) chiral scalar field $S = \int d^2x(\partial_\mu\phi)(\partial^\mu\phi)$ on 1+1 dimensional space-time, with a constraint $\partial_0\phi = -\partial_1\phi$. (such a field may arise as a chiral edge mode on the (1+1)-dimensional boundary of a condensed matter (2+1)-dimensional system, and is also used as a part of the formulation of Heterotic string theory)
- (e) any other systems of your interest.

2. Spin dependence in the s -channel scattering [B] ★

- (a) We studied the $e^- + e^+ \rightarrow \mu^- + \mu^+$ cross section during the class. What if the final state μ^- and μ^+ particles were spin-0 (scalar) particles? Now we consider a process of s -channel production of a pair of spin-0 particles ($\tilde{\mu}^-$ and $\tilde{\mu}^+$), whose Lagrangian is $\mathcal{L} = (D_\mu\Phi)^\dagger(D^\mu\Phi) - M^2|\Phi|^2$ using a complex scalar field Φ , instead of $\mathcal{L} = \bar{\Psi}_{(\mu)}(i\gamma^\mu D_\mu - M)\Psi_{(\mu)}$ using a 4-component spinor field $\Psi_{(\mu)}$. The bilinear part is $\mathcal{L}_0 = (\partial_\mu\Phi)^\dagger(\partial^\mu\Phi) - M^2|\Phi|^2$, whose canonical quantization leads to

$$\Phi_I = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(a_{\vec{p}} e^{-ip \cdot x} + b_{\vec{p}}^\dagger e^{ip \cdot x} \right), \quad (1)$$

$$\Phi_I^\dagger = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(a_{\vec{p}}^\dagger e^{ip \cdot x} + b_{\vec{p}} e^{-ip \cdot x} \right), \quad (2)$$

where $a_{\vec{p}}$ and $a_{\vec{p}}^\dagger$ are annihilation and creation operators of a spin-0 $\tilde{\mu}^-$ particle, and $b_{\vec{p}}$ and $b_{\vec{p}}^\dagger$ those of its anti-particle (spin-0 $\tilde{\mu}^+$ particle). The remnant is

$$\mathcal{L}_{\text{int}} = ieQ_{(\mu)}A_\mu[(\partial^\mu\Phi^\dagger)\Phi - \Phi^\dagger(\partial^\mu\Phi)] + (eQ_{(\mu)})^2A_\mu A^\mu|\Phi|^2. \quad (3)$$

Now, to the $e^- + e^+ \rightarrow \tilde{\mu}^- + \tilde{\mu}^+$ process, the s -channel photon exchange diagram as in Figure 1 is the only contribution. Here is a question. Do you understand that the $A_\mu\tilde{\mu}^-\tilde{\mu}^+$ vertex as in the Figure contributes a factor $-ieQ_{(\mu)}(p_3 - p_4)^\mu$, instead of the factor $-ieQ_{(\mu)}\gamma^\mu$ in the $A_\mu\mu^-\mu^+$ vertex? [skip, if trivial]

- (b) Verify that the scattering amplitude is given by

$$\begin{aligned} i\mathcal{M} (e_r^-(\vec{p}_1) + e_s^+(\vec{p}_2) \rightarrow \tilde{\mu}^-(\vec{p}_3) + \tilde{\mu}^+(\vec{p}_4)) \\ = \frac{i(Q_{(\mu)}Q_{(e)}e^2)}{(p_1 + p_2)^2 + i\epsilon} (\bar{v}_s(\vec{p}_2)\gamma^\mu u_r(\vec{p}_1))(p_3 - p_4)_\mu. \end{aligned} \quad (4)$$

- (c) Verify that the average of the |matrix element|² over the initial state spin is given by

$$\frac{1}{4} \sum_{r,s} |\mathcal{M}|^2 = \frac{(Q_{(\mu)}Q_{(e)}e^2)^2}{s^2} (+4m_{(e)}^2|\vec{p}_3|^2 - 8(\vec{p}_1 \cdot \vec{p}_3)^2 + 4|\vec{p}_3|^2(s/2 - m_{(e)}^2)) \quad (5)$$

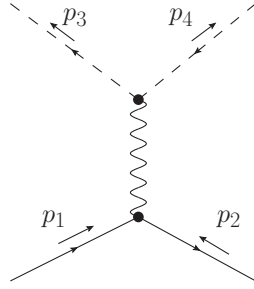


Figure 1: The Feynman diagram that contributes to the $e^-(p_1) + e^+(p_2) \rightarrow \tilde{\mu}^-(p_3) + \tilde{\mu}^+(p_4)$ scattering.

in the center of mass frame ($\vec{p}_2 = -\vec{p}_1$ and $\vec{p}_4 = -\vec{p}_3$).

- (d) We now ignore the electron mass; $m_{(e)} \sim 0.5$ MeV is much smaller than the muon mass $M_{(\mu)} \sim 106$ MeV anyway. Verify, when the center of mass energy of the $e^- + e^+$ collision is just high enough for a $\tilde{\mu}^- + \tilde{\mu}^+$ pair creation ($|\vec{p}_1| \simeq E_{\vec{p}_1} \simeq M_{(\mu)}$ but $E_{\vec{p}_1} > M_{(\mu)}$), that

$$\frac{1}{4} \sum_{r,s} |\mathcal{M}|^2 \simeq \frac{(Q_{(\mu)} Q_{(e)} e^2)^2}{2} \beta_{(\mu)}^2 \sin^2 \theta, \quad (6)$$

where $\beta_{(\mu)} := |\vec{p}_3|/M_{(\mu)}$ is the velocity of the produced $\tilde{\mu}^\pm$, and θ the scattering angle in the center of mass frame ($\vec{p}_3 \cdot \vec{p}_1 =: |\vec{p}_3| |\vec{p}_1| \cos \theta$).

