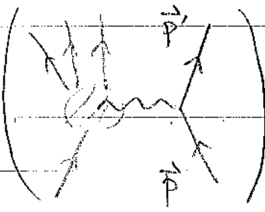
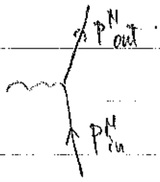
t-channel scattering by a vector field



When $\vec{p}' \sim \vec{p}$, $E_{\vec{p}'} \sim E_{\vec{p}}$ (small momentum transfer)

$$\boxed{\text{complex scalar} \quad \mathcal{L}_{int} = \left\{ (\partial_\mu \phi^*)(ieQ\phi) + (-ieQ\phi^*)(\partial_\mu \phi) \right\} A_\mu + \cdots}$$



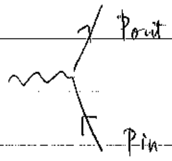
$$\Rightarrow i \times \Big\{ \underbrace{ieQ\, i p^\mu_{out}}_{(\partial_\mu \phi^*)} + (-ieQ)\underbrace{(-ip^\mu_{in})}_{(\partial_\mu \phi)} \Big\} = -ieQ\,(p_{in}+p_{out})^\mu$$

$$= -ieQ\, 2p^\mu_{out} \underset{p_{in}\sim p_{out}}{\Rightarrow} \boxed{-ieQ\, 2p^\mu}$$

$$\Rightarrow i \times \Big\{ ieQ\,\overset{\partial_\mu \phi^*}{(-ip^\mu_{in})} + (-ieQ)(ip^\mu_{out}) \Big\} = +ieQ\, 2p^\mu_{out}$$

$$\Rightarrow \boxed{+ieQ\, 2p^\mu}$$

$$\boxed{\text{Dirac spinor field.} \quad \mathcal{L}_{int} = (\bar{\Psi} i\gamma^\mu \Psi)\, ieQ A_\mu}$$



$$\Rightarrow \frac{i(\not{p}_{out}+m)}{p_{out}^2 - m^2}\, i(ieQ)\, i\gamma^\mu\, \frac{i(\not{p}_{in}+m)}{p_{in}^2-m^2} = \sum_{s,r} \frac{u_s(\vec{p}_{out})\, i}{p_{out}^2 - m^2}\, i^3 eQ \left(\bar{u}_s(\vec{p}_o)\gamma^\mu u_r(\vec{p}_i)\right) \times \frac{u_r(\vec{p}_{in})\, i}{p_{in}^2 - m^2}$$

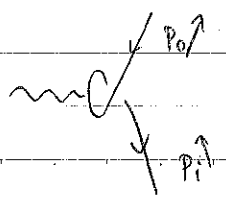
LSZ $\Rightarrow \quad i^3 eQ\left(\bar{u}_s(\vec{p}_{out})\gamma^\mu u_r(\vec{p}_{in})\right)$

When $\vec{p}_{out} = \vec{p}_{in} \parallel \hat{e}_z = p\,\hat{e}_z$

$$\Rightarrow -ieQ \Big(\underbrace{\xi^\dagger (p\cdot\sigma + p\cdot\bar{\sigma})\xi}_{\mu=0},\ \underbrace{+\xi^\dagger \left(\sqrt{p\cdot\sigma}\,\vec{\tau}\,\sqrt{p\cdot\bar{\sigma}} + \sqrt{p\cdot\bar{\sigma}}\,\vec{\tau}\,\sqrt{p\cdot\sigma}\right)\xi}_{\mu=i} \Big)$$

$$= -ieQ \left(2E_p\, \mathbb{1}_{2\times 2},\ \begin{pmatrix} -\sqrt{E-p}^{\,2} & \\ & \sqrt{E+p}^{\,2} \end{pmatrix} + \begin{pmatrix} \sqrt{E+p}^{\,2} & \\ & -\sqrt{E-p}^{\,2} \end{pmatrix} \right)$$

$$= -ieQ\left(2E_{\vec{p}},\ 2p\,\hat{e}_z\right) = \boxed{-ieQ\, 2p^\mu}$$



$$\Rightarrow -\frac{i(-\not{p}_i+m)}{p_i^2-m^2}\, i(ieQ)\, i\gamma^\mu\, \frac{i(-\not{p}_o+m)}{p_o^2-m^2} = \sum_{s,r} \frac{i\, v_r(\vec{p}_i)}{p_i^2-m^2}\, \frac{\bar{v}_s(\vec{p}_o)\, i}{p_o^2-m^2} \left(\bar{v}_r(\vec{p}_i)\, (-i^3 eQ)\gamma^\mu v_s(\vec{p}_o)\right)$$

