

Setting the initial state to the ground state (vacuum)
or to the thermal distribution.

• $\langle q_f \text{ at } t_f | q_i \text{ at } t_i \rangle = \sum_n \psi_n(q_f) \psi_n^*(q_i) e^{-iE_n(t_f - t_i)}$

$[t_i \rightarrow -\infty] \implies t_i \rightarrow -\infty(1 - i\epsilon)$ ($i\epsilon$ prescription)
 $[t_f \rightarrow +\infty] \implies t_f \rightarrow +\infty(1 - i\epsilon)$

$\times e^{-i(\Delta E) > 0} \left[(-i\epsilon) \infty \right] = e^{-\infty \cdot \epsilon \cdot (\Delta E) > 0}$

excited state contributions die out.

As a result.

$$\int_{i\epsilon\text{-prescription}} \mathcal{D}q \mathcal{D}p \quad e^{i \int dt (p \dot{q} - H)} \quad \prod_i f_i(q_i) \prod_j \tilde{f}_j(p_j)$$

Heisenberg picture states

$$\propto \langle 0, \text{future} | T \left\{ \prod_i f_i(q(t_i)) \prod_j \tilde{f}_j(p(t_j)) \right\} | 0, \text{past} \rangle$$

↑ operators in the Heisenberg picture.

• Boltzmann weight.

$Z = \sum_n e^{-\beta E_n} = \int dq \sum_n \psi_n(q) \psi_n^*(q) e^{-\beta E_n}$

$= \int d(q_i = q_f) \langle q_f \text{ at } (t_i - i\beta) | q_i \text{ at } t_i \rangle$ ($\Delta t = \frac{-i\beta}{N} = -i \frac{\beta}{N}$)
($\Delta t = \frac{\beta}{N}$)

$= \int \mathcal{D}q(\tau) \mathcal{D}p(\tau) \quad e^{i \sum_{k=1}^N \left[p_k (q_k - q_{k-1}) - (\Delta t = -i\Delta t) H(p_k, q_{k-1}) \right]}$
 $q(\tau = -i\beta) = q(\tau = 0)$

$= \int \mathcal{D}q(\tau) \mathcal{D}p(\tau) \quad e^{\int d\tau (i p \dot{q}' - H)}$
 $q(\tau = \beta) = q(\tau = 0)$

$$\frac{\partial}{\partial \tau} \Rightarrow '$$

~~time~~
 $t = -i\tau$

If $H = \frac{p^2}{2m} + V(q)$ then

$$Z \propto \int_{\delta(\tau=\beta) = \delta(\tau=0)} dq \quad e^{-\int d\tau \left\{ \frac{m}{2} (\dot{q})^2 + V(q) \right\}} \quad \text{after Gaussian integral.}$$

The exponent:

instead of $i \int dt L = i \int dt \left\{ \frac{m}{2} (\dot{q})^2 - V(q) \right\}$

now $-\int d\tau \left\{ \frac{m}{2} (\dot{q})^2 + V(q) \right\}$ negative definite.

§ 1.2 Path integral formulation for a 2-state system.

Consider a quantum mechanical system with only 2 states.

States $\leftrightarrow |0\rangle c_0 + |1\rangle c_1 \quad (c_0, c_1) \in \mathbb{C}^2$

$\{b, b^\dagger\} = 1$	$ 1\rangle = b^\dagger 0\rangle$	$b 0\rangle = 0$
	$ 0\rangle = b 1\rangle$	$b^\dagger 1\rangle = 0$

e.g. • an isolated atom with one e^- energy level

(2 states $\leftrightarrow s_z = \pm 1/2$)

• one fermionic state without $s_z = \pm 1/2$ degeneracy (e.g. $\vec{B} \neq 0$)

(2 states \leftrightarrow occupied or not)

Math preparation

Introduce a notion "Grassmann variables"

Instead of $\delta_i \delta_j = \delta_j \delta_i, \quad \partial_i \partial_j = -\partial_j \partial_i.$

$\Rightarrow \partial_i \partial_i = -\partial_i \partial_i = 0$

"functions of Grassmann variables" (by Taylor expansion.)

$$f(\theta's, \theta's) := f^{(0)}(\theta's) + \sum_i f_i^{(1)}(\theta's) \theta_i + \sum_{i < j} f_{ij}^{(2)}(\theta's) \theta_i \theta_j + \dots + f_{\text{all}}^{(n)}(\theta's) (\theta_1 \theta_2 \dots \theta_n)$$

