

§ 2 What is QFT (for)?

probably too early to ask the question.

Quantum mechanics: a vector space \mathcal{H} over \mathbb{C} with a positive definite Hermitian inner product on \mathcal{H} .
set of linear operators acting on \mathcal{H} .

Quantum Mechanics

set of tools
(procedures).
thinking framework

Quantum Field Theory

so?

systems

applicable to

models./theories.

(free particle.
harmonic oscill.
diatomic molecule
!
finite DOFs.)

(many concrete
set-ups.)

[individual QFT's]

(see below)

A couple of characterizations of QFT

* a perspective in a quantum many-body system

A quantum mechanical system of N fermions with

$$H = \sum_{i=1}^N \left(-\frac{1}{2m} \partial_{x_i}^2 + \varphi(x_i) \right) + \sum_{i \neq j}^N \frac{C}{|x_i - x_j|} \quad \text{is "equivalent" to}$$

a QFT model with

$$\begin{aligned} \psi(x) &:= \sum_n \psi_n(x) a_n & \text{ann operator} & & \psi^\dagger(x) &:= \sum_n \psi_n^*(x) a_n^\dagger & \text{creat'n op} \\ \{a_n, a_m^\dagger\} &= \delta_{n,m} & & & \{\psi(x), \psi^\dagger(y)\} &= \delta(x-y) & \end{aligned}$$

$$H = \int dx \psi^\dagger(x) \left(-\frac{\partial_x^2}{2m} + \varphi(x) \right) \psi(x) + \int dx \int dy \psi^\dagger(x) \psi^\dagger(y) \frac{C}{|x-y|} \psi(x) \psi(y).$$

dictionary :

The many body
wavefunction

$$\Psi(x_1, x_2, \dots, x_N) = \frac{1}{\sqrt{N!}}$$

$$\begin{vmatrix} \psi_{n_1}(x_1) & \dots & \psi_{n_1}(x_N) \\ \vdots & \ddots & \vdots \\ \psi_{n_N}(x_1) & \dots & \psi_{n_N}(x_N) \end{vmatrix}$$

of a $|state\rangle$
 $a_{n_1}^\dagger a_{n_2}^\dagger \dots a_{n_N}^\dagger |0\rangle$
" "

is an operator matrix element $\langle 0 | \psi(x_N) \dots \psi(x_2) \psi(x_1) | state \rangle$.

→ can think of a broader class of observables. (*1)

$\Psi(x_1, \dots, x_N)$: regarded also as a result of comparing against
a set of states $(\langle 0 | \psi(x_N) \dots \psi(x_1) \rangle)_{x_1, \dots, x_N \in (\mathbb{R}^d)^N}$.

measurement : comparison against a well-established "things"

→ can be against any basis of \mathcal{H} (the vector space
of all the states)

$\langle e_i | state \rangle$ for $\text{Span}\{|e_i\rangle\} = \mathcal{H}$. (*2)

The probability interpretation of $\Psi(x_1, \dots, x_N) = \langle 0 | \psi(x_N) \dots \psi(x_1) | state \rangle$

is available only when the particle number is conserved.

The QFT description (*1, *2) does more than

the QM description using $\Psi(x_1, \dots, x_N)$.

The QFT approach to a many-body system often focuses on

each one of particles interacting with all others.

instead of studying the entire system.

★ Quantum theory of dynamical systems where a finite number of degrees of freedom are attributed to each point in space.
(a subclass of QM systems with infinitely many DOF's)
models

• ψ : fermion. $H = \int dx \psi^\dagger \left(\frac{-\nabla^2}{2m} + V(x) \right) \psi + \int dx \int dy \frac{c}{|x-y|} (\psi^\dagger \psi)(x) (\psi^\dagger \psi)(y)$

$L = \int dx \psi^\dagger (i\partial_t \psi) - H$

• ϕ : complex scalar. $L = \int dx \phi^\dagger \left(i\partial_t + \frac{\partial^2}{2m} - V(x) \right) \phi - f(|\phi|^2)$ ↗ translational sym. on space
is not mandatory in QFT.
e.g. cold atom system in a trap potential

• Hubbard model $L = \int dx \sum_{\sigma=\uparrow,\downarrow} \psi_\sigma^\dagger (i\partial_t \psi_\sigma) - H$

$H = \int dx \left\{ \sum_{\sigma=\uparrow,\downarrow} \left(\psi_\sigma^\dagger \frac{-\nabla^2}{2m} \psi_\sigma \right) + U (\psi_\uparrow^\dagger \psi_\uparrow) (\psi_\downarrow^\dagger \psi_\downarrow) \right\}$

$\psi_{\sigma=\uparrow,\downarrow}^\dagger$: 2-component fermion

• Klein-Gordon model. $L = \int dx \left\{ \frac{1}{2} (\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2 - V(\phi) \right\}$

There is nothing wrong in the quantum theory of the Klein-Gordon model.

Just do the canonical quantization.

