§ 6 Bound States

§ 6.1 Bethe – Salpeter equation

Consider non-relativistic particles

$$\mathcal{L} = \psi_a^\dagger \left(i\partial_t + \frac{\partial_x^2}{2m_a} - eQ_a\varphi - m_a \right) \psi_a$$

(non-rela. limit of Dirac fermion or complex boson)

Think of $e^- - p^+$, $e^- - \mu^+$, $e^- - e^+$ bound states.

(~~bound states of heavy quark : much the same~~)

(Cooper pair : much the same ; different in details)

Consider

$$\iiiint \langle \Omega | T\{ \psi_a(x_3) \psi_b(x_4) \psi_b^\dagger(x_2) \psi_a^\dagger(x_1) \} | \Omega \rangle \, e^{-ip_1\cdot x_1} e^{-ip_2\cdot x_2} e^{ip_3\cdot x_3} e^{ip_4\cdot x_4} \, d^4x_1\, d^4x_2\, d^4x_3\, d^4x_4$$

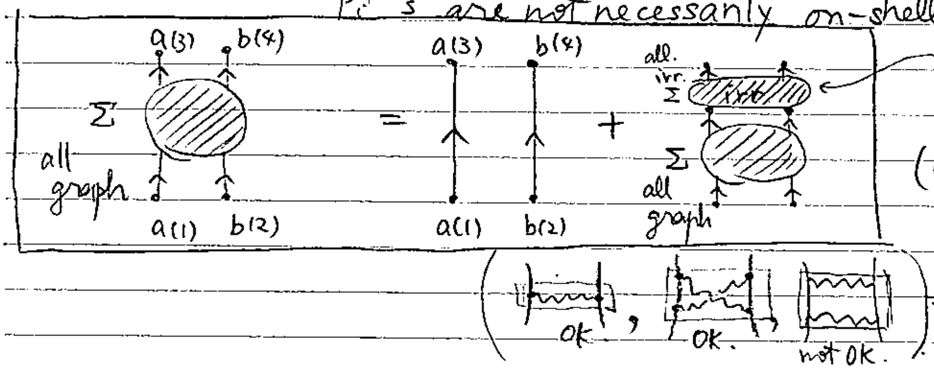
$$=: (2\pi)^4 \delta^4(p_1+p_2-p_3-p_4)\; G(P_{CM}^\mu ; p^\mu, p'^\mu)$$

Relabel the momenta:

$$\begin{cases} p_1^\mu = p^\mu + \eta_a P_{CM}^\mu \\ p_2^\mu = -p^\mu + \eta_b P_{CM}^\mu \end{cases} \qquad \begin{cases} p_3^\mu = p'^\mu + \eta_a P_{CM}^\mu \\ p_4^\mu = -p'^\mu + \eta_b P_{CM}^\mu . \end{cases}$$

$$\left(\eta_a = \frac{m_a}{m_a+m_b} \qquad \eta_b = \frac{m_b}{m_a+m_b} \right)$$

p_i^μ's are not necessarily on-shell.



Two-particle irreducible graphs

(def here)

- ~~external legs are not included.~~
- graphs where {a(3) or b(4)} remains connected to {a(1) or b(2)} even after arbitrary 2 propagators are cut

$$\boxed{\begin{aligned} G(P_{CM}^\mu ; p^\mu, p'^\mu) &= (2\pi)^4 \delta^4(p-p')\, D_a(P_{CM}, p')\, D_b(P_{CM}, p') \\ &+ D_a(P_{CM}, p')\, D_b(P_{CM}, p') \int \frac{d^4p''}{(2\pi)^4} K_{irr}(P_{CM} ; p', p'')\, G(P_{CM}^\mu ; p^\mu, (p'-p'')^\mu) \end{aligned}}$$

(Bethe – Salpeter eq.)

non-rela parametrization

$$p^0 \Rightarrow \omega \qquad (p')^0 = \omega' \quad ; \quad P_{CM}^0 = (m_a + m_b) + (\Delta E).$$

