

# QFT II/QFT

“Category D” homework problems (ver. 1))

- Reports on these homework problems are supposed to be submitted through the U Tokyo LMS. We request that the file name includes the problem number, such as II-1\*\*\*.pdf or \*\*\*\*-IV-2-IX-1.jpeg.

[D-2] see homework E-4 (fractional quantum Hall effect / Chern–Simons theory).

## [D-3] $R_\xi$ Gauge

Read appropriate resources to learn how to deal with massive vector fields in a non-abelian gauge theory with spontaneous symmetry breaking (key word:  $R_\xi$ -gauge), and write a summary as a report. For example, [Peskin Schroeder] textbook §20.1, §20.2 and §21.1 (based on §9.4 and 16.2) cover this subject. This is what you need when computing scattering amplitudes involving the W-bosons and Z-boson in the Standard Model.

[D-4] see homework V-1 (quantization of constrained systems).

[D-5] see homework X-1 (hyperfine structure and Lamb shift).

## [D-6] Batalin–Vilkovisky formalism

If you have already learned the Fadeev–Popov and BRST quantization somewhere (e.g., in §9.4, §16.2 and §16.4 of [Peskin Schroeder], or Kugo’s textbook (in Japanese) on QFT), then you might also be interested in taking one more step to learn a little more general version of this. Read

- Marc Henneaux, “Lectures on the antifield-BRST formalism for gauge theories” in Nucl. Phys.B proceedings Supplements **18** (1990) pp.47–105. (To download this lecture note, click [here] within the online network within U Tokyo, and further click the link “DOI: 10.1016/0920-5632(90)90647-D” on that page. In case you cannot download, please let me know.)

and submit a summary of what you have learned or a copy of notes that you wrote while you were reading this lecture note. Weinberg’s textbook QFT vol. II (§15.9) may also be used as a side reader for this purpose.

## [D-1] Spinor Helicity formalism

### Conventions

In this note, we use the following convention.<sup>1</sup>

$$\gamma^\mu = \begin{pmatrix} & \sigma^\mu \\ \bar{\sigma}^\mu & \end{pmatrix}, \quad \sigma^\mu = (\mathbf{1}_{2 \times 2}, -\vec{\tau}), \quad \bar{\sigma}^\mu = (\mathbf{1}_{2 \times 2}, \vec{\tau}), \quad (1)$$

so that  $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} = 2\text{diag}(+, -, -, -)$ . A four component spinor  $\Psi$  is split into two component spinors as

$$\Psi = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix}, \quad \bar{\Psi} := \Psi^\dagger \gamma^0 = (\bar{\psi}_{\dot{\alpha}}, \chi^\alpha), \quad \alpha = 1, 2, \quad \dot{\alpha} = 1, 2. \quad (2)$$

Individual entries of the  $2 \times 2$  matrices  $\sigma^\mu$  and  $\bar{\sigma}^\mu$  are denoted by  $(\sigma^\mu)_{\alpha\dot{\alpha}}$  and  $(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha}$ .

The two component spinors,  $\psi_\alpha, \bar{\chi}^{\dot{\alpha}}$  etc. are in separate irreducible spinor representations of  $\text{Spin}(1, 3)$ . There are two irreducible spinor representations of  $\text{Spin}(1, 3)$ ,  $\mathbf{2}_R$  and  $\mathbf{2}_L$ . When the Lorentz transformation of the spinor  $\psi_\alpha$  is given by  $\psi_\alpha \rightarrow \psi'_\alpha = (\rho(g))_\alpha^\beta \psi_\beta$ , the Lorentz transformation of the spinor  $\chi^\alpha$  is  $\chi^\alpha \rightarrow (\chi')^\alpha = \chi^\beta \rho(g)_\beta^\alpha$ . In the sense that we can use the same  $2 \times 2$  matrix (representation), we say that the spinors  $\psi_\alpha$  and  $\chi^\alpha$  are in the representation,  $\mathbf{2}_R$ . Similarly,  $\bar{\psi}^{\dot{\alpha}}$  and  $\bar{\chi}_{\dot{\alpha}}$  are in the other spinor representation  $\mathbf{2}_L$ .