$$+ieQ\left(\bar{v}_r(\vec{p}_i)\gamma^\mu v_s(\vec{p}_0)\right) = +ieQ\left(\xi_r^\dagger\begin{pmatrix}\sqrt{p_i\cdot\sigma}\sqrt{p_0\cdot\sigma} \\ +\sqrt{p_i\cdot\bar{\sigma}}\sqrt{p_0\cdot\bar{\sigma}}\end{pmatrix}\xi_s,\ \xi_r^\dagger\begin{pmatrix}-\sqrt{p_i\cdot\sigma}\ \vec{\tau}\ \sqrt{p_0\cdot\sigma} \\ +\sqrt{p_i\cdot\bar{\sigma}}\ \vec{\tau}\ \sqrt{p_0\cdot\bar{\sigma}}\end{pmatrix}\xi_s\right)$$

if $\vec{p}_i = \vec{p}_0 = p\hat{e}_z$

$$\Rightarrow +ieQ\left(\xi_r^\dagger\, 2E_p\, \xi_s,\ \xi_r^\dagger \xi_s\, 2p\hat{e}_z\right)$$

$$\boxed{+ieQ\ 2p^\mu\ \mathbb{1}_{2\times2}}$$

$\boxed{\text{Consider the } \boxed{\text{non-relativistic limit}} \text{ of the current in a scattering.}}$

$$\left[\bar{u}_r(\vec{p}\,')\gamma^\mu u_s(\vec{p})\right]$$ where the 4-momenta $p^\mu = (E_{\vec{p}}, \vec{p})$ and $p'^\mu = (E_{\vec{p}'}, \vec{p}\,')$ are both non-relativistic.

Now, in the basis where $\gamma^\mu = \begin{pmatrix} & \sigma^\mu \\ \bar{\sigma}^\mu & \end{pmatrix}$ $\quad \sigma^\mu = (\mathbb{1}, \vec{\tau})$, $\bar{\sigma}^\mu = (\mathbb{1}, -\vec{\tau})$ as in P.S.

$$u_s(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi_s \\ \sqrt{p\cdot\bar{\sigma}}\,\xi_s \end{pmatrix} \cong \begin{pmatrix} \sqrt{\left(M + \frac{\vec{p}^{\,2}}{2M} - \vec{p}\cdot\vec{\tau}\right)}\,\xi_s \\ \sqrt{\left(M + \frac{\vec{p}^{\,2}}{2M} + \vec{p}\cdot\vec{\tau}\right)}\,\xi_s \end{pmatrix}$$

$$\cong \sqrt{M} \begin{pmatrix} \left(1 - \frac{\vec{p}\cdot\vec{\tau}}{2M} + \frac{(\vec{p})^2}{8M^2}\right)\xi_s \\ \left(1 + \frac{\vec{p}\cdot\vec{\tau}}{2M} + \frac{(\vec{p})^2}{8M^2}\right)\xi_s \end{pmatrix}.$$

So,

- $$\left[\bar{u}_r(\vec{p}\,')\gamma^{\overset{\text{time}}{0}} u_s(\vec{p})\right] \cong M \times \left(\begin{array}{l} \xi_r^\dagger \left(1 + \frac{\vec{p}\,'\cdot\vec{\tau}}{2M} + \frac{(\vec{p}\,')^2}{8M^2}\right)\cdot \mathbb{1}_{2\times2}\cdot\left(1 + \frac{\vec{p}\cdot\vec{\tau}}{2M} + \frac{(\vec{p})^2}{8M^2}\right)\xi_s \\ + \xi_r^\dagger\left(1 - \frac{\vec{p}\,'\cdot\vec{\tau}}{2M} + \frac{(\vec{p}\,')^2}{8M^2}\right)\cdot\mathbb{1}_{2\times2}\cdot\left(1 - \frac{\vec{p}\cdot\vec{\tau}}{2M} + \frac{(\vec{p})^2}{8M^2}\right)\xi_s \end{array}\right)$$