• electron in a periodic potential (crystal)

$\psi_\sigma(x) = \sum_n \psi_n(x) a_n \rightarrow \int_{\text{Brillouin Zone}} \frac{d^d k}{(2\pi)^d} \sum_{b \in \text{bands}} \psi_b(x; k) a_{b,k,\sigma}$ $\psi_b(x; k)$: Bloch wavefunction of a state in the band "b" with the momentum $k \in \text{BZ}$.
 σ : spin \uparrow, \downarrow (B.Z.)

multiple bands: may define multiple field operators

$\psi_{b,\sigma}(x) := \int_{\text{Brillouin Zone}} \frac{d^d k}{(2\pi)^d} \psi_b(x; k) a_{b,k,\sigma}$
(multiple outer shell e's in a unit cell)

• electron-phonon system

• spin system

the Standard model of particle physics.

(1+3+8) vector fields. 45 2-component fermion fields
2 complex scalar fields.

• Quantum Electrodynamics

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \sum_j \bar{\Psi}_j \left\{ i \gamma^\mu (\partial_\mu + i e Q_j A_\mu) - m_j \right\} \Psi_j$$

Ψ_j : 4-component fermion field

• nucleon-meson system

⋮

Think of those models as dynamical systems with some DOF's labeled by points in space. Then follow the ordinary procedure of quantization. (use Dirac brackets if necessary: hw VI-1)

For those models (systems), we will wish to work out:

- what the ground state is like
- what the excitation spectrum is like
- how the system/states evolve in time
- how various operator expectation values are correlated

would be nice to have a tool box / thinking framework that can be applied to a broad class of models.

★ a tool (procedure) to compute the "rates" of

processes where the number of particles / particle species is not preserved

$$\text{e.g. } \left(\begin{array}{l} e^+ e^- \rightarrow \gamma \quad e^+ e^- \rightarrow q + \bar{q} \text{ (quark-antiquark)} \\ q + \bar{q} \rightarrow h + t + \bar{t} \\ (\text{proton}) + (\text{proton}) \rightarrow (\text{many hadrons}) \\ \text{photo-emission / absorption. } \dots \end{array} \right.$$

* a general theory constraining quantum systems on (space \times (time))
with basic principles.

The Hilbert space ^(physicists dialect) (= the vector space of all the states in a quantum system in consideration.)

does not necessarily look like a Fock space.

(a vector space spanned by
{ $|\text{vacuum}\rangle$, 1-particle excitat'n on $|\text{vac}\rangle$, 2-particle excitat'n on $|\text{vac}\rangle$, })

only in a system that can be regarded as that of weakly interacting particles.

• A Lagrangian is not always available.

• basic principles :

locality causality, unitarity ...

How far can we go with them?

Supplementary notes on spinor fields on $\mathbb{R}^{d,1} = \mathbb{R}^{3,1}$ Minkowski space-time.

SA

For any representation ρ of the group $SO(3,1)$, we can think of a field

$\phi(x^\mu)$ whose Lorentz transformation is given by

$$\phi(x^\mu) \longrightarrow \rho(g) \cdot \phi((g^{-1}x)^\mu) \quad g \in SO(3,1)$$

scalar fields: $\phi(x^\mu) \longrightarrow \phi(y^\mu)$ $\rho = \text{trivial repr.}$

vector fields: $\phi_\lambda(x^\mu) \longrightarrow g_\lambda{}^\kappa \phi_\kappa(y^\mu)$ $\rho = \text{defining repr.}$

$$y^\mu = (g^{-1})^\mu{}_\nu x^\nu$$

There are other representations of $Spin(3,1)$.

\rightarrow there are other kinds of fields.

Let $\eta^{\mu\nu} = \text{diag}(+, -, -, -)$ and $\gamma^{\mu\nu} = \begin{pmatrix} 0 & 1_{2 \times 2} \\ 1_{2 \times 2} & 0 \end{pmatrix}$ $\Rightarrow \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$

$$\gamma^{\mu\nu} = \begin{pmatrix} 0 & \tau^i \\ -\tau^i & 0 \end{pmatrix}$$

Then for $\Lambda^\rho{}_\sigma = \begin{pmatrix} 0 & \beta^1 & \beta^2 & \beta^3 \\ \beta^1 & 0 & 0^3 & 0^2 \\ \beta^2 & 0^3 & 0 & 0^1 \\ \beta^3 & 0^2 & 0^1 & 0 \end{pmatrix}$

$$(g)^\kappa{}_\lambda = (\exp[\Lambda])^\kappa{}_\lambda$$

$$\left(\Rightarrow \frac{1}{4} \Lambda_{\rho\sigma} [\gamma^\rho, \gamma^\sigma] = \begin{pmatrix} -\vec{\beta} \cdot \vec{\tau} + i\vec{0} \cdot \vec{\tau} & 0 \\ 0 & \vec{\beta} \cdot \vec{\tau} + i\vec{0} \cdot \vec{\tau} \end{pmatrix} \right)$$