"derivative w.r.p.t. θ_i "

$$f(\theta, \theta, \theta') = f^{(0)}(\theta) + f_1^{(1)}(\theta) \theta + f_2^{(1)}(\theta) \theta' + f_n^{(2)}(\theta) \theta \theta'$$

$$\Rightarrow \left(\frac{\partial f}{\partial \theta}\right) = f_1^{(1)}(\theta) - f_n^{(2)}(\theta) \theta' \quad \left(\frac{\partial \theta}{\partial \theta}\right) = f_2^{(1)}(\theta) + f_n^{(2)}(\theta) \theta'$$

(pull out θ to the right)
use $\theta \theta' = -\theta' \theta$

(pull out θ to the left)

"integrate w.r.p.t. θ_i "

$$\int d\theta f(\theta, \theta) = \int d\theta (f^{(0)}(\theta) + f^{(1)}(\theta) \theta) := f^{(1)}(\theta)$$

(so $\int d\theta \left(\frac{\partial}{\partial \theta} f(\theta, \theta)\right) = 0$ for any $f(\theta, \theta)$)) verify this by yourself PLUS

$$\int d\theta_1 d\theta_2 \left(f^{(0)}(g) + f_1^{(1)}(g)\theta_1 + f_2^{(1)}(g)\theta_2 + f_{12}^{(2)}(g)\theta_1\theta_2 \right) \\ = \int d\theta_2 \left(f_1^{(1)}(g) + f_2^{(1)}(g)\theta_2 \right) = f_{12}^{(2)}(g)$$

set $d\theta_1 d\theta_2 = -d\theta_2 d\theta_1$ so that the same result $f_{12}^{(2)}(g)$ is obtained when doing the $d\theta_2$ integration first.

The wavefunction of a state $|state\rangle = |0\rangle c_0 + |1\rangle c_1$ ($c_{0,1} \in \mathbb{C}$)

can be expressed as $\Psi_{state}(\bar{\theta}) = c_0 + c_1 \bar{\theta}$

by using a Grassmann variable $\bar{\theta}$.

Analogy of $\Psi_{state}(g) = \langle g | state \rangle$ in a bosonic (ordinary) QM system,

$$\Psi_{state}(\bar{\theta}) = \langle 0 | + \bar{\theta} \langle 1 | | state \rangle$$

Analogy of $|state\rangle = \int dg |g\rangle \Psi_{state}(g)$,

$$|state\rangle = \int d\bar{\theta} d\theta (|0\rangle + |1\rangle \theta) e^{-\bar{\theta}\theta} \Psi_{state}(\bar{\theta})$$

← verify by yourself that the RHS is $|0\rangle c_0 + |1\rangle c_1$.

Insertion of a complete system of states:

$$\mathbb{1}_{2x2} = \int d\bar{\theta} d\theta (|0\rangle + |1\rangle \theta) e^{-\bar{\theta}\theta} (\langle 0 | + \bar{\theta} \langle 1 |)$$

analogous to $\mathbb{1} = \int dg |g\rangle \langle g|$

The inner product

$$\langle state' | state \rangle = (c_0')^* c_0 + (c_1')^* c_1 \in \mathbb{C}$$

$$= \int d\bar{\theta} d\theta (c_0'^* + c_1'^* \theta) (c_0 + c_1 \bar{\theta}) e^{-\bar{\theta}\theta} = \int d\bar{\theta} d\theta \Psi_{state'}^*(\bar{\theta}) \Psi_{state}(\bar{\theta}) e^{-\bar{\theta}\theta}$$

$$= \int d\bar{\theta} d\theta ((c_0')^* + (c_1')^* \theta) e^{-\bar{\theta}\theta} (c_0 + c_1 \bar{\theta}) = \int d\bar{\theta} d\theta \Psi_{state'}^*(\bar{\theta}) e^{-\bar{\theta}\theta} \Psi_{state}(\bar{\theta})$$

$$(c_0 + c_1 \bar{\theta})^* := (c_0^* + c_1^* \theta)$$

(θ and $\bar{\theta}$: independent Grassmann variables)
(The relat'n $\bar{\theta}^* = \theta$ is necessary only for PLUS the $\int \Psi^* \Psi$ expression for $\langle state' | state \rangle$.)

Now, we have enough preparation.

② Express time-evolution in terms of path integral.

