QFT II/QFT homework XIII

- Reports on these homework problems are supposed to be submitted through the U Tokyo LMS. We request that the file name includes the problem number, such as II-1***.pdf or ****-IV-2-IX-1.jpeg. The LMS will show who had submitted the file (student ID and name), so the file name will not have to contain your name or ID number.

- course credit guaranteed by woking on $1\#[A]+1.5\#[B]+2\#[C]+4\#[D]+8\#[E] \geq 9$ problems.

- A sample solution is available as a pdf file for problems with the mark $\star$.

1. **Imaginary time tau-ordered propagator: off-shell form** [B] $\star$
   Presumably at the end of Week 14 lecture, we have derived the imaginary time tau-ordered propagator

$$\langle T_{au}\left\{\phi(\vec{x}+\vec{y},\tau+\tau_0)\phi(\vec{y},\tau_0)\right\}\rangle_\beta, \qquad 0\leq \tau_0, \tau_0+\tau\leq\beta \tag{1}$$

in a free real scalar field theory on $(d+1)$-dimensional space-time:

$$D(\vec{x},\tau)|_{0<\tau<\beta} = \int \frac{d^dk}{(2\pi)^d}\frac{1}{2E_k}\left(e^{i\vec{k}\cdot\vec{x}-E_k\tau}\frac{1}{1-e^{-E_k\beta}} + e^{-i\vec{k}\cdot\vec{x}+E_k\tau}\frac{e^{-E_k\beta}}{1-e^{-E_k\beta}}\right). \tag{2}$$

An expression for $\tau<0$ is omitted here. Now, let us think of expressing this propagator in the full Fourier expansion, not just in the d space directions but also in the imaginary time direction. That is to introduce $\tilde{D}$ that fits into

$$D(\vec{x},\tau)|_{0<\tau<\beta} =: T\sum_{m\in\mathbb{Z}}\int\frac{d^3k}{(2\pi)^d}e^{i\vec{k}\cdot\vec{x}}e^{i\tau(2\pi/\beta)m}\tilde{D}(\vec{k},(2\pi/\beta)m). \tag{3}$$

Verify that

$$\tilde{D}(\vec{k},(2\pi/\beta)m) = \frac{1}{((2\pi/\beta)m)^2+E_k^2}; \tag{4}$$

Tihs concise result may well be regarded as analytic contiunation of the Minkowski space propagator $k^0\to i(2\pi/\beta)m$. Also, this expression is intuitively acceptable, because the partition function of a free scalar field theory with imaginary time is the Gaussian integral with the exponent proportional to $((2\pi/\beta)m)^2+\vec{k}^2+m^2$.

2. **Non-rela effective theories via path integration** [B]

(a) We have dealt with the process of deriving the effective theory of non-relativisitic two component fermion from Dirac Lagrangian in a couple of different perspectives so far. This homework problem provides one more take on this phenomenon. Let us use the gamma matrices of the form

$$\gamma^0 = \begin{pmatrix} \mathbf{1}_{2\times2} & \\ & -\mathbf{1}_{2\times2} \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} & \vec{\tau} \\ -\vec{\tau} & \end{pmatrix}, \tag{5}$$

and the four component Dirac fermion be split into the upper two and lower two components in this frame,

$$\Psi =: e^{-iMt} \begin{pmatrix} \psi \\ \bar{\chi} \end{pmatrix}. \tag{6}$$

i. Now, rewrite the Dirac Lagrangian

$$\mathcal{L}_{\text{Dirac}} = \Psi^\dagger \gamma^0 \left(i\gamma^\mu (\partial_\mu + ieQ_e A_\mu) - M \right) \Psi \tag{7}$$

in terms of ψ and χ. [You will find that the mass parameter M cancels in the coefficient of $\psi^\dagger \psi$, while it does not in the coefficient of $\chi\chi^\dagger$.]

ii. Complete a square with respect to χ–$\chi^\dagger$, and carry out the Gaussian integral with respect to

$$\mathcal{D}\chi^\dagger \mathcal{D}\chi \subset \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}\bar{\chi} \mathcal{D}\chi = \mathcal{D}\Psi \mathcal{D}\bar{\Psi}. \tag{8}$$

[The remnant of completion of a square in $\mathcal{L}_{\text{Dirac}}$ must be the effective theory Lagrangian of the non-relativisitic two component fermion.]

