QFT II/QFT homework XIV

- Reports on these homework problems are supposed to be submitted through the U Tokyo LMS. We request that the file name includes the problem number, such as II-1***.pdf or ****-IV-2-IX-1.jpeg. The LMS will show who had submitted the file (student ID and name), so the file name will not have to contain your name or ID number.
- course credit guaranteed by woking on $1\#[A] + 1.5\#[B] + 2\#[C] + 4\#[D] + 8\#[E] \geq 9$ problems.
- A sample solution is available as a pdf file for problems with the mark $\star$.

1. **Real-time formalism propagators and Fluctuation Dissipation theorem** [C] $\star$

   (a) Consider a harmonic oscillator, where the unit excitation energy (frequency) is E. $\phi(t) = (ae^{-iEt} + a^\dagger e^{iEt})/\sqrt{2E}$, $p(t) = \sqrt{E/2}(ae^{-iEt} - a^\dagger e^{+iEt})/i$. Now, compute

$$\Delta^<(t) := \frac{\mathrm{Tr}\left[e^{-\beta H_0}\phi(0)\phi(t)\right]}{\mathrm{Tr}\left[e^{-\beta H_0}\right]}, \qquad \Delta^>(t) := \frac{\mathrm{Tr}\left[e^{-\beta H_0}\phi(t)\phi(0)\right]}{\mathrm{Tr}\left[e^{-\beta H_0}\right]}, \tag{1}$$

   and find their expressions that use the Bose–Einstein distribution

$$n_E := \frac{1}{(e^{\beta E} - 1)}. \tag{2}$$

   (b) Note that the τ-ordered propagator in the imaginary time formalism corresponds to

$$\begin{cases} \Delta^<(t \to -i\tau), & \text{if } \tau < 0, \\ \Delta^>(t \to -i\tau), & \text{if } \tau > 0. \end{cases} \tag{3}$$

   (c) Verify, by using $e^{\beta H_0}\phi(t)e^{-\beta H_0} = \phi(t - i\beta)$, that $\Delta^<(t) = \Delta^>(t - i\beta)$.

   (d) Now, we examine relations among those propagators in their Fourier-transformed version. As a preparation, verify that

$$\Theta(t) = \int \frac{d\omega}{2\pi} e^{-i\omega t} \frac{i}{\omega + i\epsilon}, \qquad -\Theta(-t) = \int \frac{d\omega}{2\pi} e^{-i\omega t} \frac{i}{\omega - i\epsilon}. \tag{4}$$

(e) Let

$$\langle 0|\,[\phi(t),\phi(0)]\,|0\rangle =: \int \frac{d\omega}{2\pi} e^{-i\omega t}\rho(\omega), \tag{5}$$

$$\Delta^R(t) = \Theta(t)\langle 0|\,[\phi(t),\phi(0)]\,|0\rangle =: \int \frac{d\omega}{2\pi} e^{-i\omega t}\widetilde{\Delta}^R(\omega), \tag{6}$$

$$\Delta^A(t) = -\Theta(-t)\langle 0|\,[\phi(t),\phi(0)]\,|0\rangle =: \int \frac{d\omega}{2\pi} e^{-i\omega t}\widetilde{\Delta}^A(\omega). \tag{7}$$

Verify that

$$\widetilde{\Delta}^R(\omega) = \int \frac{d\omega'}{2\pi}\frac{i}{\omega-\omega'+i\epsilon}\rho(\omega'), \qquad \widetilde{\Delta}^A(\omega) = \int \frac{d\omega'}{2\pi}\frac{i}{\omega-\omega'-i\epsilon}\rho(\omega'), \tag{8}$$

$$\mathrm{Re}\left(\widetilde{\Delta}^R(\omega)\right) = \frac{1}{2}\rho(\omega). \tag{9}$$

An example: in the case of a harmonic oscillator, $\langle 0|\,[\phi(t),\phi(0)]\,|0\rangle = (e^{-iEt} - e^{iEt})/(2E)$, so $\rho(\omega)$ is the following:

$$\rho(\omega) = \frac{2\pi}{2E}\left(\delta(\omega-E) - \delta(\omega+E)\right), \tag{10}$$

(f) Using the fact that $\Delta^R(t) = \Theta(t)(\Delta^>(t) - \Delta^<(t))$, and the fact that $\Delta^<(t) = \Delta^>(t-i\beta)$, derive the following relations on the Fourier transforms of $\Delta^<$ and $\Delta^<$:

$$\widetilde{\Delta}^<(\omega) = e^{-\beta\omega}\widetilde{\Delta}^>(\omega), \tag{11}$$

$$\rho(\omega) = \widetilde{\Delta}^>(\omega) - \widetilde{\Delta}^<(\omega) = (1-e^{-\beta\omega})\widetilde{\Delta}^>(\omega). \tag{12}$$

(g) (remark, not a homework problem) Combining all the results we have derived above, we see that

$$\mathrm{Re}\left(\widetilde{\Delta}^R(\omega)\right) = \frac{1}{2}\rho(\omega) = \frac{1-e^{-\beta\omega}}{1+e^{-\beta\omega}}\frac{\left(\widetilde{\Delta}^> + \widetilde{\Delta}^<\right)}{2} = \tanh(\beta\omega/2)\frac{\left(\widetilde{\Delta}^> + \widetilde{\Delta}^<\right)}{2}. \tag{13}$$

This relation is an example of the Fluctuation–dissipation theorem; in fact, this relation holds not just for fields ϕ that are used for perturbative computations in a quantum field theory system, but also for any kinds of operators $\mathcal{O}$. Two point functions $\Delta^<(t)$ and $\Delta^>(t)$ are defined as in (1) by simply replacing ϕ by $\mathcal{O}$. The statement $\Delta^<(t) = \Delta^>(t - i\beta)$ still holds true. In the expressions (5–7), $[\phi(t), \phi(0)]$ is replaced by $[\mathcal{O}(t), \mathcal{O}(0)]$, and the vacuum expectation value by thermal average, because $[\mathcal{O}(t), \mathcal{O}(0)]$ is an operator in general, rather than a $\mathbb{C}$-number. The algebra that we have gone through, (4, 8–12), still holds true, and hence the relation (13) follows. The combination $(\widetilde{\Delta}^> + \widetilde{\Delta}^<)(\omega)$ in the right-hand side is the power spectrum (Fourier transform) of the thermal average of the fluctuation of the operator at the quadratic order, $\{\mathcal{O}(t), \mathcal{O}(0)\}$. On the other hand, $\widetilde{\Delta}^R$ on the left-hand side determines (linear) response of the system under an external time-varying field coupled to the operator $\mathcal{O}$; its real part[1] corresponds to the dissipation in the $\mathcal{O}$–$\mathcal{O}$ channel

[1]The imaginary part of the response function is not dissipative in nature. Imagine a free electron moving in an AC electric field. The current due to the oscillating electron motion tracks the oscillating electric field, but with the delay in phase by $\pi/2$, so the response function is pure imaginary.