QFT II/QFT homework X

- submission via the U Tokyo LMS. We request that the file name includes the problem number(s), such as II-1***.pdf or ****-IV-2-IX-1.jpeg. The LMS will show who had submitted the file (student ID and name), so the file name will not have to contain your name or ID number.
- A sample solution is available as a pdf file for problems with the mark ⋆.

1. **Partial wave decomposition at work** [B] ⋆

   Let us get the feeling how the partial wave decomposition works in practice, using the results of perturbative computations of 2body to 2body scattering amplitudes.

   (a) We begin with the easiest example. Let us think of a 2-body to 2-body scattering in the s-channel, where a pari of scalar particles $\Phi^-(p_1)$ and $\Phi^+(p_2)$ coupled to a photon annihilates in pair and produce another pair of scalar particles $\Phi'^-(p_3)$ and $\Phi'^+(p_4)$. For simplicity, we only deal with the case where the center of mass energy is much higher than their rest mass (so that the mass parameters are negligible). The scattering amplitude is

$$\mathcal{M} = (-e^2 Q_\Phi Q_{\Phi'})\frac{(p_1-p_2)\cdot(p_3-p_4)}{s} \simeq (-e^2 Q_\Phi Q_{\Phi'})\frac{u-t}{2s} \simeq (-e^2 Q_\Phi Q_{\Phi'})\frac{\cos\theta}{2}, \tag{1}$$

   where θ is the scattering angle in the center of mass frame. Verify, by fitting the result above into the following expansion,

$$\frac{\mathcal{M}}{2(4\pi)^2} \simeq \mathcal{M}_{\rm red} = \sum_{\ell=0}^{\infty} Y_{\ell,m}(\hat{\mathbf{p}}_3)\,[\mathcal{M}_\ell(s)]\,(Y_{\ell,m}(\hat{\mathbf{p}}_1))^{\rm cc}, \tag{2}$$

$$Y_{\ell,m=0}(\hat{n}) = P_\ell(\cos\theta)\sqrt{\frac{2\ell+1}{4\pi}}, \tag{3}$$

   that only the $\ell = 1$ partial wave is non-zero in this scattering, and that

$$\mathcal{M}_{\ell=1}(s) \simeq \frac{-\alpha(Q_\Phi Q_{\Phi'})}{12}. \tag{4}$$

   [So, in this example, $\mathcal{M}_{\ell=1}$ turns out to be independent of the center of mass energy $\sqrt{s}$, at this tree level calculation. The S-matrix in this $\ell = 1$ partial wave is $S_{\ell=1} \simeq 1 + i(-\alpha QQ')/12 \simeq e^{-i\alpha QQ'/12}$, while $S_{\ell\neq 1} = 1$ in all other partial waves.]

(b) (If you are also interested in working on this...) Let us now consider a little more complicated case, where the initial state is not a pair of scalar $\Phi^- + \Phi^+$, but a pair of spin-1/2 fermions, $e^-(p_1) + e^+(p_2)$. We still consider the case where the final state is a pair of scalars $\Phi'^-(p_3) + \Phi'^+(p_4)$. We know that the scattering amplitude (at the center of mass frame, in the relativisitic limit, $\hat{\mathbf{p}}_1 = -\hat{\mathbf{p}}_2 = \hat{e}_z$) is given by

$$\mathcal{M} = (e^2 Q_e Q_{\Phi'}) \operatorname*{tr}_{2\times 2}\left[\begin{pmatrix} 0 & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & \end{pmatrix}\left(\xi_{e^-} \otimes \xi^\dagger_{e^+}\right)\right] \tag{5}$$

where θ and ϕ indicate the direction of the momentum $\hat{\mathbf{p}}_3$ after the scattering (in the center of mass frame). A 2×2 matrix

$$\xi_{e^-} \otimes \xi^\dagger_{e^+} = \begin{pmatrix} \frac{s^0_0 - s^1_0}{\sqrt{2}} & s^1_+ \\ -s^1_- & \frac{s^0_0 + s^1_0}{\sqrt{2}} \end{pmatrix} \tag{6}$$

is a spin wavefunction; basis elements $|s^z_1, s^z_2\rangle$ correspond to

$$|1/2, 1/2\rangle \to \xi \circ \xi^\dagger = \begin{pmatrix} 0 & 1 \\ & \end{pmatrix}, \quad |1/2, -1/2\rangle \to \xi \circ \xi^\dagger = \begin{pmatrix} -1 & 0 \\ & \end{pmatrix},$$
$$|-1/2, 1/2\rangle \to \xi \circ \xi^\dagger = \begin{pmatrix} & \\ 0 & 1 \end{pmatrix}, \quad |-1/2, -1/2\rangle \to \xi \circ \xi^\dagger = \begin{pmatrix} & \\ -1 & 0 \end{pmatrix}.$$

We wish to consider the partial wave decomposition of this amplitude now.