$$\rho_{\text{spin}}(g) := \exp\left(\frac{1}{4} \Lambda_{\rho\sigma} [\gamma^\rho, \gamma^\sigma]\right) = \begin{pmatrix} \exp[-\frac{\vec{\beta} \cdot \vec{\tau} + i\vec{0} \cdot \vec{\tau}}{2}] & 0 \\ 0 & \exp[\frac{\vec{\beta} \cdot \vec{\tau} + i\vec{0} \cdot \vec{\tau}}{2}] \end{pmatrix}$$

$$=: \begin{pmatrix} \rho_L(g) & 0 \\ 0 & \rho_R(g) \end{pmatrix}$$

$\rho_{\text{spin}}, \rho_L, \rho_R$ are all representations of $Spin(3,1)$.

2-component	\mathbb{R} -component (Dirac) spinor fields	$\rho = \rho_{\text{spin}}$	$\psi(x) \rightarrow \rho_{\text{spin}}(g) \psi(y)$
	left handed spinor fields: $\rho = \rho_L$		$\chi(x) \rightarrow \rho_L(g) \chi(y)$
	right handed spinor fields: $\rho = \rho_R$		$\chi(x) \rightarrow \rho_R(g) \chi(y)$

$$(y^\mu = (g^{-1})^\mu{}_\nu x^\nu)$$

SB

The Dirac Lagrangian.

*D1

$$\mathcal{L}_{\text{Dirac}} = \bar{\Psi} (i\gamma^\mu \partial_\mu - m) \Psi$$

is known to be invariant under the Lorentz transformation.

$$(\bar{\Psi} := \Psi^\dagger \gamma^0)$$

γ^μ 's: 4×4 matrices.

$\Psi(x)$: 4-component spinor fields.

It is also possible to use a notation

$$\Psi = \begin{pmatrix} \psi \\ \chi \end{pmatrix} \begin{matrix} \text{both} \\ \text{2-component} \\ \uparrow \\ \text{4-component} \end{matrix} \quad \bar{\Psi} = (\psi^\dagger, \chi^\dagger) \begin{pmatrix} \mathbb{1}_{2 \times 2} \\ \mathbb{1}_{2 \times 2} \end{pmatrix} = (\chi^\dagger, \psi^\dagger)$$

*D2

$$\mathcal{L}_{\text{Dirac}} = \psi^\dagger i \overline{\sigma}^\mu \partial_\mu \psi + \chi^\dagger i \sigma^\mu \partial_\mu \chi - m \chi^\dagger \psi - m \psi^\dagger \chi$$

All the 4 terms are known to be individually Lorentz invariant.

$$\overline{\sigma}^\mu = (\mathbb{1}_{2 \times 2}, -\vec{\sigma})$$

$$\sigma^\mu = (\mathbb{1}_{2 \times 2}, \vec{\sigma})$$

$\mu=0, 1, 2, 3$

$\mu=0, 1, 2, 3$

lower-left block of γ^μ

upper-right block of γ^μ .

There is another way to write $\mathcal{L}_{\text{Dirac}}$.

Mathematically

$$[P_R(g)]^{c.c.} = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} P_L(g) \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}$$

(complex conjugate on each matrix element)

$$[P_L(g)]^{c.c.} = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix} P_R(g) \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

as 2×2 matrices.

\Rightarrow for a right handed spinor field χ

$$\begin{pmatrix} -1 & \\ & 1 \end{pmatrix} \chi^* =: \psi_\chi \text{ is a left-handed spinor field.}$$

for a left-handed spinor field ψ .

$$\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \psi^* =: \chi_\psi \text{ is a right-handed spinor field.}$$

So, it is also possible to use a notation

$$\psi = \begin{pmatrix} \psi \\ (-1) \psi_x^* \end{pmatrix} \quad \bar{\psi} = (\psi_x^T (-1), \psi^\dagger)$$

$$\boxed{\mathcal{L}_{\text{Dirac}} = \psi^\dagger i \bar{\sigma}^\mu (\partial_\mu \psi) + \psi_x^T i (-1) \sigma^\mu (-1) (\partial_\mu \psi_x^*)}$$

$$-m \psi_x^T (-1) \psi - m \psi^\dagger (-1) \psi_x^*$$

$$= \psi^\dagger i \bar{\sigma}^\mu (\partial_\mu \psi) + \psi_x^T i (\bar{\sigma}^\mu)^T (\partial_\mu \psi_x^*)$$

$$-m \psi_x^T (-1) \psi - m \psi^\dagger (-1) \psi_x^*$$

$$= \psi^\dagger i \bar{\sigma}^\mu (\partial_\mu \psi) + \psi_x^T i \bar{\sigma}^\mu (\partial_\mu \psi_x)$$

$$-m \psi_x^T (-1) \psi - m \psi^\dagger (-1) \psi_x^*$$

*D3

All the 4 terms are individually Lorentz invariant.