Using  $\text{Spin}(1, 3)$  invariant anti-symmetric tensors<sup>2</sup>  $\epsilon_{\alpha\beta}$  and  $\epsilon_{\dot{\alpha}\dot{\beta}}$ ,

$$\epsilon_{12} = \epsilon_{1\dot{2}} = -1, \quad \epsilon_{21} = \epsilon_{2\dot{1}} = +1, \quad \epsilon_{\alpha\beta}\epsilon^{\beta\gamma} = \delta_\alpha^\gamma, \quad \epsilon_{\dot{\alpha}\dot{\beta}}\epsilon^{\dot{\beta}\dot{\gamma}} = \delta_{\dot{\alpha}}^{\dot{\gamma}}, \quad (3)$$

spinor indices are raised and lowered by

$$\psi^{\alpha\beta} = \epsilon^{\alpha\beta} \psi_\beta, \quad \psi_\alpha = \epsilon_{\alpha\beta} \psi^\beta, \quad \bar{\psi}_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}} \bar{\psi}^{\dot{\beta}}, \quad \bar{\psi}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}} \bar{\psi}_{\dot{\beta}}. \quad (4)$$

The fact that both  $\bar{\Psi}\Psi$  and  $\bar{\Psi}\gamma_5\Psi$  are Lorentz invariant is equivalent to a statement that both

$$\chi\psi := \chi^\alpha \psi_\alpha = \epsilon_{\alpha\beta} \chi^\alpha \psi^\beta, \quad \bar{\psi}\bar{\chi} := \bar{\psi}_{\dot{\alpha}} \bar{\chi}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}} \bar{\psi}_{\dot{\alpha}} \bar{\chi}_{\dot{\beta}} \quad (5)$$

are Lorentz invariant.

Later on, we will use the following relations,

$$\bar{\sigma}^{\mu\dot{\alpha}\alpha} = \epsilon^{\dot{\alpha}\dot{\beta}} \epsilon^{\alpha\beta} \sigma_{\beta\dot{\beta}}^\mu, \quad \sigma_{\alpha\dot{\alpha}}^\mu = \epsilon_{\alpha\beta} \epsilon_{\dot{\alpha}\dot{\beta}} \bar{\sigma}^{\mu\dot{\beta}\beta}, \quad (6)$$

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<sup>1</sup>This choice of  $\sigma^\mu$  and  $\bar{\sigma}^\mu$  is to follow the convention of a textbook “*Supersymmetry and Supergravity*,” by J. Wess and J. Bagger.

<sup>2</sup>The tensor  $\epsilon_{\alpha\beta}$  can be regarded as the Clebsh–Gordon coefficient of  $\mathbf{2}_R \otimes \mathbf{2}_R \cong \mathbf{1} \oplus \mathbf{3}$  for the  $\mathbf{1}$  component.

$$\sigma_{\alpha\dot{\alpha}}^{\mu}\bar{\sigma}_{\mu}^{\dot{\beta}\beta} = 2\delta_{\alpha}^{\beta}\delta_{\dot{\alpha}}^{\dot{\beta}} \quad \text{and} \quad \sigma_{\alpha\dot{\alpha}}^{\mu}\bar{\sigma}^{\nu\dot{\alpha}\alpha} = 2\eta^{\mu\nu}. \quad (7)$$

### On shell states (recap)

Solution to the Dirac equation

$$(i\gamma^{\mu}\partial_{\mu} - m)e^{-ip\cdot x}u(p) = 0, \quad (i\gamma^{\mu}\partial_{\mu} - m)e^{-ip\cdot x}v(p) = 0, \quad (8)$$

are in the form of

$$u(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi \\ \sqrt{p\cdot\bar{\sigma}}\xi \end{pmatrix}, \quad v(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\eta \\ -\sqrt{p\cdot\bar{\sigma}}\eta \end{pmatrix}, \quad p^2 = m^2 \quad (9)$$

for some  $\text{SO}(3) \subset \text{SO}(1, 3)$  spinors  $\xi$  and  $\eta$ . There are two positive energy solutions (due to the choice of  $\xi$ ) and two negative energy solutions (due to the choice of  $\eta$ ).