$$\begin{aligned} \Psi(\bar{\theta}_N @ t_N) &= (\langle 0 | + \bar{\theta}_N \langle 1 |) e^{-iHT} | \text{state} @ t_0 \rangle \\ &= (\langle 0 | + \bar{\theta}_N \langle 1 |) e^{-iH\Delta t} \int d\bar{\theta}_{N-1} d\theta_{N-1} (|0\rangle + |1\rangle \theta_{N-1}) e^{-\bar{\theta}_{N-1} \theta_{N-1}} (\langle 0 | + \bar{\theta}_{N-1} \langle 1 |) \\ &\quad e^{-iH\Delta t} \int d\bar{\theta}_{N-2} d\theta_{N-2} (|0\rangle + |1\rangle \theta_{N-2}) e^{-\bar{\theta}_{N-2} \theta_{N-2}} (\langle 0 | + \bar{\theta}_{N-2} \langle 1 |) \\ &\quad \vdots \\ &\quad e^{-iH\Delta t} \int d\bar{\theta}_2 d\theta_2 (|0\rangle + |1\rangle \theta_2) e^{-\bar{\theta}_2 \theta_2} (\langle 0 | + \bar{\theta}_2 \langle 1 |) \\ &\quad e^{-iH\Delta t} \int d\bar{\theta}_0 d\theta_0 (|0\rangle + |1\rangle \theta_0) e^{-\bar{\theta}_0 \theta_0} \Psi_{\text{state}}(\bar{\theta}_0) \end{aligned}$$

$$\begin{aligned} \text{Let } \begin{pmatrix} \langle 0 | H | 0 \rangle & \langle 0 | H | 1 \rangle \\ \langle 1 | H | 0 \rangle & \langle 1 | H | 1 \rangle \end{pmatrix} &= \begin{pmatrix} H_{00} & H_{01} \\ H_{10} & H_{11} \end{pmatrix} \Rightarrow H = H_{00} b b^\dagger + H_{01} b + b^\dagger H_{10} + H_{11} b^\dagger b \\ &= H_{00} + H_{01} b + b^\dagger H_{10} + (H_{11} - H_{00}) b^\dagger b \end{aligned}$$

Evaluate the matrix elements (mod $(\Delta t)^2$ terms.)

$$\begin{aligned} &(\langle 0 | + \bar{\theta}_k \langle 1 |) e^{-iH\Delta t} (|0\rangle + |1\rangle \theta_{k-1}) \\ &\cong (1 + \bar{\theta}_k \theta_{k-1}) - i(\Delta t) (H_{00} + H_{01} \theta_{k-1} + \bar{\theta}_k H_{10} + H_{11} \bar{\theta}_k \theta_{k-1}) \\ &\cong \exp \left[\bar{\theta}_k \theta_{k-1} - i(\Delta t) \{ H_{00} + H_{01} \theta_{k-1} + \bar{\theta}_k H_{10} + (H_{11} - H_{00}) \bar{\theta}_k \theta_{k-1} \} \right] \\ &= \exp \left[\bar{\theta}_k \theta_{k-1} - i(\Delta t) H(b \rightarrow \theta_{k-1}, b^\dagger \rightarrow \bar{\theta}_k) \right] \end{aligned}$$

$$\begin{aligned} \text{So, } \Psi(\bar{\theta}_N @ t_N) &\cong \int d\bar{\theta}_{N-1} d\theta_{N-1} \dots d\bar{\theta}_1 d\theta_1 d\bar{\theta}_0 d\theta_0 \\ &\quad \exp \left[\sum_{k=0}^{N-1} (\bar{\theta}_{k+1} \theta_k) - \sum_{k=0}^{N-1} (\bar{\theta}_k \theta_k) - i(\Delta t) \sum_{k=0}^{N-1} H \left(\begin{matrix} b \rightarrow \theta_k \\ b^\dagger \rightarrow \bar{\theta}_{k+1} \end{matrix} \right) \right] \Psi_{\text{state}}(\bar{\theta}_0) \\ &\cong \int d\bar{\theta}_N \sim d\theta_0 \exp \left[i \int dt \{ (-i\partial_t \bar{\theta}) \theta - H(b \rightarrow \theta, b^\dagger \rightarrow \bar{\theta}) \} \right] \Psi_{\text{state}}(\bar{\theta}_0) \\ &= \int d\bar{\theta}_N \sim d\theta_0 \exp \left[i \int dt \bar{\theta} (i\partial_t \theta) - H(b \rightarrow \theta, b^\dagger \rightarrow \bar{\theta}) \right] \Psi_{\text{state}}(\bar{\theta}_0) \\ &\quad \left(\text{of } L = \bar{\theta} (i\partial_t \theta) - H. \right) \end{aligned}$$

⊙ How to set the bary condition in path integral after evolution.

just combine the "inner product" and "time evolution" p.4 & p.5.

$$\langle \text{state}' @ t_N | e^{-iHT} | \text{state} @ t_0 \rangle$$

$$= \int d\bar{\theta}_N d\theta_N d\bar{\theta}_{N-1} \sim d\theta_0 \Psi_{\text{state}' }^*(\theta_N) e^{i \sum_{k=0}^{N-1} (\bar{\theta}_{k+1} \theta_k - \bar{\theta}_k \theta_k) - i \Delta t \sum_{k=0}^{N-1} H(b \rightarrow \theta_k, b^\dagger \rightarrow \bar{\theta}_{k+1})} \Psi_{\text{state}}(\bar{\theta}_0)$$

⊙ More general operator matrix elements in path integral formulation.