This Dirac Lagrangian (*D2) = (*D2) = (*D3) describes physics

of e^- and e^+ (or μ^- and μ^+) when $m = 511 \text{ keV}$ ($m = 105 \text{ MeV}$)

13C

The neutrino kinetic term (Majorana scenario)

Just use ^{one} 2-component spinor field χ

$$\boxed{\mathcal{L}_\nu = \chi^\dagger i \bar{\sigma}^\mu (\partial_\mu \chi) - \frac{m}{2} \chi^T (-1) \chi - \frac{m}{2} \chi^\dagger (-1) \chi^*}$$

*D1

$(-1) \chi^*$ is a right-handed spinor field.

(*D1) is obtained from (*D3) by setting $\chi_x = \chi$ and

rescaling $\chi \rightarrow \frac{1}{\sqrt{2}} \chi$.

(field redefinition)

e^+ and e^- are different so $\chi_x \neq \chi$.

there is just $\nu \rightarrow$ set $\chi_x = \chi$.

There is another way to write \mathcal{L}_D .

$$\bar{\Psi}_\nu := \begin{pmatrix} \psi \\ (i^{-1})\psi^* \end{pmatrix} \quad \bar{\Psi} = (\psi^\dagger \quad (-1)^2 \psi^\dagger)$$

$$\boxed{\mathcal{L}_D = \frac{1}{2} \bar{\Psi}_\nu (i \gamma^\mu \partial_\mu \Psi_\nu) - \frac{m}{2} \bar{\Psi}_\nu \Psi_\nu} \quad (*D2)$$

(*D2) is obtained from (*D1) by

imposing a condition

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \bar{\Psi}^* = \bar{\Psi} \quad \leftarrow \text{(Majorana condition)}$$

$$\Leftrightarrow \psi_x = \psi \Leftrightarrow \chi = (i^{-1})\psi^*$$

and rescaling the field

$$\left(\bar{\Psi} \text{ w/ Maj. cond.} \right) \rightarrow \frac{1}{\sqrt{2}} \bar{\Psi}_\nu$$

§D

The non-relativistic limit of \mathcal{L}_{Dirac}

→ described by a 2-component spinor field χ_e .

(not the same as ψ or χ on χ .)

$$\text{in } \bar{\Psi} = \begin{pmatrix} \psi \\ \chi \end{pmatrix}$$

→ (see week 07 lecture note
hw VII-1. and XIV-1.)

§E

In condensed matter applications,

electrons might be dealt with by multiple fields (see p.3 this week)

low-dim modes (e.g. surface modes) may or may not carry full $SO(3) \subset SO(3,1)$ spin DOF.

S-matrix etc. and how to compute them.

3.1

S-matrix, decay rate and cross section

Think of a case where interactions are turned on after $t=T_-$ and switched off by $t=T_+$.

H : the full Hamiltonian

H_0 : the bilinear part of H_0 $H = H_0 + V$

example (QED) $H_0 = -\bar{\Psi}(i\gamma^i \partial_i - m)\Psi + (\text{photon})$

$V = g\bar{\Psi}(\gamma^i e A_i + \gamma^0 e A_0)\Psi$

The Hilbert space of the free H_0 theory is generated by Fock states

$\{ |0\rangle, a_{n,\vec{p}}^\dagger |0\rangle, a_{n_1,\vec{p}_1}^\dagger a_{n_2,\vec{p}_2}^\dagger |0\rangle, \dots \}$ at $t=T_+$

For the interacting theory ($H_0 \rightsquigarrow H \rightsquigarrow H_0$), we can use

the in-states

$e^{-i\int_{T_-}^{T_+} dt H} e^{-iH_0(T_- - T_+)} \text{ on } \{ |0\rangle, a_{n,\vec{p}}^\dagger |0\rangle, a_{n_1,\vec{p}_1}^\dagger a_{n_2,\vec{p}_2}^\dagger |0\rangle, \dots \}$

as a basis of the Hilbert space.

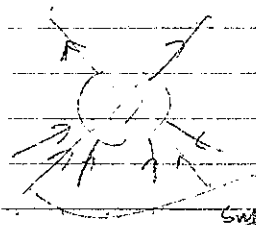
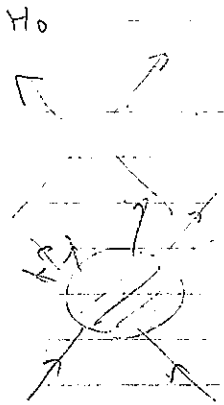
The "in-states" are denoted by

$|0\rangle, a_{n,\vec{p}}^\dagger |0\rangle, a_{n_1,\vec{p}_1}^\dagger a_{n_2,\vec{p}_2}^\dagger |0\rangle$
 $\propto |n,\vec{p}\rangle^{\text{in}} \quad \propto |n_1,\vec{p}_1; n_2,\vec{p}_2\rangle^{\text{in}}$