When we consider the case the mass parameter vanishes,  $m = 0$ , things simplify a lot. The mode function  $u(p)$  vanishes in the lower two components in the positive energy states with helicity  $h = +1/2$ , while the upper two components of  $u(p)$  vanish in the positive energy states with helicity  $h = -1/2$ . An anti-particle state with momentum  $p^{\mu} = (E, \vec{p})$  with helicity  $h = \pm 1/2$  corresponds to  $v(p)$  with the two component spinors  $\sqrt{p\cdot\sigma}\eta$  and  $\sqrt{p\cdot\bar{\sigma}}\eta$  in the  $\mp 1/2$  eigenspace of  $\vec{p}\cdot\vec{\tau}$  [as explained in the lecture]. For this reason, the two component spinor field  $\psi$  is the annihilation operator of right-handed ( $h = +1/2$ ) particles and creation operator of left-handed ( $h = -1/2$ ) anti-particles, while  $\bar{\chi}$  is the annihilation operator of left-handed ( $h = -1/2$ ) particles and creation operator of right-handed ( $h = +1/2$ ) anti-particles.

### Step 1

Now, let us introduce a treatment of spinors for **massless** and **on-shell** states that is much more simple than keeping track of all the  $\sqrt{p\cdot\sigma}$ 's and  $\sqrt{p\cdot\bar{\sigma}}$ 's above. Note first that the on-shell massless condition  $p^2 = 0$  for a four momentum  $p^{\mu}$  is equivalent to a condition that the following  $2 \times 2$  matrix

$$p_{\alpha\dot{\alpha}} := p_{\mu}\sigma_{\alpha\dot{\alpha}}^{\mu} \quad (10)$$

has a vanishing determinant. This matrix is rank-1, unless  $p^{\mu} = 0$ . So, there must be<sup>3</sup> some two-component spinors  $\lambda_{\alpha}$  and  $\tilde{\lambda}_{\dot{\alpha}}$  for a given on-shell massless  $p^{\mu}$  so that

$$p_{\alpha\dot{\alpha}} = \lambda_{\alpha}\tilde{\lambda}_{\dot{\alpha}}. \quad (11)$$

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<sup>3</sup>The choice of  $\lambda$  and  $\tilde{\lambda}$  is not unique for a given four momentum  $p^{\mu}$ ; we can replace  $\lambda$  and  $\tilde{\lambda}$  by  $\lambda \times a$  and  $\tilde{\lambda}/a$ , respectively.

Lorentz transformation on  $p^\mu$  in the vector representation of  $\text{SO}(1,3)$  is equivalent to the Lorentz transformation on  $\lambda_\alpha$  and  $\tilde{\lambda}_{\dot{\alpha}}$  that are in the  $\mathbf{2}_R$  and  $\mathbf{2}_L$  representations combined.

Solutions to the massless Dirac equation are given by

$$e^{-ip \cdot x} \begin{pmatrix} \lambda_\alpha \\ 0 \end{pmatrix}, \quad e^{-ip \cdot x} \begin{pmatrix} 0 \\ \tilde{\lambda}_{\dot{\alpha}} \end{pmatrix}, \quad e^{ip \cdot x} \begin{pmatrix} 0 \\ \tilde{\lambda}_{\dot{\alpha}} \end{pmatrix}, \quad e^{ip \cdot x} \begin{pmatrix} \lambda_\alpha \\ 0 \end{pmatrix} \quad (12)$$

for  $h = \pm 1/2$  particle states and  $h = \pm 1/2$  anti-particle states, respectively. To see this, it is enough, in the case of  $h = +1/2$  particle state as an example, to see that<sup>4</sup>

$$(p \cdot \bar{\sigma})^{\dot{\alpha}\alpha} \lambda_\alpha = \tilde{\lambda}^{\dot{\alpha}} \lambda^\alpha \lambda_\alpha = 0. \quad (13)$$