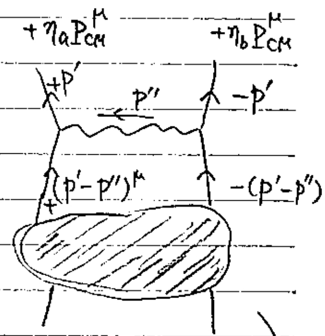
$$p_3'^{\mu} \Rightarrow \left(m_a + \eta_a (\Delta E) + \omega' , \eta_a \vec{P}_{CM} + \vec{p}' \right)$$

$$p_4'^{\mu} \Rightarrow \left(m_b + \eta_b (\Delta E) - \omega' , \eta_b \vec{P}_{CM} - \vec{p}' \right)$$

$$D_a^{(\text{Tree})} = \frac{i}{\left[\eta_a (\Delta E) + \omega' - \frac{(\eta_a \vec{P}_{CM} + \vec{p}')^2}{2 m_a} + i\varepsilon \right]}$$

$$D_b^{(\text{tree})} = \frac{i}{\left[\eta_b (\Delta E) - \omega' - \frac{(\eta_b \vec{P}_{CM} - \vec{p}')^2}{2 m_b} + i\varepsilon \right]}.$$

LO approximation to K_{irr}



in a photon exchange

$$K_{irr} = (-ieQ_a)(-ieQ_b) \frac{(-i)}{[(p'')^2 = (\omega'')^2 - (\vec{p}'')^2]}.$$

in a phonon exchange

$$\left(K_{irr} = \left(\frac{ig}{\Lambda} \right)^2 \frac{(+i) \quad (\vec{p}'' \cdot \vec{p}'')}{[(\omega'')^2 - v_s^2 (\vec{p}'')^2 + i\varepsilon]} . \longleftarrow \mathcal{L}_{int} = \frac{g}{\Lambda} (\vec{\partial} \cdot \vec{\varphi}) \psi^\dagger \psi \right)$$

Now, think of a case there are contributions of the form

$$G(P_{CM}^{\mu} ; p^{\mu}, p'^{\mu}) = \sum_n \left\{ \chi_n(p') \frac{i}{(P_{CM})^2 - M_n^2 + i\varepsilon} \chi_n^*(p ; \vec{P}_{CM}) \right\} + \left(\text{non-pole terms} \right).$$

$\Longleftrightarrow$ $\exists$ bound states

$$\langle \Omega | T \{ \psi_a(-p_3) \psi_b(-p_4) \} | n ; \vec{P}_{CM*} \rangle$$

$$= (2\pi)^4 \delta^3 (\vec{P}_{CM} - \vec{P}_{CM*}) \, \delta \left(m_a + m_b + (\Delta E) - E_{n, \vec{P}_{CM*}} \right) \cdot \chi_n(p')$$

$\vec{P}_{CM,*}$

Q: Verify that

$$\begin{cases} \left[\langle \Omega | \psi(x) \psi(x) \psi^\dagger(x) \psi^\dagger(x) | \Omega \rangle \right] = +6 \\ \left[G(P_{CM}^{\mu} ; p^{\mu}, p'^{\mu}) \right] = -6 \\ \left[\chi_n(p') \right] = -2. \end{cases}$$

$$\begin{cases} \left[\langle \Omega | \psi(x) \psi(x) | \text{state} \rangle \right] = +2 \\ \left[\langle \Omega | \psi(p) \psi(p) | \text{state} \rangle \right] = -6 \end{cases}$$

So, both the RHS & LHS of the eqn above have mass-dim -6. (sanity check)

Comparing the residue on a pole in the BS equation, we obtain

$$\chi_n(p') \cong D_a(\underline{\underline{\Delta E}}, \vec{P}_{CM}; p')\, D_b(\underline{\underline{\Delta E}}, \vec{P}_{CM}; p') \times \int \frac{d^3 p''}{(2\pi)^3} \int \frac{d\omega''}{2\pi} \frac{i(eQ_a\, eQ_b)}{(\omega'')^2 - |\vec{p}''|^2} \chi_n(p' - p'') \qquad \text{———} (*)$$