$$\cong 2M\left\{\xi_r^\dagger\xi_s\left(1 + \frac{(\vec{p}\,')^2 + (\vec{p})^2}{8M^2}\right) + \xi_r^\dagger\frac{(\vec{p}\,'\cdot\vec{\tau})(\vec{p}\cdot\vec{\tau})}{4M^2}\xi_s\right\} \qquad + \mathcal{O}(|\vec{p}|^3)$$

$$= 2M\left\{\xi_r^\dagger\xi_s\left(1 + \frac{(\vec{p}_{av})^2}{2M^2}\right) + \frac{i(\vec{p}\,' - \vec{p})^i(\vec{p}_{av})^j}{2M^2}\,\varepsilon^{ijk}\left(\xi_r^\dagger\frac{\tau^k}{2}\xi_s\right)\right\} \qquad \vec{p}_{av} := \frac{\vec{p}\,' + \vec{p}}{2}$$

- $$\left[\bar{u}_r(\vec{p}\,')\gamma^{\underset{\text{space}}{i}} u_s(\vec{p})\right] \cong M\left\{ \begin{array}{l} \xi_r^\dagger\left(1 + \frac{\vec{p}\,'\cdot\vec{\tau}}{2M} + \cdots\right)\cdot\tau^i\cdot\left(1 + \frac{\vec{p}\cdot\vec{\tau}}{2M} + \cdots\right)\xi_s \\ - \xi_r^\dagger\left(1 - \frac{\vec{p}\,'\cdot\vec{\tau}}{2M} + \cdots\right)\cdot\tau^i\cdot\left(1 - \frac{\vec{p}\cdot\vec{\tau}}{2M} + \cdots\right)\xi_s \end{array}\right\}$$

$$\cong 2M\,\xi_r^\dagger\frac{(\vec{p}\,'\cdot\vec{\tau})\tau^i + \tau^i(\vec{p}\cdot\vec{\tau})}{2M}\xi_s$$

$$= 2M\left\{(\xi_r^\dagger\xi_s)\cdot\frac{p_{av}^i}{M} - i\varepsilon^{ijk}\frac{(\overset{(\vec{p}\,'-\vec{p})}{\Delta p})^j}{M}\left[\xi_r^\dagger\left(\frac{\tau^k}{2}\right)\xi_s\right]\right\}$$

$\Rightarrow$ In a t-channel scattering, a non-rela particle rarely changes its spin.

$\Rightarrow$ $[\bar{u}\gamma^\mu u]$ in the quantum scattering amplitude is approximately the classical $\boxed{2M \times u^\mu = 2\cdot p_{av}^\mu}$ apart from spin dep. corrections suppressed by $\left(\frac{\Delta p}{M}\right)$.

☆ Let us take the target (μ^+ now) mass M to be much larger than the incoming electron energy. (in the CM frame).

Then $\left\{\begin{array}{ll}[\bar{u}_{r'}(\vec{p}_{\mu,out})\gamma^\mu u_r(\vec{p}_{\mu,in})] & \text{target } \mu^- \\ [\bar{v}_r(\vec{p}_{\mu,in})\gamma^\mu v_{r'}(\vec{p}_{\mu,out})] & \text{taarget } \mu^+\end{array}\right\}$ is approximated by $2M_\mu\times(1,0,0,0)$,

and the photon propagator in the Feynmangauge

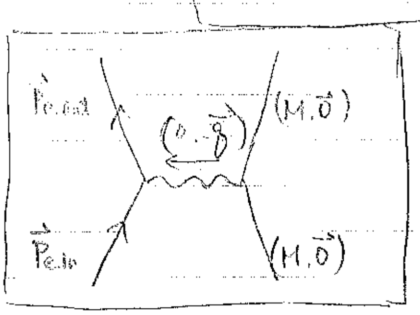
$$\left(\frac{-i\eta_{\mu\nu}}{t}=\frac{-i\eta_{\mu\nu}}{-|\vec{q}|^2}\right)$$

coupled with the current $\propto Q_\mu e\cdot(1,\vec{0})^\nu$

gives rise to the Fourier transform of the Coulomb potential

(as we are familiar with in Quantum Mechanics).

$\hookrightarrow$ regardless of the spin of the target particle.