Proof for the three other solutions are the same. The following short-hand notation is used frequently in the following:

$$|p\rangle := \begin{pmatrix} \lambda_\alpha \\ 0 \end{pmatrix}, \quad |p] := \begin{pmatrix} 0 \\ \tilde{\lambda}_{\dot{\alpha}} \end{pmatrix}, \quad \langle p| := (\lambda^\alpha, 0), \quad [p| := (0, \tilde{\lambda}_{\dot{\alpha}}). \quad (14)$$

### Exercise

1. If you find any statements that are non-trivial for you, verify them, whenever you can. [If you understand the meaning of the statements, but do not understand why, it is an option to accept the statement and move on.] If you find a statement whose meaning is not clear, ask others what it means. .... This ‘‘Exercise’’ applies also to the rest of this note.
2. Verify, in the simplest example of  $p^\mu = (E, 0, 0, E)$  that the solutions derived in the two different ways—one is those using  $u(p)$  and  $v(p)$ , and the other using  $\lambda_\alpha$  and  $\tilde{\lambda}_{\dot{\alpha}}$ —are the same. [Here, we need to fix an ambiguity in choosing  $\lambda_\alpha$  and  $\tilde{\lambda}_{\dot{\alpha}}$  for a given momentum  $p^\mu$ . We do so as in the next exercise problem.]
3. Compute explicitly the following:

$$\lambda_\alpha = \tilde{\lambda}_{\dot{\alpha}} = \exp \left[ i \frac{\varphi}{2} \tau_3 \right] \exp \left[ i \frac{\theta}{2} \tau_2 \right] \begin{pmatrix} \sqrt{2E} \\ 0 \end{pmatrix}, \quad (15)$$

which is for a massless on-shell four momentum  $p^\mu = E(1, \sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$ .

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<sup>4</sup>Spinors  $\lambda_\alpha$  and  $\tilde{\lambda}_{\dot{\alpha}}$  are  $\mathbb{C}$ -valued, just like  $u(p)$  and  $v(p)$  are. The fermion fields  $\psi_\alpha$  and  $\bar{\chi}^{\dot{\alpha}}$  are Grassmann, because they are given by those solutions multiplied by fermion annihilation/creation operators.

	$m \neq 0$	$h = +1/2$	$h = -1/2$
particle incoming	$u(p)$	$ \lambda\rangle$	$ \lambda\rangle$
anti-particle incoming	$\bar{v}(p)$	$\langle\lambda $	$[\lambda $
particle outgoing	$\bar{u}(p)$	$[\lambda $	$\langle\lambda $
anti-particle outgoing	$v(p)$	$ \lambda\rangle$	$ \lambda\rangle$

Table 1: Feynman rule assigns to fermion external states four component spinors shown in the second column, in QED with  $m \neq 0$  fermions. When the fermions are massless (when the massless approximation can be made), however, we can use the ones in the 3rd and 4th columns instead.

## Step 2

The Feynman rule for QED is given by

$$\frac{i\not{p}}{p^2 + i\epsilon}, \quad \frac{-i\eta_{\mu\nu}}{p^2 + i\epsilon} \quad (16)$$

for a fermion and photon propagator, and

$$-i\gamma^\mu e Q_\Psi \quad (17)$$

for a vertex,<sup>5</sup> as usual. For fermion external states, see Table ??

**Step 3:** First example,  $s$ -channel scattering.

Consider  $e^+ + e^-$  scattering processes in the  $s$ -channel, as in Figure ?? (a). The particle 1 is an electron with incoming momentum  $p_1^\mu$  and the particle 2 a positron with momentum  $p_2^\mu$ . As a first application of the spinor-helicity formalism, we consider this  $s$ -channel process that goes to  $\mu^+ + \mu^-$ . The particle 3 is  $\mu^-$  with outgoing momentum  $p_3^\mu$  and the particle 4  $\mu^+$  with momentum  $p_4^\mu$ . The Feynman rule given above immediately leads us to the following expression for the scattering amplitude of  $e_{h=+1/2}^- + e_{h=-1/2}^+ \longrightarrow \mu_{h=+1/2}^- + \mu_{h=-1/2}^+$  at energy scale  $\sqrt{s}$  much higher than the muon mass  $m_\mu$ :