Suppose that we can ignore $(\omega'')^2$ against $|\vec{p}''|^2$ in the dominant region of the integral (verified later). —— (※)

Then we see that the $[\omega' = (p')^0]$-dependence in the RHS of (*) comes only from $D_a \cdot D_b$. So,

$$\chi_n(\omega', \vec{p}'; \vec{P}_{CM}) = \frac{i}{\left[\eta_a (\underline{\underline{\Delta E}})_n + \omega' - \frac{(\eta_a \vec{P}_{CM} + \vec{p}')^2}{2m_a} + i\varepsilon\right]} \frac{i \times \overset{\equiv}{\underline{\underline{\chi}}_n}(\vec{p}'; \vec{P}_{CM})}{\left[\eta_b (\underline{\underline{\Delta E}}_n) - \omega' - \frac{(\eta_b \vec{P}_{CM} - \vec{p}')^2}{2m_b} + i\varepsilon\right]},$$

$$\psi_n(\vec{p}'; \vec{P}_{CM}) := \int \frac{d\omega'}{2\pi} \chi_n(\omega', \vec{p}'; \vec{P}_{CM})$$

$$= \frac{-\underline{\underline{\chi}}_n(\vec{p}'; \vec{P}_{CM})}{2\pi} \frac{1}{\left[(\underline{\underline{\Delta E}})_n - \frac{\vec{P}_{CM}^2}{2(m_a+m_b)} - \frac{|\vec{p}'|^2}{2\mu_{ab}}\right]} \int d\omega' \left(\frac{1}{\left[\eta_a \underline{\underline{\Delta E}}_n + \omega' - \frac{(\ )^2}{2m_a} + i\varepsilon\right]} + \frac{1}{\left[\eta_b (\underline{\underline{\Delta E}}_n) - \omega' - \frac{(\ )^2}{2m_b} + i\varepsilon\right]} \right)$$

$$= \frac{-(-2\pi i)}{2\pi} \frac{\underline{\underline{\chi}}_n(\vec{p}'; \vec{P}_{CM})}{[\quad]} = \frac{(+i)}{\left[(\underline{\underline{\Delta E}})_n - \frac{\vec{P}_{CM}^2}{2(m_a+m_b)} - \frac{\vec{p}'^2}{2\mu_{ab}}\right]} \underline{\underline{\chi}}_n(\vec{p}', \vec{P}_{CM}).$$

Thus, the eqn (*) can be rewritten as

$$\int \frac{d^3 p''}{(2\pi)^3} \frac{(eQ_a\, eQ_b)}{|\vec{p}''|^2} \underset{\text{RHS}}{\psi_n(\vec{p}' - \vec{p}''; \vec{P}_{CM})} \overset{(*)}{\cong} \underset{\text{LHS}}{\frac{i}{D_a\, D_b} \chi_n(p')} = i\, \underline{\underline{\chi}}_n = \left(\underline{\underline{\Delta E}}_n - \frac{\vec{P}_{CM}^2}{2(m_a+m_b)} - \frac{(\vec{p}')^2}{2\mu_{ab}}\right) \psi_n(\vec{p}'; \vec{P}_{CM})$$

$$\Big\Downarrow \text{ Fourier transform in } \vec{p}' \to \vec{r}$$

$$\boxed{\left((\underline{\underline{\Delta E}})_n - \frac{\vec{P}_{CM}^2}{2(m_a+m_b)}\right) \tilde{\psi}_n(\vec{r}; \vec{P}_{CM}) = \left(\frac{-\vec{\partial}_r^{\,2}}{2\mu_{ab}} + \frac{e^2 Q_a Q_b}{4\pi r}\right) \tilde{\psi}_n(\vec{r}; \vec{P}_{CM})}$$