If we sum over the spin of the final state e^-, and average over the spin of the initial state e^-,

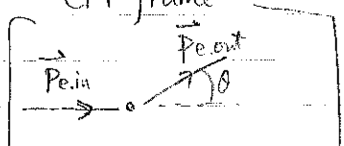
then

$$\frac{1}{2}\sum_{s,s'}\left|\mathcal{M}(e^-\mu^+\to e^-\mu^+)\right|^2 \cong \frac{(Q_\mu Q_e e^2)^2}{(|\vec{q}|^2)^2}\frac{(2M_\mu)^2}{2}\,\mathrm{Tr}_{\gamma's}\left[\gamma^0(\not{p}_{e,in}+m_e)\gamma^0(\not{p}_{e,out}+m)\right]$$

$$\cong \frac{e^4}{|\vec{q}|^4}(2M_\mu)^2\,2\left[p^0_{e,in}p^0_{e,out}+\vec{p}_{e,in}\cdot\vec{p}_{e,out}+m_e^2\right]$$

$$=\frac{e^4\,(2M_\mu)^2}{|\vec{p}_e|^4\left(2\sin\left(\frac{\theta}{2}\right)\right)^4}\cdot 4\left[E_e^2-|p_e|^2\sin^2\left(\frac{\theta}{2}\right)\right]$$

CM frame



$E_{e,in}\simeq E_{e,out}$
$|\vec{p}_{e,in}|\simeq|\vec{p}_{e,out}|$
((recoil of the target negligible))

The final state phase space is

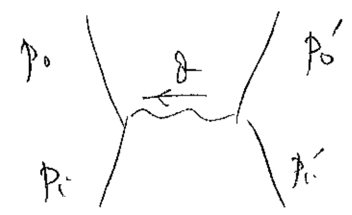
$$\frac{1}{2M\cdot 2E_e\,(p/E_e-0)}\int\frac{d^3\vec{p}_{e,out}}{(2\pi)^3}\frac{1}{2E_e}\frac{1}{2M_\mu}(2\pi)\,\delta(E_{e,out}-E_{e,in})$$

$$\cong\frac{d^2\Omega}{(4\pi)^2}\frac{1}{4M_\mu^2},$$

(Rutherford scattering)

So, $\boxed{\frac{d\sigma}{d\Omega}=\frac{\alpha_e^2}{4|p_e|^4\sin^4(\theta/2)}\left[E_e^2-p_e^2\sin^2(\theta/2)\right]}\ \overset{e\text{ non rela}}{\Longrightarrow}\ \boxed{\frac{\alpha_e^2\,m_e^2}{4|p_e|^4\sin^4(\theta/2)}}$

In the limit where both particles are non-relativistic...



$$i\mathcal{M}\,(2\pi)^4\delta^4(p_0+p_0'-p_i-p_i')$$

$$\cong 2m_e\, 2M_p\,(ieQ_e)(ieQ_p)\int d^4z\, d^4y \int e^{+i(p_0-q-p_i)\cdot z}\; e^{i(p_0'+q-p_i')\cdot y}\;\frac{(-i)}{q^2}\frac{d^4q}{(2\pi)^4}$$

(do $d^4\left(\frac{z+y}{2}\right)$) then...

$$i\mathcal{M}\,(2\pi)^4\delta^4(p_0+p_0'-p_i-p_i')$$

$$\cong (2\pi)^4\delta^4(p_0+p_0'-p_i-p_i')\int d^4z\int e^{i(p_0-p_i-q)\cdot z}\left(2m_e\, 2M_p\; ieQ_e\, ieQ_p\right)\frac{(-i)}{(q^0)^2-|\vec{q}|^2}\frac{d^4q}{(2\pi)^4}$$

center of mass frame $\Rightarrow \vec{p}_i+\vec{p}_i'=\vec{0} \Rightarrow \vec{p}_0+\vec{p}_0'=\vec{0}$.

$\Downarrow$ $\delta(p^0\text{'s})$ $\quad$ $(\delta^3(\vec{p}\text{'s}))$

non 0 only if $(p_0)^0=(p_i)^0$.

(do both $dz^0\,\frac{dq^0}{2\pi}$) $\qquad dt_z\, e^{-iq^0\cdot t_z}=(2\pi)\delta(q^0)$.