We can also take another basis:

the out-states

$e^{i\int_{T_+}^{T_-} dt H} e^{-iH_0(T_+ - T_-)} \text{ on } \{ |0\rangle, a_{n,\vec{p}}^\dagger |0\rangle, a_{n_1,\vec{p}_1}^\dagger a_{n_2,\vec{p}_2}^\dagger |0\rangle, \dots \}$



The "in-states" and "out-states" are normalized so that

$$\begin{aligned} \langle n_1, \vec{p}_1 | n_2, \vec{p}_2 \rangle^{in} &= \delta_{n_1, n_2} (2E_{\vec{p}_1}) \delta^3(\vec{p}_1 - \vec{p}_2) (2\pi)^3 \\ \langle n_1, \vec{p}_1 | n_2, \vec{p}_2 \rangle^{out} &= \delta_{n_1, n_2} (2E_{\vec{p}_1}) \delta^3(\vec{p}_1 - \vec{p}_2) (2\pi)^3 \end{aligned}$$

in relativistic situations.

(if non-rela. $\langle \vec{p}_1 | \vec{p}_2 \rangle^{in} = \delta^3(\vec{p}_1 - \vec{p}_2) (2\pi)^3$ is more conventional.)

The S'-matrix is defined by

$$S'_{\beta\alpha} := \langle \beta | \alpha \rangle,$$

α, β : Fock states

which measures the difference between the in-state basis and the out-state basis.

The truly scattering part of the S'-matrix

is in. $S'_{\beta\alpha} = \mathbb{1}_{\beta\alpha} + (2\pi)^4 \delta^4(p_{out} - p_{in}) i \mathcal{M}_{\beta\alpha}$

For a single particle : decay rate

$$d\Gamma = \frac{1}{(2E_{in})} \prod_{i=1}^{N_f} \left[\frac{d^3\vec{p}_i}{(2\pi)^3} \frac{1}{2E_{\vec{p}_i}} \right] (2\pi)^4 \delta^4(p_f - p_{in}) |\mathcal{M}|^2$$

Lorentz invariant

A boosted particle has a longer lifetime.

For a pair of colliding particles : cross section

$$d\sigma = \frac{1}{(2E_1)(2E_2)(|\vec{v}_1 - \vec{v}_2|)} \prod_{i=1}^{N_f} \left[\frac{d^3\vec{p}_i}{(2\pi)^3} \frac{1}{2E_{\vec{p}_i}} \right] (2\pi)^4 \delta^4(p_f - p_{in}) |\mathcal{M}|^2$$

$$|\vec{v}_1 - \vec{v}_2| := \left| \frac{\vec{p}_1}{E_1} - \frac{\vec{p}_2}{E_2} \right|$$

boost invariant.

when the two particles are moving in the same direction along the z-axis.

Q: Verify that $[P] = +1$ and $[\sigma] = -2$ for arbitrary N_f by dimension counting.

Q: Verify that $E_1 E_2 |\vec{v}_1 - \vec{v}_2|$ is boost invariant (along the collision axis).

Notes

$\frac{d^3p}{(2\pi)^3} \frac{1}{2E_p}$ is boost invariant.
 This is because $\begin{pmatrix} p'_z \\ E' \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} p_z \\ E \end{pmatrix}$, and $\frac{dp'_z}{E'} = \frac{dp_z \gamma (1 + \frac{p_z}{E}\beta)}{E'} = \frac{dp_z}{E}$.
 $(2E_p^3) \delta^3(\vec{p} - \vec{q})$ is also invariant. $\gamma = 1/\sqrt{1-\beta^2}$

A side remark

π^+ : $m \approx 140 \text{ MeV}$, $\tau \approx 2.6 \times 10^{-8} \text{ s}$. $\pi^+ \rightarrow \mu^+ + \nu_\mu$ (2-body)
 μ^+ : $m \approx 105 \text{ MeV}$, $\tau \approx 2.2 \times 10^{-6} \text{ s}$. $\mu^+ \rightarrow e^+ + \nu_e + \bar{\nu}_\mu$ (3-body)

§ 3.2. LSZ reduction formula

relates time-ordered correlations with the S -matrix.

§ 3.2.1 Källén - Lehmann spectral representation (+ Umezawa - Kamefuchi)

When a QFT (model) is built from ($\mathcal{H} \approx$ Fock space) + interactions

then \exists a clear relation between fields and 1-particle states.
(operators)

$$\varphi(x^{\mu}) = \sum_n \varphi_n(x^{\mu}) a_n \quad \langle 0 | \varphi(x) | n \rangle = \varphi_n(x)$$

When a QFT model is given first, on the other hand,

we do not know a priori which operators are associated with
a one particle state (even when a Fock space structure
is in the Hilbert space).

Example: QED: easy

$$\begin{aligned} \bar{\psi} &\leftrightarrow e^- \\ \psi^\dagger &\leftrightarrow e^+ \\ A_\mu &\leftrightarrow \gamma \end{aligned}$$

QCD: given by gluon field A_μ^a , quark fields $\bar{\psi}, \psi$
but $\pi^\pm, \pi^0, \rho, \dots$ come out as 1-particle
states at low-energy.