$$\begin{aligned} i\mathcal{M} \left( e_{+1/2}^-(p_1) + e_{-1/2}^+(p_2) \longrightarrow \mu_{+1/2}^-(p_3) + \mu_{-1/2}^+(p_4) \right) \\ = \frac{-i}{s + i\epsilon} (-ieQ_e[2|\gamma^\mu|1\rangle) (-ieQ_\mu[3|\gamma_\mu|4\rangle). \end{aligned} \quad (18)$$

Here,  $|i\rangle$ ,  $[i]$  etc. stand for  $|p_i\rangle$ ,  $[p_i]$  etc. It is straightforward to write down the expression

<sup>5</sup>Convention: the covariant derivative for  $\Psi$  is  $D_\mu = \partial_\mu + ieQ_\Psi A_\mu$ , where  $Q_\Psi$  is the electric charge of  $\Psi$ .

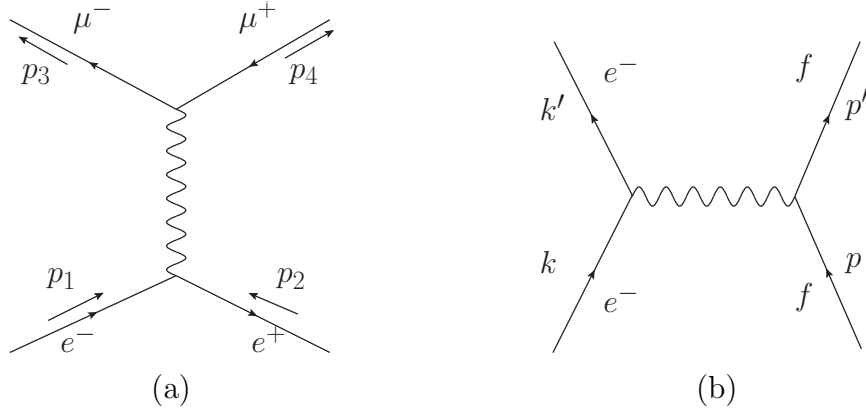


Figure 1: (a) is the only one Feynman diagram contributing to the  $e^- + e^+ \rightarrow f^- + f^+$  scattering (at tree level), whereas (b) is the one contributing to the  $e^- + f \rightarrow e^- + f$  scattering (at tree level).

for three other processes

$$\begin{aligned}
 e_{+1/2}^- + e_{-1/2}^+ &\rightarrow \mu_{-1/2}^- + \mu_{+1/2}^+, \\
 e_{-1/2}^- + e_{+1/2}^+ &\rightarrow \mu_{+1/2}^- + \mu_{-1/2}^+, \\
 e_{-1/2}^- + e_{+1/2}^+ &\rightarrow \mu_{-1/2}^- + \mu_{+1/2}^+.
 \end{aligned}$$

The scattering amplitude vanishes, when the initial states are in the helicity configuration of  $e_{+1/2}^+ + e_{+1/2}^+$  or  $e_{-1/2}^- + e_{-1/2}^-$ . Similarly, the final states never come out in the helicity configuration of  $(+1/2, +1/2)$  or  $(-1/2, -1/2)$ . Let us now work on the amplitude (??).

### Exercise

1. Verify  $[p_i | \gamma^\mu | p_j \rangle = \langle p_i | \gamma^\mu | p_j ]$ . Use (??).
2. We will use this notation heavily in the following:

$$\langle ij \rangle := \langle p_i p_j \rangle := \lambda(p_i)^\alpha \lambda(p_j)_\alpha, \quad [ij] := [p_i p_j] := \tilde{\lambda}(p_i)_{\dot{\alpha}} \tilde{\lambda}(p_j)^{\dot{\alpha}}. \quad (19)$$