$$=(2\pi)^4\delta^3(\vec{p}_0'+\vec{p}_0)\,\delta(E_{p_0}-E_{p_i})\; 2m_e\, 2M_p\,(ieQ_e\; ieQ_p)$$

$$\int d^3\vec{z}\int\frac{d^3\vec{q}}{(2\pi)^3}\,\frac{i}{|\vec{q}|^2}\; e^{-i(\vec{p}_0-\vec{p}_i)\cdot\vec{z}}\; e^{i\vec{q}\cdot\vec{z}}$$

$$\boxed{\int\frac{d^3\vec{q}}{(2\pi)^3}\,\frac{e^{i\vec{q}\cdot\vec{z}}}{|\vec{q}|^2}=\frac{1}{4\pi|\vec{z}|}}$$

$$\boxed{\frac{\mathcal{M}}{2m_e\, 2M_p}=\int\frac{ieQ_e\; ieQ_p}{4\pi|\vec{z}|}\; e^{-i(\vec{p}_0-\vec{p}_i)\cdot\vec{z}}\, d^3\vec{z}}$$

$$\int d^3\vec{z}\begin{pmatrix}\text{out-state}\\ \text{wave fcn}\\ e^{i\vec{p}_0\cdot\vec{z}}\end{pmatrix}^{*}\begin{pmatrix}\text{the potential term}\\ \text{in Quant. Mech.}\\ \text{Hamiltonian}\end{pmatrix}\begin{pmatrix}\text{in-state}\\ \text{wave fcn}\\ e^{i\vec{p}_i\cdot\vec{z}}\end{pmatrix}$$

as in quantum mechanics!!

relativistic normalization

$$\boxed{\langle\vec{q}_{in}\,|\,\vec{p}_{in}\rangle=(2\pi)^3\delta^3(\vec{q}_{in}-\vec{p}_{in})\,2E_{\vec{p}}}$$

non-relativistic normalization

$$\boxed{\langle\vec{q}_{in}\,|\,\vec{p}_{in}\rangle=(2\pi)^3\delta^3(\vec{q}_{in}-\vec{p}_{in})}$$

$$\left(\int d^3\vec{z}\,\left(e^{+i\vec{q}\cdot\vec{z}}\right)^{*} e^{i\vec{p}\cdot\vec{z}}=(2\pi)^3\delta^3(\vec{q}-\vec{p})\right)$$

rescaling for both e^- and p^+ (or μ^+).

What if use a polarized e^--beam? (also non-rela.)

(The target μ^+ (or ion$^+$) is not necessarily polarized.)

[The e^--spin dependent term is found in

$$\left[\bar{u}_r \gamma^0 u_{r'}\right] \simeq 2m_e\left[\xi_r^\dagger \xi_{r'}\,(1+\cdots) + i\frac{(\vec{q}\times\vec{p}_{av})}{2m_e^2}\,\underbrace{\xi_r^\dagger\left(\frac{\vec{\sigma}}{2}\right)\xi_{r'}}_{\text{matrix element of the spin operator}} + \cdots\right].$$

But this term alone leads only to spin-dependent complex phase; after taking $|\mathcal{M}|^2$, it is gone.]

This $\xi_r^\dagger\left(\frac{\vec{\sigma}}{2}\right)\xi_{r'}$ term : $\vec{L}\cdot\vec{S}$ coupling.

The leading order term :

$$\Delta(i\mathcal{M}) = (2m_e)\cdot(2m_{(\mu)})\,(ieQ_{(e)})(-ieQ_{(\mu)})\,\xi_r^\dagger\xi_{r'}\cdot\frac{-i}{|\vec{q}|^2}$$

factor out $(2m_e)(2m_{(\mu)})$ to get the amplitude in the non-rela QM.

$$\Rightarrow \Delta(i\mathcal{M})_{\text{non-rela}}(\vec{q}) = -i\cdot\frac{(-e^2Q_{(e)}Q_{(\mu)})}{|\vec{q}|^2} \xrightarrow[\text{F.T.}\left(\int\frac{d^3\vec{q}}{(2\pi)^3}e^{i\vec{q}\cdot\vec{r}}\right)]{} -i\,\frac{e^2Q_{(e)}\cdot(-Q_{(\mu)})}{4\pi r} = -i\,\underbrace{V(r)}_{\text{Coulomb potential}}$$

Now.

$$\Delta(i\mathcal{M}) = (2m_e)(2m_{(\mu)})(ieQ_{(e)})(-ieQ_{(\mu)})\,\frac{-i}{|\vec{q}|^2}\,i\,\frac{(\vec{q}\times\vec{p}_{av})\cdot\langle\vec{S}\rangle}{2m_e^2}$$

$$\Rightarrow \Delta(i\mathcal{M})_{\text{non-rela}}(\vec{q}) = -i\,\frac{(-e^2Q_{(e)}Q_{(\mu)})}{|\vec{q}|^2}\cdot\frac{(i\vec{q}\times\vec{p}_{av})\cdot\langle\vec{S}\rangle}{2m_e^2} \xRightarrow{\text{F.T}}$$