Quantum Hall system: similar issues.

A necessary condition for one particle states $\in \mathcal{H}$. (Hilbert space)

a family of states $|X(\vec{p})\rangle$; labeled by $\vec{p} \in \mathbb{R}^3$. (assuming $SO(3,1)$ symmetry
in this discussion)

$$\text{with } H |X(\vec{p})\rangle \approx \sqrt{(m_X^2 + \vec{p}^2)} |X(\vec{p})\rangle$$

for some m_X (indep. of \vec{p}) $\in \mathbb{R}_{\geq 0}$

An operator in a QFT model is

said to be "associated with" one particle states $|X(\vec{p})\rangle$

when ... (next page)

Consider $\int d^3y \int d^3x e^{ip \cdot y} e^{-iq \cdot x} \langle \Omega | T \{ A(y) B(x) \} | \Omega \rangle$. — (**)

$|\Omega\rangle$: the vacuum state in a interacting QFT model.

$A(x), B(x)$: operators in that QFT model.

Assume translational symmetry on $\mathbb{R}^{1,3}$

\Leftrightarrow Noether charge P^μ well-defined on \mathcal{H} .
and $P^\mu |\Omega\rangle = 0$.

$$\begin{aligned} &\langle \Omega | T \{ A(y) B(x) \} | \Omega \rangle \\ &= \langle \Omega | e^{-ip \cdot a} T \{ A(y) B(x) \} e^{ip \cdot a} | \Omega \rangle \\ &= \langle \Omega | T \{ e^{-ip \cdot a} A(y) e^{ip \cdot a} e^{-ip \cdot a} B(x) e^{ip \cdot a} \} | \Omega \rangle \\ &= \langle \Omega | T \{ A(y-a) B(x-a) \} | \Omega \rangle \quad \text{for any } a^\mu \in \mathbb{R}^{1,3}. \end{aligned}$$

So $\langle \Omega | T \{ A(y) B(x) \} | \Omega \rangle$ depends only on $(x-y)^\mu$.

$$\Rightarrow (**) = (2\pi)^4 \delta^4(p-q) \int d^3z e^{ip \cdot z} \langle \Omega | T \{ A(z) B(0) \} | \Omega \rangle.$$

How does $\int d^3z e^{ip \cdot z} \langle \Omega | T \{ A(z) B(0) \} | \Omega \rangle$ depend on p^μ ?

(***)

A partial contribution to (***)

from integration over $z^0 \in (\text{positive}, +\infty)$ region
with $|X(\vec{p})\rangle \langle X(\vec{p})|$ insertion.

$$\begin{aligned} \Delta_X(***) &= \int_{0^+}^{+\infty} dz^0 \int d^3z \int \frac{d^3k}{(2\pi)^3} \frac{1}{2E_{\vec{k}}} e^{ip \cdot z} \langle \Omega | A(z) | X(\vec{k}) \rangle \langle X(\vec{k}) | B(0) | \Omega \rangle \\ &= \int_{0^+}^{+\infty} dz^0 \int d^3z \int \frac{d^3k}{(2\pi)^3} \frac{1}{(2E_{\vec{k}}(z))} e^{ip \cdot z} \langle \Omega | e^{-ip \cdot z} A(0) e^{-ip \cdot z} | X(\vec{k}) \rangle \langle X(\vec{k}) | B(0) | \Omega \rangle \\ &= \int_{0^+}^{+\infty} dz^0 \int d^3z \int \frac{d^3k}{(2\pi)^3} \frac{1}{(2E_{\vec{k}})} e^{ip \cdot z} e^{-ik \cdot z} \langle \Omega | A(0) | X(\vec{k}) \rangle \langle X(\vec{k}) | B(0) | \Omega \rangle \end{aligned}$$

Noether charge P^μ operator.

$k^0 = \sqrt{m^2 + \vec{k}^2}$ (\vec{k} : on-shell condition imposed) PLUS

d^3z d^3k integrat'n

$$\Delta_X(\times\times) = \dots = \int_{0+}^{+\infty} dz^0 \frac{e^{i(p^0 - E_{\vec{p}})z^0}}{(2E_{\vec{p}})} \langle \Omega | A(0) | X(\vec{p}) \rangle \langle X(\vec{p}) | B(0) | \Omega \rangle$$

The $(i\epsilon)$ prescription (week-02 lecture)

time integration chosen in the direction $(-\infty, +\infty) \times (1-i\epsilon)$

to obtain $\langle \Omega | T \{ \dots \} | \Omega \rangle$.

for real valued $p^0 < E_{\vec{p}}$ the z^0 integration is convergent.

$$= \frac{i}{(2E_{\vec{p}})(p^0 - E_{\vec{p}})} \langle \Omega | A(0) | X(\vec{p}) \rangle \langle X(\vec{p}) | B(0) | \Omega \rangle$$