- (e) Verify in the high-energy limit ($|\vec{p}_1| \gg M_{(\mu)}$) that

$$\frac{1}{4} \sum_{r,s} |\mathcal{M}|^2 \simeq \frac{(Q_{(\mu)} Q_{(e)} e^2)^2}{2} \sin^2 \theta. \quad (7)$$

- (f) (not as a part of the homework problem) The nearly massless initial state pair $e^- + e^+$ turns into a virtual photon with polarization transverse to the axis of the $e^- + e^+$ collision (lecture note for Week 05 and 06). This virtual photon couples to the p -wave relative wavefunction of the final state $\tilde{\mu}^- + \tilde{\mu}^+$ pair. This is why the pair of spin-0 particles are more likely to come out in the direction transverse ($\sin^2 \theta \sim 1$ than ~ 0) to the axis of collision. The pair creation processes of spin-1/2 particles and spin-0 particles have different θ -dependence in their differential cross section (cf. Week 05 lecture). This can be used to determine the spin of pair-produced particles experimentally.

3. Forward-backward Asymmetry [B] ★

- (a) Consider an s -channel process of $e^-(p_1) + e^+(p_2) \rightarrow V^* \rightarrow f^-(p_3) + f^+(p_4)$ going through a vector field V to a pair of spin-1/2 fermions $f^- + f^+$. This process is similar to the QED process $e^- + e^+ \rightarrow \gamma^* \rightarrow \mu^- + \mu^+$, but now we think of a theory where the vector field V_μ couples to the 4-component spinor fields for e^\mp and f^\mp by

$$\mathcal{L}_{\text{int}} = -g_L^{(i)} \bar{\Psi}_i \gamma^\mu V_\mu \left(\frac{1 - \gamma_5}{2} \right) \Psi_i - g_R^{(i)} \bar{\Psi}_i \gamma^\mu V_\mu \left(\frac{1 + \gamma_5}{2} \right) \Psi_i, \quad (8)$$

where $i = e, f$, and

$$\gamma_5 := \begin{pmatrix} -\mathbf{1}_{2 \times 2} & \\ & \mathbf{1}_{2 \times 2} \end{pmatrix}, \quad (9)$$

so $P_L = (1 - \gamma_5)/2$ and $P_R = (1 + \gamma_5)/2$ retain only the left-handed and right-handed spinor components, respectively.

Compute i) $|\mathcal{M}(e_\downarrow^- + e_\uparrow^+ \rightarrow f_\downarrow^- + f_\uparrow^+)|^2$, and ii) $|\mathcal{M}(e_\downarrow^- + e_\uparrow^+ \rightarrow f_\uparrow^- + f_\downarrow^+)|^2$ in the relativistic limit ($|\vec{p}_{1,2,3,4}| \simeq E$ in the center of mass frame and $m_{(e)}$ and $m_{(f)}$ negligible when compared with E). Here, the states $|n^\pm, \vec{p}, \uparrow\rangle$ and $|n^\pm, \vec{p}, \downarrow\rangle$ for $n = e, f$ are characterized by

$$\vec{p} \cdot \vec{s} |n^\pm, \vec{p}, \uparrow\rangle = +\frac{E}{2} |n^\pm, \vec{p}, \uparrow\rangle, \quad \vec{p} \cdot \vec{s} |n^\pm, \vec{p}, \downarrow\rangle = -\frac{E}{2} |n^\pm, \vec{p}, \downarrow\rangle; \quad (10)$$

\vec{s} is the spin operator.

- (b) Draw a sketchy graph of those $|\mathcal{M}|^2$'s as a function of $\cos \theta \in [-1, +1]$; $\vec{p}_1 \cdot \vec{p}_3 =: |\vec{p}_1| |\vec{p}_3| \cos \theta$.
- (c) Even if the initial state e^\pm are not polarized, and we do not detect the spin of the final state f^\pm , we may still define the forward-backward asymmetry

$$A_{FB} = \frac{\sigma(e^- e^+ \rightarrow f^- f^+ \mid \theta \leq \pi/2) - \sigma(e^- e^+ \rightarrow f^- f^+ \mid \theta \geq \pi/2)}{\sigma(e^- e^+ \rightarrow f^- f^+ \mid \theta \leq \pi/2) + \sigma(e^- e^+ \rightarrow f^- f^+ \mid \theta \geq \pi/2)}, \quad (11)$$

if it is possible to measure the charge of the final state particles (and distinguish f^- from f^+) experimentally ($A_{FB} > 0$ means that f^- comes out more often in the forward direction of the e^- beam than in the backward direction of the e^- beam).

Verify that A_{FB} is non-zero if $g_L^{(e)} \neq g_R^{(e)}$ and $g_L^{(f)} \neq g_R^{(f)}$.

(d) (this is not a problem) The coupling of the Z -boson in the Standard Model has this property indeed. The experimental data of $e^- + e^+ \rightarrow \mu^- + \mu^+$ shown in the Week 05–06 lectures clearly have a signal of the forward-backward asymmetry.¹ Further reading: Phys. Rept. **271** (1996) 181–266 “Experimental tests of the standard model in $e^+e^- \rightarrow f\bar{f}$ at the Z resonance” or arXiv:hep-ex/0509008 “Precision Electroweak Measurements on the Z Resonance.”

¹We should not use $|\mathcal{M}_{\text{photon}}|^2 + |\mathcal{M}_Z|^2$, but $|\mathcal{M}_{\text{photon}} + \mathcal{M}_Z|^2$ in computing the cross section, in fact. So, the standard model prediction for A_{FB} is not as simple a formula as what is obtained in part (c).