Because

$$\langle 0 | + \bar{\theta}_k \langle 1 | e^{-iH\Delta t} b | 0 \rangle + | 1 \rangle \theta_{k-1} \rangle \underset{\text{mod } (\Delta t)^2}{\cong} \exp[-i(\Delta t)(H_{00} + \bar{\theta}_k H_{10})] \theta_{k-1}$$

$$= \exp[\bar{\theta}_k \theta_{k-1} - i(\Delta t) H(b \rightarrow \theta_{k-1}, b^\dagger \rightarrow \bar{\theta}_k)] \theta_{k-1}$$

$$\langle 0 | + \bar{\theta}_k \langle 1 | b^\dagger e^{-iH\Delta t} (| 0 \rangle + | 1 \rangle \theta_{k+1}) \rangle \underset{\text{mod } (\Delta t)^2}{\cong} \bar{\theta}_k \exp[-i(\Delta t)(H_{00} + H_{01} \theta_{k+1})]$$

$$= \bar{\theta}_k \exp[\bar{\theta}_k \theta_{k+1} - i(\Delta t) H(b \rightarrow \theta_{k+1}, b^\dagger \rightarrow \bar{\theta}_k)]$$

$$\langle \text{state}' @ t_N | T \left\{ e^{-i \int dt H} b_{at t_i}^\dagger b_{at t_j} \right\} | \text{state} @ t_0 \rangle \quad (\text{Schrodinger picture})$$

$$= \langle \text{state}' | T \left\{ b^\dagger(t_i) b(t_j) \right\} | \text{state} \rangle \quad (\text{Heisenberg picture})$$

$$= \int d\bar{\theta}_N \sim d\theta_0 \Psi_{\text{state}' }^*(\theta_N) e^{i \int dt \{ \bar{\theta}(\partial_t \theta) - H(b \rightarrow \theta, b^\dagger \rightarrow \bar{\theta}) \}} \bar{\theta}_i \theta_j \Psi_{\text{state}}(\bar{\theta}_0)$$

§ 1.3 Path integral formulation for quantum field theories

Example 1: a free scalar boson on $(d+1)$ -dimensional space-time

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi) (\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2 \iff \mathcal{H} = \frac{1}{2} \pi^2 + \frac{1}{2} (\partial_i \phi) (\partial_i \phi) + \frac{1}{2} m^2 \phi^2$$

$$[\phi(\vec{x}, t), \pi(\vec{y}, t)] = i \delta^d(\vec{x} - \vec{y})$$

This is just a quantum mechanics of infinitely many harmonic oscillators.

Individual oscillators are labeled by \vec{k} ($\omega = \sqrt{k^2 + m^2}$).

$$\begin{aligned} & \langle \Omega | T \{ \phi(x_1^+) \phi(x_2^+) \dots \pi(y_1^+) \pi(y_2^+) \dots \} | \Omega \rangle \\ &= \frac{\int \mathcal{D}\phi \mathcal{D}\pi e^{i \int dt \int d^d x (\pi \dot{\phi} - \mathcal{H} = \mathcal{L})} (\phi(x_1) \phi(x_2) \dots \pi(y_1) \pi(y_2) \dots)}{\int \mathcal{D}\phi \mathcal{D}\pi e^{i \int dt \int d^d x \mathcal{L}}} \end{aligned}$$

→ takes care of the normalization of the vacuum state $|\Omega\rangle$

Example 1': add interaction terms to \mathcal{H} . (eg. $+ = \frac{\lambda}{4!} \phi^4$)

→ just replace $(\pi \dot{\phi} - \mathcal{H}_0)$ by $(\pi \dot{\phi} - \mathcal{H})$

Example 2: Dirac theory $\mathcal{L} = \bar{\Psi} (i \gamma^\mu \partial_\mu - m) \Psi$

$$\begin{aligned} & \langle \Omega | T \{ \Psi(x_1) \bar{\Psi}(x_2) \dots \Psi(y_1) \bar{\Psi}(y_2) \dots \} | \Omega \rangle \\ &= \frac{\int \mathcal{D}\bar{\Psi} \mathcal{D}\Psi e^{i \int dt \int d^d x \mathcal{L}} (\Psi(x_1) \bar{\Psi}(x_2) \dots \Psi(y_1) \bar{\Psi}(y_2) \dots)}{\int \mathcal{D}\bar{\Psi} \mathcal{D}\Psi e^{i \int dt \int d^d x \mathcal{L}}} \end{aligned}$$

where we integrate over infinitely many Grassmann coordinates.
↳ (labeled by $\vec{k} \in \mathbb{R}^3$, $\Psi, \bar{\Psi}$ and time slices.)