Similarly the integration over $z^0 \in (-\infty, \text{negative})$ contributes by

$$\begin{aligned} \Delta_X(\times\times) &= \int_{-\infty}^{0-} dz^0 \int d^3z \int \frac{d^3k}{(2\pi)^3} \frac{1}{(2E_{\vec{k}})} e^{ip \cdot z} e^{ik \cdot z} \langle \Omega | B(0) | X(\vec{k}) \rangle \langle X(\vec{k}) | A(0) | \Omega \rangle \\ &= \int_{-\infty}^{0-} dz^0 \frac{e^{i(p^0 + E_{\vec{p}})z^0}}{(2E_{\vec{p}})} \langle \Omega | B(0) | X(-\vec{p}) \rangle \langle X(-\vec{p}) | A(0) | \Omega \rangle \end{aligned}$$

The (z^0) integration toward $-\infty(1-i\epsilon)$ is convergent for real valued $(-E_{\vec{p}}) < p^0$.

$$= \frac{i}{(-2E_{\vec{p}})(p^0 + E_{\vec{p}})} \langle \Omega | B(0) | X(-\vec{p}) \rangle \langle X(-\vec{p}) | A(0) | \Omega \rangle$$

Källén-Lehmann spectral representation:

The time-ordered correlation function of two operators

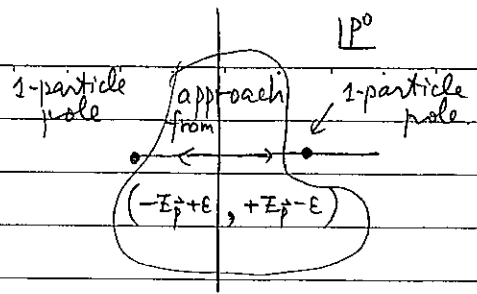
in the Fourier space $(\times\times)$ is expressed as a sum over excited states "associated" w/ the operators.

$$\left(\langle \text{excited state} \stackrel{\text{|| (def)}}{=} | A(0) | \Omega \rangle \neq 0 \right)_{\text{or } B}$$

A pole in the p^0 -plane \Leftrightarrow ^{an} associated one particle state

branch cuts \Leftrightarrow multi particle states

A technical note



We have arrived at the picture of finding poles on the real p^0 axis.

when approached from the open interval $(-E_p + \epsilon, +E_p - \epsilon)$.
[in the derivation in the previous 2 pages]

That is roughly right, but we can be a little more precise:

the inverse Fourier transform of $\Delta_X(\ast, \ast)$ should reproduce the

$$\left\{ \begin{array}{l} e^{-iE_p z^0} \quad (z^0 > 0) \\ e^{+iE_p z^0} \quad (z^0 < 0) \end{array} \right\}$$

contributions.

For that property to be satisfied,

pole of (\ast, \ast) at	$\left\{ \begin{array}{l} p^0 = +E_p - i\epsilon \\ p^0 = -E_p + i\epsilon \end{array} \right\}$	inv. F.T. contour	$(p^0) \in \mathbb{R}$.
or	pole of (\ast, \ast) at	$\left\{ \begin{array}{l} p^0 = +E_p \\ p^0 = -E_p \end{array} \right\}$	inv. F.T. contour $(p^0) \in \mathbb{R} \times (1+i\epsilon)$

When the operators $A(x)$, $B(y)$ are associated with a one-particle state $|X(\vec{p})\rangle$.

* the pole of $(***)$ comes with a residue $Z_X = \langle \Omega | A(\omega) | X(\vec{p}) \rangle \langle X(\vec{p}) | B(\omega) | \Omega \rangle$

The states $B(\omega) | \Omega \rangle / \sqrt{Z_X}$ contain one particle states $|X(\vec{p})\rangle$
 $\langle \Omega | A(\omega) / \sqrt{Z_X}$ $\langle X(\vec{p}) |$

with the appropriate normalization.

* Examples:

o free scalar particle: $A(x) = B(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{(2E_k)} \left(a_{\vec{k}} e^{-ik \cdot x} + a_{\vec{k}}^\dagger e^{ik \cdot x} \right)$

$$\Rightarrow Z = 1.$$

o weakly interacting fermions

$$L = \int d^3x \bar{\psi} \left(i \partial_t + \frac{\nabla^2}{2m} \right) \psi + \int d^3x d^3y (\bar{\psi} \psi)(x) \frac{c}{|x-y|} (\bar{\psi} \psi)(y)$$

$$\psi(x) = z_n \psi_n(x) a_n \quad \text{if } c=0.$$

When $c \neq 0$ but small.

F.T. $(\langle \Omega | T \{ \bar{\psi}(x) \psi(y) \} | \Omega \rangle)$ will still have contribution from fermionic one particle states (as in the case of $c=0$) but we should not expect $Z=1$.

o QCD.

$$\langle \rho \text{ meson} | \sum_f \bar{\psi}_f \gamma^\mu \psi_f | \Omega \rangle \neq 0.$$

= u.d.

$$\langle \pi^a \text{ meson} | \sum_{f, f'=u,d} (\bar{\psi}_f \gamma_5 (\tau^a)_{ff'} \psi_{f'}) | \Omega \rangle \neq 0.$$

↑
quark bilinear operators of QCD.