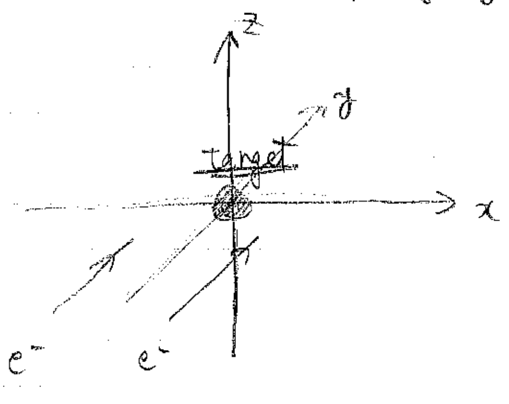
$$(\Delta V) = \frac{(\vec{\partial}_r\times\vec{p}_{av})\cdot\langle\vec{S}\rangle}{2m_e^2}\left(\frac{e^2Q_{(e)}(-Q_{(\mu)})}{4\pi r}\right)$$

$$= -\frac{\alpha\,Q_{(e)}(-Q_{(\mu)})}{2m_e^2\,r^3}(\vec{r}\times\vec{p}_{av})\cdot\langle\vec{S}\rangle$$

$$\Rightarrow +\frac{\alpha Z}{2m_e^2\,r^3}\,\vec{L}\cdot\langle\vec{S}\rangle$$

{ $Q_{(e)} = -1$.
$-Q_{(\mu)}$ = target charge $\Rightarrow Z > 0$ }

This $\vec{L}\cdot\vec{S}$ coupling gives rise to $\langle\vec{S}\rangle$ dependent potential.



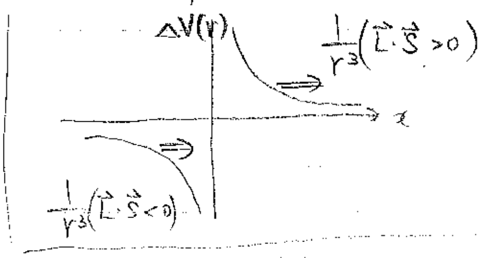
e^- approaching the target in the $0<x$ region

$\Rightarrow \vec{L}=(\vec{r}\times\vec{p}_{av})$ points to the $\hat{e}_z$ direction

e^- approaching the target in the $x<0$ region

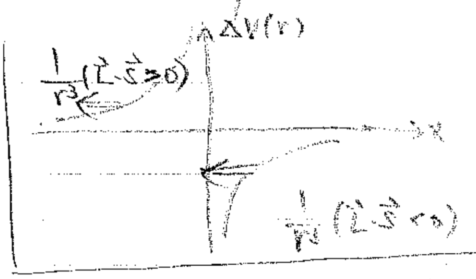
$\Rightarrow \vec{L}=(\vec{r}\times\vec{p}_{av})$ points to the $(-\hat{e}_z)$ direction.

If e^- has spin $\vec{S}\parallel\hat{e}_z$



the e^- receives a kick to the $\hat{e}_x$ direction whether it is in the $0<x$ region or in the $x<0$ region.

If e^- has spin $\vec{S}\parallel-\hat{e}_z$



the e^- receives a kick to the $-\hat{e}_x$ direction regardless of $0<x$ or $x<0$.

The asymmetry in the scattering of a polarized e^- beam in the Mott scattering is generated at 1-loop order (and beyond).

$iM = i(R+iI)$
$\Rightarrow |M|^2 = R^2+I^2$
sgn(I) irrelevant

The tree-level (Born approximation) is not enough.

$iM + \alpha(iM)^2 \sim (iM)(1+\alpha iM)$

$\hookrightarrow (1-\alpha I + i\alpha R) \leftarrow$ relevant here

$\boxed{\text{Exploited in spin-resolved photoemission spectroscopy.}}$

For more about this computation, see.

"Electron Scattering without Atomic or Nuclear Excitation"
by J.W. Motz, H. Olsen and H.W. Koch
Rev. Mod. Phys. 36 (1964) 881.