(b) If one hopes to write down a quantum field theory for a spin-0 particle whose number conserves (imagine an alkali atom), the field theory of a complex scalar field will usually be the first choice. If you are interested only in using it in a non-relativisitc regime, however, it is not necessary to retain both the particle and its anti-particle in your theory; there must be a low-energy effective theory where you only retain the particle without its anti-particle. What is the process of deriving this low-energy effective theory like? Here is how.

i. The path integral of a theory of a complex scalar field Φ is carried out over the space with the coordinates $(\Phi(k^0, \vec{k}), \Phi^*(k^0, \vec{k}))$. Let $\phi_+(k^0, \vec{k})$ and $\phi_-(k^0, \vec{k})$

for $\omega > 0$ be the positive and negative frequency parts of $\Phi(k^0, \vec{k})$; similarly, we put the positive and negative frequency parts of Φ^* as ϕ^*_- and ϕ^*_+, respectively. The path integral measure is now

$$\mathcal{D}\phi_+ \mathcal{D}\phi_- \mathcal{D}\phi^*_- \mathcal{D}\phi^*_+. \tag{9}$$

When you start from

$$\mathcal{L}_{\rm KG} = (\partial_\mu \Phi)^* (\partial^\nu \Phi) - M^2 |\Phi|^2, \tag{10}$$

verify that this Lagrangian is already in the form of a sum of a square (so we do not need to complete a square), and that we are ready to integrate out ϕ_- and ϕ^*_-.

ii. (this is a remark, not a problem) Now you are left with $\mathcal{D}\phi_+(k^0_{\geq 0}, \vec{k})$ and $\mathcal{D}\phi^*_+(k^0_{\leq 0}, \vec{k})$. Quantization of this field theory leads to ϕ_+ containing just the annihilation operators of a particle, without the creation opertor of its anti-particle, because the $\phi_+(k^0_{\geq 0}, \vec{k})$ field does not admit a negative frequency solution to the equation of motion.

iii. Rewrite the effective theory Lagrangian of ϕ_+–ϕ^*_+ in terms of a new pair of fields $\phi_+ =: e^{-iMt} \underline{\phi}_+$ and $\phi^*_+ =: e^{+iMt} \underline{\phi}^*_+$. Once you have done that, you will presumably feel motivated to make a furtuer redefinition, $\underline{\phi}_+ := \underline{\underline{\phi}}_+ / \sqrt{2M}$. Verify then that the effective theory Lagrangian written in terms of $\underline{\underline{\phi}}_+$ and its Hermitian conjugate contains terms that look like the action for the Schroedinger equation.

iv. Suppose that we are interested in using this effective theory only in circumstances where $|\vec{k}| \ll M$ (that is, in non-relativistic situations). This is translated into the presence of a small parameter $\lambda \ll 1$ so that $|\vec{k}| \sim \lambda \times \mathcal{O}(M)$. Then the operator $\vec{\partial}^2/M$ is given a scaling behavior $M \times \lambda^2$. This means that we should assign the same scaling behavior $M \times \lambda^2$ to the operator ∂_t. What is the scaling dimension of the operator $(\partial_t)^2/M$, then? [That fact that this operator is assigned a higher scaling dimension (simply the power of λ) than the two others justifies to drop this $(\partial_t)^2/M$ operator from consideration (or to treat this operator as a correction term).]

v. If you are interested, repeat the same procedure for a complex sclar field theory, but now with an interaction term $-\frac{\kappa}{4}|\Phi|^4$ term added to $\mathcal{L}_{\rm KG}$.