That is how one particle states in interacting models are characterized.

(intuition backed up by experiments) translated to (poles in op-op. correlath fns)

52. Plots of Cross Sections and Related Quantities

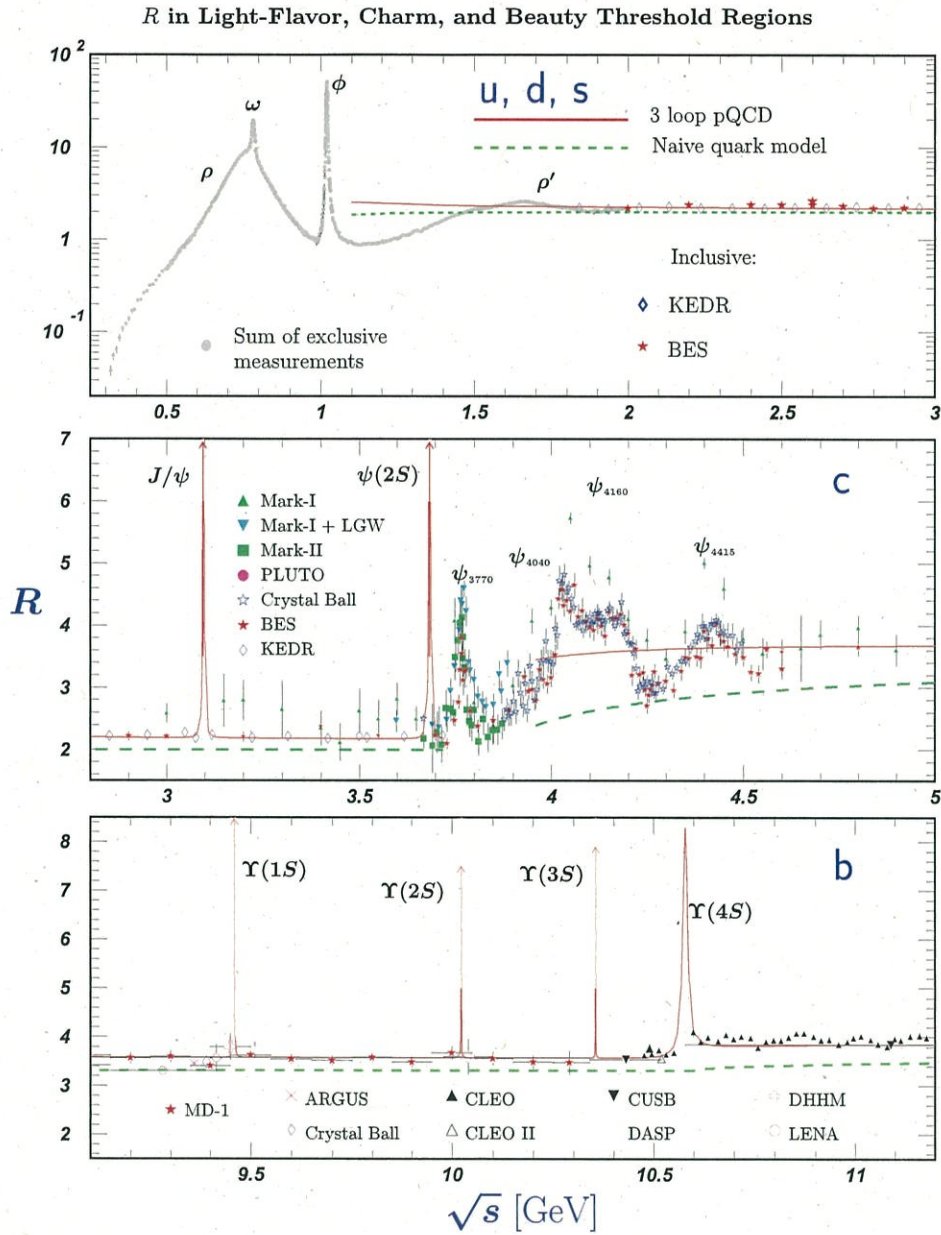


Figure 52.3: R in the light-flavor, charm, and beauty threshold regions. Data errors are total below 2 GeV and statistical above 2 GeV. The curves are the same as in Fig. 52.2. **Note:** CLEO data above $\Upsilon(4S)$ were not fully corrected for radiative effects, and we retain them on the plot only for illustrative purposes with a normalization factor of 0.8. The full list of references to the original data and the details of the R ratio extraction from them can be found in [100]. The computer-readable data are available at <http://pdg.lbl.gov/current/xsect/>. (Courtesy of the COMPAS (Protvino) and HEPDATA (Durham) Groups, August 2019.)