Instead of the decomposition (2) for a 2-to-2 scattering between two spin-0 particles, we need to use

$$\begin{aligned}\mathcal{M}_{\text{red}} = & \sum_{j, j_z} \sum_{\ell, m, s3z, s4z} C_{s3,s4,\ell}(j, j_z; s_{3z}, s_{4z}, m)\ (Y_{\ell,m}(\theta,\phi))\ [\mathcal{M}_j(s)] \\ & \qquad \sum_{\ell', m', s1z, s2z} C_{s1,s2,\ell'}(j, j_z; s^z_1, s^z_2, m')\ (Y_{\ell',m'}(\theta',\phi'))^{\text{cc}} \\ & \qquad\qquad |s^z_3, s^z_4\rangle\langle s^z_1, s^z_2|, \end{aligned} \tag{7}$$

where $C_{s1,s2,\ell}(j, j_z; s_{1z}, s_{2z}, m)$ is the Clebsch–Gordan coefficient describing the irreducible decomposition $(\text{spin} s_1) \otimes (\text{spin} s_2) \otimes (\text{spin} \ell) \simeq \cdots \oplus (\text{spin} j) \oplus \cdots$. The partial wave amplitude $[\mathcal{M}_j]$ for a given total angular momentum j is not just a complex number (for a given center of mass energy $\sqrt{s}$) but a matrix, because there may be multiple ways to add spins s_1, s_2 [resp. s_3, s_4] and the angular momentum ℓ' [resp. ℓ] of some relative wavefunction of $\hat{\mathbf{p}}_1$ [resp. $\hat{\mathbf{p}}_3$] to obtain j.

i. (this is not a problem) In this part (b), we still consider the case the final state particles Φ'^- and Φ'^+ are spin-0 particles ($s_3 = s_4 = 0$), so we only need to use $C_{0,0,\ell}(j, j_z; 0, 0, m) = \delta_{\ell,j}\delta_{j_z,m}$. Looking at the (θ, ϕ) dependence of the amplitude (5) and using the spehreical harmonics

$$Y_{\ell=1}^{m=\pm1} = \sqrt{\frac{3}{4\pi}}\frac{\mp 1}{\sqrt{2}}\sin\theta e^{\pm\phi}, \qquad Y_{\ell=1}^{m=0} = \sqrt{\frac{3}{4\pi}}\cos\theta, \tag{8}$$

we find that only the $j = \ell = 1$ term should be retained in the expansion (7).

$[\mathcal{M}_{j=1}]$ is a 1×4 matrix. The "$1\times$" part must already be obvious. The "$\times 4$" part is also understood as follows. For the initial state, $s_1 = s_2 = 1/2$. So, $s_{\rm tot} = 1$ or 0. For $s_{\rm tot} \otimes \ell'$ to contain $j = 1$, the only possibilities are $(s_{\rm tot}, \ell') = (0, 1), (1, 0), (1, 1), (1, 2)$. The 1×4 entries of the matrix $[\mathcal{M}_{j=1}]$ are denoted by $([\mathcal{M}_1^{(0,1)}], [\mathcal{M}_1^{(1,0)}], [\mathcal{M}_1^{(1,1)}], [\mathcal{M}_1^{(1,2)}])$, using $(s_{\rm tot}, \ell')$ as the label.

ii. (still this is not a problem) The scattering amplitude is

$$\mathcal{M}_{\rm red} = \frac{\alpha Q_\Phi Q_{\Phi'}}{8\pi}\left(Y_1^1(\theta,\phi)\sqrt{\frac{8\pi}{3}}(-s_+^1) + Y_1^{-1}(\theta,\phi)\sqrt{\frac{8\pi}{3}}(-s_-^1)\right), \tag{9}$$

or equivalently,

$$\mathcal{M}_{\rm red} = \frac{\alpha Q_\Phi Q_{\Phi'}}{\sqrt{24\pi}}\left(Y_1^1(\theta,\phi)\begin{pmatrix} & \\ -1 & 0\end{pmatrix} + Y_1^{-1}(\theta,\phi)\begin{pmatrix} 0 & 1 \\ & \end{pmatrix}\right), \tag{10}$$

$$= -\frac{\alpha Q_\Phi Q_{\Phi'}}{\sqrt{24\pi}}\left(Y_1^1(\theta,\phi)\langle 1/2, 1/2| + Y_1^{-1}(\theta,\phi)\langle -1/2, -1/2|\right). \tag{11}$$