Note that  $\langle p_i p_i \rangle$  and  $[p_i p_i]$  vanish, because the spinors  $\lambda_\alpha(p_i)$  and  $\tilde{\lambda}_{\dot{\alpha}}(p_i)$  are not Grassmann. Now, verify the following relation (using (??)):

$$\langle ij \rangle [ji] = 2p_i \cdot p_j. \quad (20)$$

3. Verify, using (??), that

$$[i | \gamma^\mu | j \rangle [k | \gamma_\mu | l \rangle = [i | \gamma^\mu | j \rangle \langle l | \gamma_\mu | k \rangle = -2 \langle jl \rangle [ik]. \quad (21)$$

4. Derive, for the amplitude (??), that

$$i\mathcal{M} = \frac{i(e^2 Q_e Q_\mu)}{s} 2[23]\langle 41 \rangle, \quad |\mathcal{M}|^2 = 4(e^2 Q_e Q_\mu)^2 \frac{u^2}{s^2} = 4(e^2 Q_e Q_\mu)^2 \cos^4\left(\frac{\theta}{2}\right). \quad (22)$$

5. Verify that  $i\mathcal{M}$  for the process  $e_{+1/2}^- + e_{-1/2}^+ \rightarrow \mu_{-1/2}^- + \mu_{+1/2}^+$  is given by (??) with [23]⟨41⟩ replaced by [24]⟨31⟩.

If there is a complex scalar field  $\Phi$  with QED charge  $Q_\Phi$ , the scattering amplitude of  $e_{+1/2}^-(p_1) + e_{-1/2}^+(p_2) \rightarrow \Phi^-(p_3) + \Phi^+(p_4)$  at high energy ( $E \gg m_\Phi$ ) is given by

$$i\mathcal{M} = \frac{-i}{s + i\epsilon} (-ieQ_e [2|\gamma^\mu|1]) (-ieQ_\Phi (p_3 - p_4)_\mu). \quad (23)$$

So,

$$i\mathcal{M} = \frac{i(e^2 Q_e Q_\Phi)}{s} [2|(\not{p}_3 - \not{p}_4)|1] = \frac{i(e^2 Q_e Q_\Phi)}{s} [2|(2\not{p}_3 - \not{p}_1 - \not{p}_2)|1] = \frac{i(e^2 Q_e Q_\Phi)}{s} 2[23]\langle 31 \rangle, \quad (24)$$

and we arrive at the result

$$|\mathcal{M}|^2 = (e^2 Q_e Q_\Phi)^2 \frac{4ut}{s^2} = (e^2 Q_e Q_\Phi)^2 \sin^2(\theta), \quad (25)$$

from which we can recover the result of homework IV-2 by taking average over the initial state spin configuration.

All the computations above only reproduce the results that we have already been familiar with (either in the lecture, or in homework IV-2). You will have noticed, however, that once you start to use the formula (??, ??), it is much easier and faster in this spinor–helicity formalism to compute the scattering amplitudes without summing/averaging over the spin configuration. This spinor–helicity formalism is only for massless states, but still this is very useful.

**Step 3.5** forward–backward asymmetry ( $\simeq$  homework IV-3)

There is a vector boson—called  $Z$ -boson—in the Standard Model in addition to photon, and it couples to a Dirac fermion  $f$  (such as  $f = e, \mu, \tau, u, d, \dots$ ) as in

$$\mathcal{L}_{\text{int}} \ni -\psi^\dagger \bar{\sigma}^\mu Z_\mu g^R \psi - \chi \sigma^\mu Z_\mu g^L \bar{\chi}. \quad (26)$$

In the case of photon, the coupling constants satisfies the relation  $g^R = g^L = eQ_f$ , but the coupling constants of the  $Z$  boson does not satisfy this condition.

