QFT II/QFT homework IX

- submission via the U Tokyo LMS. We request that the file name includes the problem number(s), such as II-1***.pdf or ****-IV-2-IX-1.jpeg. The LMS will show who had submitted the file (student ID and name), so the file name will not have to contain your name or ID number.
- A sample solution is available as a pdf file for problems with the mark $\star$.

1. **M1, E2 transitions etc.** [C] $\star$
   Electric dipole emission (E1 transition) is not the only possible mechanisms of transitions between atomic energy eigenstates. Explore more about those higher order effects, following your intellectual curiosity. References include ....
   - Landau Lifshitz vol 4 *Quantum Electrodynamics*, §45–50,
   - TAKAYANAGI, Kazuo *Genshi-bunshi Butsuri-gaku* (Asakura Publ. Co) written in Japanese, §4.4.2–4.4.5
   - look up online

2. **Positronium Decay** [B (or C)]
   Let us work out how to use Bethe–Salpeter wavefunction to compute the decay rate of a positronium (a bound state of a pair of e^-e^+) to two photons. Here, we need to note that each photon carries energy that is approximately m_e (in the rest frame of the initial bound state). The photon momenta, or derivatives acting on a photon field in the Lagrangian, is therefore not smaller than m_e. So, $\vec{\partial}/m_e$-expansion is not particularly useful in computing the matrix element for the decay rate. For this computation, it is better to use the frame of four-component spinor where

$$\gamma^0 = \begin{pmatrix} \mathbf{1} & \\ & -\mathbf{1} \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} & \vec{\tau} \\ -\vec{\tau} & \end{pmatrix}. \tag{1}$$

   The Bethe–Salpeter wavefunction $\chi(p)$ in (here, $p^\mu := (p_1 - p_2)^\mu/2 =: (\omega, \vec{p})$)

$$\langle \Omega | T \left\{ \Psi(p_1)\overline{\Psi}(p_2) \right\} | s_{\rm tot}, s_z; n, l, m \rangle = (2\pi)^4 \delta^4(p_1 + p_2 - p_{\rm CM}) \left[\chi_{nls;s_z m}(p) \right]_{4\times 4} \tag{2}$$

is a 4×4 matrix valued function, and is approximately given by

$$[\chi_{nls;s_z m}(\omega,\vec{p})]_{4\times4} \simeq \begin{pmatrix} \mathbf{1}_{2\times2} \\ \frac{\vec{p}\cdot\vec{\tau}}{2m_e} \end{pmatrix} [P(s_{\rm tot}, s_z)]_{2\times2} \left(-\frac{\vec{p}\cdot\vec{\tau}}{2m_e},\ -\mathbf{1}_{2\times2}\right) \chi_{nlm}(\omega,\vec{p}), \tag{3}$$

$$[P(s_{\rm tot}, s_z)]_{2\times2} = \begin{cases} \mathbf{1}_{2\times2} & s_{\rm tot} = 0 \\ \tau^3 & s_{\rm tot} = 1,\ s_z = 0, \\ (\tau^1 \pm i\tau^2)/\sqrt{2} & s_{\rm tot} = 1,\ s_z = \pm 1 \end{cases} \tag{4}$$

$$\int \frac{d\omega}{2\pi} \chi_{nlm}(\omega,\vec{p}) \simeq \sqrt{4m_e}\,\psi^{\rm NRQM}_{nlm}(\vec{p}), \tag{5}$$

using the Fourier transform of the wavefunction of a state $|n,l,m\rangle$ (with the redueced mass $m_e/2$) in the non-relativistic quantum mechanics (that is, $\psi^{\rm NRQM}_{nlm}(\vec{p})$). In the rest of this problem, we set $\vec{p}_{\rm CM} = \vec{0}$.

(a) Verify that the matrix element of positronium $\to \gamma + \gamma$ is given by

$$i\mathcal{M} \simeq \int \frac{d\omega}{2\pi} \int \frac{d^3\vec{p}}{(2\pi)^3} (-ieQ_e)^2\ \epsilon^*_\mu(\vec{k})\epsilon^*_\nu(-\vec{k}) \tag{6}$$
$$\mathrm{tr}_{4\times4}\left[\left(\frac{\gamma^\nu i[\omega\gamma^0 - (\vec{p}-\vec{k})^i\gamma^i + m_e]\gamma^\mu}{\omega^2 - (\vec{p}-\vec{k})^2 - m_e^2} + \frac{\gamma^\mu i[\omega\gamma^0 - (\vec{p}+\vec{k})^i\gamma^i + m_e]\gamma^\nu}{\omega^2 - (\vec{p}+\vec{k})^2 - m_e^2}\right)[\chi_{4\times4}]\right],$$

if this expression is not trivial for you.

(b) Because $|\vec{k}| = m_e + (\Delta E)/2 \approx \mathcal{O}(m_e)$, while $\vec{p}$ is typically $\mathcal{O}(m_e\alpha)$ and ω even less, it makes sense to drop all of ω and $\vec{p}$ (and retain only m_e and $\vec{k}$) from the vertex-and-propagator $(\cdots + \cdots)$ part in the expression above. You will then notice that $d\omega$ integral can be carried out, and χ turns into $\psi^{\rm NRQM}$. Now, carry out the rest of the computation to find the decay rate of the positronium $(n,l,m,s_{\rm tot},s_z) = (1,0,0;0,0)$ state. [It is not as important to get the O(1) coefficient precisely as to get the right power of m_e and α.] You will also be able to confirm that the two outgoing photons have oppsite angular momentum.

(c) Peskin–Schroeder Problem 5.4 (at the end of chapter 5) contains more information. If you are interested, you might think of exploring more. (category [C] then)