The operator form $\mathcal{M}_{\rm red}^{\rm op}$ in (10, 11) becomes the amplitude $\mathcal{M}_{\rm red}^{\rm amp}$ in (9) when we evaluate the former on the spin wavefunction (6); ${\rm tr}_{2\times2}[\mathcal{M}_{\rm red}^{\rm op}(\xi \circ \xi^\dagger)] = \mathcal{M}_{\rm red}^{\rm amp}$. The following translation is understood in the equality between (10) and (11).

$$\langle 1/2, 1/2| \Leftrightarrow \begin{pmatrix} & \\ 1 & 0\end{pmatrix}, \quad \langle 1/2, -1/2| \Leftrightarrow \begin{pmatrix} -1 & 0 \\ & \end{pmatrix},$$
$$\langle -1/2, 1/2| \Leftrightarrow \begin{pmatrix} & \\ 0 & 1\end{pmatrix}, \quad \langle -1/2, -1/2| \Leftrightarrow \begin{pmatrix} 0 & -1 \\ & \end{pmatrix}.$$

iii. Now, complete the partial wave decomposition. That can be done by setting

$$\frac{\alpha Q_\Phi Q_{\Phi'}}{\sqrt{24\pi}}(-s_+^1) \tag{12}$$
$$= [\mathcal{M}_1^{(0,1)}]s_0^0 \times 0 + [\mathcal{M}_1^{(1,0)}][Y_0^0]_0 s_+^1 + [\mathcal{M}_1^{(1,1)}][Y_1^0]_0\frac{s_+^1}{\sqrt{2}} + [\mathcal{M}_1^{(1,2)}][Y_2^0]_0\frac{s_+^1}{\sqrt{10}},$$

$$0 \tag{13}$$
$$= [\mathcal{M}_1^{(0,1)}][Y_1^0]_0 s_0^0 + [\mathcal{M}_1^{(1,0)}][Y_0^0]_0 s_0^1 + [\mathcal{M}_1^{(1,1)}][Y_1^0]_0 \times 0 + [\mathcal{M}_1^{(1,2)}][Y_2^0]_0 \frac{-2s_0^1}{\sqrt{10}},$$

and

$$\frac{\alpha Q_\Phi Q_{\Phi'}}{\sqrt{24\pi}}(-s_-^1) \tag{14}$$
$$= [\mathcal{M}_1^{(0,1)}]s_0^0 \times 0 + [\mathcal{M}_1^{(1,0)}][Y_0^0]_0 s_-^1 + [\mathcal{M}_1^{(1,1)}][Y_1^0]_0 \frac{-s_-^1}{\sqrt{2}} + [\mathcal{M}_1^{(1,2)}][Y_2^0]_0 \frac{s_-^1}{\sqrt{10}},$$

for $j_z = +1, 0$ and -1, respectively. Here, in writing down the right hand sides, we used the Clebsch–Gordan coefficients relating $|j = 1, j_z\rangle^{(s_{\rm tot},\ell')}$ and $|s_{\rm tot}; s_{\rm tot}^z\rangle|\ell'; m'\rangle$:

$$\begin{aligned}
(s_{\rm tot} \otimes \ell') \supset (j = 1) \ni |1, +\rangle^{(1,1)} &= \frac{|1;+\rangle|1;0\rangle - |1;0\rangle|1;+\rangle}{\sqrt{2}}, \\
|1, 0\rangle^{(1,1)} &= \frac{|1;+\rangle|1;-\rangle - |1;-\rangle|1;+\rangle}{\sqrt{2}}, \\
|1, -\rangle^{(1,1)} &= \frac{|1;0\rangle|1;-\rangle - |1;-\rangle|1;0\rangle}{\sqrt{2}},
\end{aligned}$$

and

$$\begin{aligned}
|1, +\rangle^{(1,2)} &= \frac{|1;+\rangle|2;0\rangle - \sqrt{3}|1;0\rangle|2;1\rangle + \sqrt{6}|1;-\rangle|2;2\rangle}{\sqrt{10}}, \\
|1, 0\rangle^{(1,2)} &= \frac{\sqrt{3}|1;+\rangle|2;-1\rangle - 2|1;0\rangle|2;0\rangle + \sqrt{3}|1;-\rangle|2;1\rangle}{\sqrt{10}}, \\
|1, -\rangle^{(1,2)} &= \frac{\sqrt{6}|1;+\rangle|2;-2\rangle - \sqrt{3}|1;0\rangle|2;-1\rangle + |1;-\rangle|2;0\rangle}{\sqrt{10}},
\end{aligned}$$

and also used the fact that

$$Y_{\ell'}^{m'}(\cos\theta = 1, {}^\forall\phi) = \delta_{m'}\sqrt{\frac{2\ell'+1}{4\pi}} =: [Y_{\ell'}^0]_0 \delta_{m'}. \tag{15}$$