"Theoretical Study of $e^\pm$ scattering by the Au Atom"
by M. Khatun et al.
Results in Physics 28 (2021) 108742

J. Physics B: Atomic Molecular and Optical Physics.
vol 25 4281 (1992). M. Dümmler et al.

(data + theory)

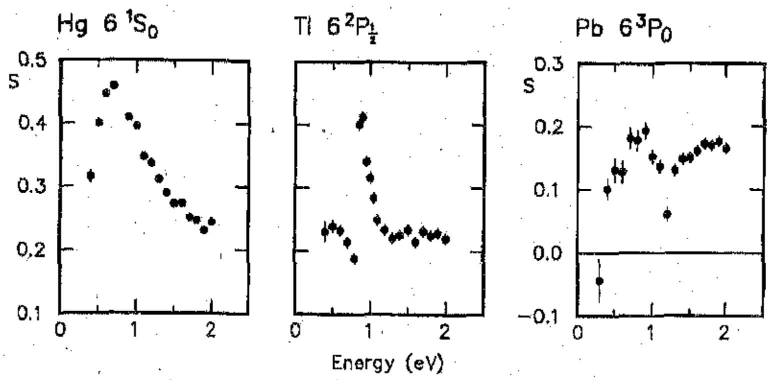


Figure 2. Energy scan of the Sherman function S for elastic collisions of electrons with Hg, Tl and Pb atoms. The scattering angle is $\theta = 60°$.

asymmetry (energy dependence)

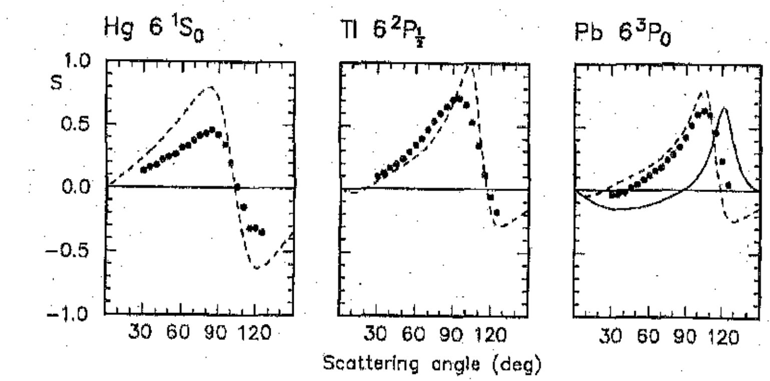
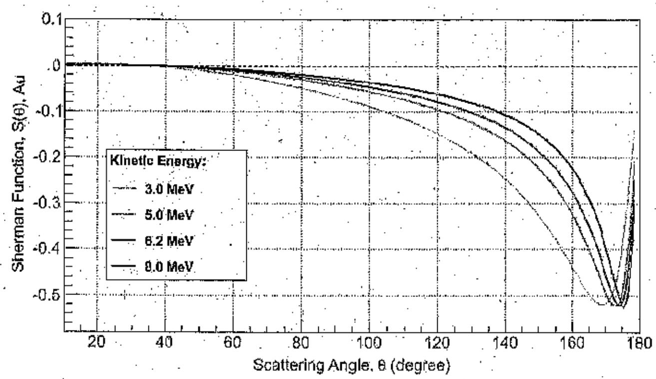
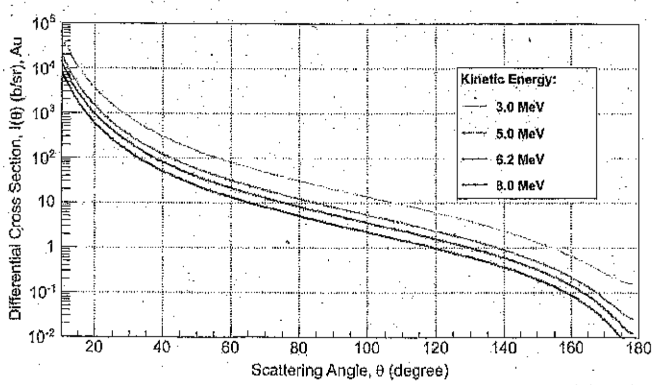


Figure 3. Sherman function S plotted against scattering angle for elastic collisions of electrons with Hg, Tl and Pb atoms at $E = 1.0$ eV. ●, experiment; ——, theory (Bartschat *et al* 1990a, b, Goeiss *et al* 1991a, b); – – –, theory (Fritsche *et al* 1992).

asymmetry (angle dependence)





taken from a presentation slide
by Riad Suleiman (2013 July)
(theory only (e⁻ around Au not included))