**Exercise**

1. Verify that the scattering amplitude for  $e_{+1/2}^- + e_{-1/2}^+ \rightarrow f_{+1/2} + \bar{f}_{-1/2}$  is given by

$$|\mathcal{M}|^2 = u^2 \left| \frac{Q_e Q_f e^2}{s} + \frac{g_e^R g_f^R}{s - m_Z^2} \right|^2, \quad (27)$$

and the amplitude for  $e_{+1/2}^- + e_{-1/2}^+ \rightarrow f_{-1/2} + \bar{f}_{+1/2}$  by

$$|\mathcal{M}|^2 = t^2 \left| \frac{Q_e Q_f e^2}{s} + \frac{g_e^R g_f^L}{s - m_Z^2} \right|^2, \quad (28)$$

when the center of mass energy  $\sqrt{s}$  of these scattering processes is much higher than the mass of  $e^\pm$  and  $f, \bar{f}$ . The distribution of the final states as a function of the scattering angle  $\theta$  therefore depends, in a very non-trivial way, on the coupling constants  $g_f^R, g_f^L$  of the final state fermions, and on the energy scale  $\sqrt{s}$  relatively to the mass of the  $Z$  boson.

#### Step 4: $t$ -channel scattering

Consider, now, another 2-body to 2-body scattering between  $e^-$  and another fermion  $f$  with an electric charge  $Q_f$  at energy scale much higher than the masses of  $e^\pm$  and  $f, \bar{f}$ . [we ignore  $Z$ -boson exchange in this Step 4.] Feynman diagram in Figure ?? (b) allows us to write down the amplitude right away.

$$i\mathcal{M} \left( e_{+1/2}^-(k) + f_{+1/2}(p) \rightarrow e_{+1/2}^-(k') + f_{+1/2}(p') \right) = \frac{-i}{t + i\epsilon} (-ieQ_e \langle k' | \gamma^\mu | k \rangle) (-ieQ_f \langle p' | \gamma_\mu | p \rangle), \quad (29)$$

$$i\mathcal{M} \left( e_{-1/2}^-(k) + f_{+1/2}(p) \rightarrow e_{-1/2}^-(k') + f_{+1/2}(p') \right) = \frac{-i}{t + i\epsilon} (-ieQ_e \langle k' | \gamma^\mu | k \rangle) (-ieQ_f \langle p' | \gamma_\mu | p \rangle). \quad (30)$$

The amplitudes with  $f_{-1/2}$  in the initial (and final) states are also obtained in a similar way.

#### Exercise

1. Verify, in the two scattering processes above, that

$$i\mathcal{M} = \frac{i(e^2 Q_e Q_f)}{t} \begin{cases} 2[k'p'] \langle pk \rangle, \\ 2\langle k'p \rangle [p'k], \end{cases} \quad |\mathcal{M}|^2 = 4(e^2 Q_e Q_f)^2 \begin{cases} s^2/t^2, \\ u^2/t^2, \end{cases} \quad (31)$$

respectively.

2. Verify, in the  $e_{h_e=\pm 1/2}^-(k) + \Phi(p) \longrightarrow e_{h_e=\pm 1/2}^-(k') + \Phi(p')$  scattering, that

$$i\mathcal{M} = i\frac{e^2 Q_e Q_\Phi}{t} \begin{cases} 2[k'p]\langle pk \rangle, \\ 2\langle k'p \rangle [pk], \end{cases} \quad |\mathcal{M}|^2 = 4(e^2 Q_e Q_\Phi)^2 \frac{s(-u)}{t^2}. \quad (32)$$

**Remark** The matrix elements have different dependence on the kinematical variables depending on whether the target particle has spin 1/2 or spin 0. This aspect was crucial in determining the spin of “quarks” within a hadron. We just need to measure the energy and momentum distribution of an outgoing electron in a electron–hadron collision experiment; in this kind of experiments called deep inelastic scattering, it is conventional to introduce a kinematical parameter

$$y := \frac{s+u}{s} = 1 + \frac{u}{s}. \quad (33)$$

In terms of this  $y$ ,

$$\frac{s^2+u^2}{t^2} = \frac{t^2}{s^2} [1 + (1-y)^2], \quad \frac{s(-u)}{t^2} = \frac{t^2}{s^2} [1-y]. \quad (34)$$

The  $y$ -dependence was determined in deep inelastic scattering experiments, and the spin of quarks was determined to be 1/2.

**Remark** See references in the homework [E-1] for advantages of the spinor-helicity formalism in uncovering theoretical aspects of scattering amplitudes in general.