2. **Neutrino Mass and Partial Wave Unitarity** [A]
   There are two different ways so neutrinos have non-zero masses. One of the two ways is the Majorana scenario, where

$$\mathcal{L} \supset \bar\psi_{\dot\alpha} i\bar\sigma_\mu^{\dot\alpha\alpha} D^\mu \psi_\alpha + \mathcal{L}_{\rm dim5}, \qquad \mathcal{L}_{\rm dim5} = \frac{1}{2M}\epsilon^{\alpha\beta}(\psi_\alpha\phi_0)(\psi_\beta\phi_0) + \text{h.c.}. \tag{16}$$

ψ is a 2-component Grassmann field (see lecture note Week 03 for more details) with the two components labeled by $\alpha = 1, 2$. ϕ_0 is a complex scalar field. $\epsilon^{\alpha\beta}$ is the totally anti-symmetric tensor with $\epsilon^{12} = +1$. When the field ϕ_0 develops non-zero vacuum expectation value, the dimension-5 operator in $\mathcal{L}_{\text{dim5}}$ contains a term that is bilinear in the fluctuation (ψ), which is regarded as a mass term of the fluctuation.

(a) Verify that the scattering amplitude of $\nu(\vec{p}_1) + \nu(\vec{p}_2) \to \bar{\phi}_0(\vec{p}_3) + \bar{\phi}_0(\vec{p}_4)$ is

$$i\mathcal{M}^{\text{LO}} = i\frac{\sqrt{2E_1 2E_2}}{M} = i\frac{\sqrt{s}}{M} \tag{17}$$

at the tree level leading order. Use the relativistic approximation, and the center of mass frame.

(b) Verify that only the $j = 0$ partial wave is present in this scattering amplitude, and that

$$\begin{aligned} &Y_0^0(\theta,\phi)\ [\mathcal{M}_{j=0}]^{\text{LO}} C_{(1/2,1/2,\ell')}(0,0;s_1^z,s_2^z,m')([Y_{\ell'}^{m'}]_0)^{\text{c.c.}}\langle s_1^z, s_2^z| \\ &\quad = Y_0^0(\theta,\phi)\frac{1}{(8\pi)}\frac{\sqrt{s}}{M}\left((Y_0^0)_0^{\text{cc}}\frac{(\langle\downarrow\uparrow| - \langle\uparrow\downarrow|)}{\sqrt{2}} + \frac{1}{\sqrt{3}}(Y_1^0)_0^{\text{cc}}\frac{(\langle\downarrow\uparrow| + \langle\uparrow\downarrow|)}{\sqrt{2}}\right). \end{aligned} \tag{18}$$

Here, $(Y_{\ell'}^{m'=0})_0 := Y_{\ell'}^{m'}(\theta' = 0, \phi') = \delta_{m'}\sqrt{(2\ell'+1)/(4\pi)}$.

(c) According to the tree level leading order result, the $j = 0$ partial wave $1 + i[\mathcal{M}_{j=0}]^{\text{LO}}$ has an imaginary part larger than $\mathcal{O}(1)$ for large $\sqrt{s}$. The partial wave unitarity will be restored, presumably due to presence of extra contributions to $[\mathcal{M}_{j=0}]$ that are at least just as large as $[\mathcal{M}_{j=0}]^{\text{LO}}$ at energy scale

$$\sqrt{s} \gtrsim 8\pi M. \tag{19}$$

That is an indication that the Standard Model with $\mathcal{L}_{\text{dim5}}$ is not going to be a good approximation of the interactions of elementary particles at least at such high energy scale.

To account for the atmospheric neutrino oscillation in the Majorana scenario, we need

$$\frac{(\langle\phi_0\rangle)^2}{M} \gtrsim \sqrt{|\Delta m^2_{\text{atm}}|} \simeq \sqrt{2 \times 10^{-3}}\ \text{eV}, \tag{20}$$

where $\langle\phi_0\rangle \simeq 174\text{GeV}$ is the Higgs boson vacuumm expectation value. So, extra contributions can be negligible only in the energy range

$$\sqrt{s} \lesssim 8\pi M \lesssim 8\pi \frac{(\langle\phi_0\rangle)^2}{\sqrt{|\Delta m^2_{\mathrm{atm}}|}}. \tag{21}$$

Work out the upperbound on the right-hand side.