Schrödinger equation

$$\left(\frac{1}{\mu_{ab}} := \frac{1}{m_a} + \frac{1}{m_b} = \left(\frac{m_a m_b}{(m_a + m_b)}\right)^{-1} \text{ reduced mass} \right)$$

$\boxed{\chi_n(p') \text{ "is" the matrix element } \langle \text{vac} | T\{\psi_a \psi_b\} | \text{bound state}_n \rangle}$

(notation: χ_n, ψ_n as in LL4 or Takahashi; $\underline{\underline{\chi}}_n \sim \Gamma_{\text{two subscripts}}$ in LL4)

§ 6.2 Hydrogen atom spectroscopy in QED

Schrödinger eq : $\Delta E_n = -\frac{m_e \alpha^2}{2n^2}$

But.... • QED corrections to $\psi^\dagger_{(e)}\left(i\partial_t - m + \frac{\vec{\partial}^2}{2m} - eQ_e\varphi\right)\psi_{(e)}$ $\longrightarrow$ fine structure.

• proton also moves $\longrightarrow$ hyperfine structure.

• Kirr is not just  $\longrightarrow$ Lamb shift

§ 6.2.1 Fine structure.

electron Lagrangian

$-iEt + i p\cdot x$

$$\mathcal{L} = \bar{\Psi}\left\{i\gamma^\mu(\partial_\mu + ieQ_eA_\mu) - m_{e}\right\}\Psi, \quad \gamma^0 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} & \tau^i \\ -\tau^i & \end{pmatrix} \Rightarrow i\not{\partial} \sim \begin{pmatrix} E & -\vec{p}\cdot\vec{\tau} \\ \vec{p}\cdot\vec{\tau} & -E \end{pmatrix}$$

$\Rightarrow$ diagonalize by $\vec{p}$-dependent field redefinition.

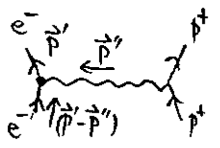
$$\Rightarrow \mathcal{L} \cong (\psi^\dagger_{e^-}, \psi^\dagger_{e^+})\begin{bmatrix} i\partial_t - m - eQ_e\varphi + \frac{(\vec{\partial})^2}{2m_e} + \frac{eQ_e(\vec{\partial}\cdot\vec{E})}{8m_e^2} + \frac{eQ_e(\vec{E}\times(-i\vec{\partial}))\cdot\vec{\tau}}{4m_e^2} + \frac{(\vec{\partial})^4}{8m_e^3} + \cdots & ☆ \\ ☆ & i\partial_t + m - eQ_e\varphi - \frac{(\vec{\partial})^2}{2m_e} + \frac{eQ_e(\vec{\partial}\cdot\vec{E})}{8m_e^2} - \frac{eQ_e(\vec{E}\times(-i\vec{\partial}))\cdot\vec{\tau}}{4m_e^2} + \cdots \end{bmatrix}$$

$$\times \begin{pmatrix} \psi_{e^-} \\ \psi_{e^+} \end{pmatrix} \qquad (\psi_{e^-}, \psi_{e^+} : \text{both 2-component spinor fields.})$$

[see homework VII-1 for more details] $\left(☆\text{'s} \sim \mathcal{O}\left(\frac{1}{m_e^2}\right)\right)$

Corrections : $$D_{e^-} = \frac{i}{\left[\eta_e(\Delta E) + \omega' - \frac{(\eta_a\vec{P}_{CM} + \vec{p}\,')^2}{2m_e} - \underbrace{\frac{(\eta_a\vec{P}_{CM} + \vec{p}\,')^4}{8m_e^3}} + i\varepsilon\right]}$$

$$K_{irr.} = (-ieQ_{p^+})(-ieQ_{e^-})\left\{1 + \underbrace{\frac{-|\vec{p}\,''|^2}{8m_e^2}} + \underbrace{\frac{i(\vec{p}\,''\times(\vec{p}\,'-\vec{p}\,''))\cdot\left(\frac{\vec{\tau}}{2}\right)}{2m_e^2}} + \cdots\right\}$$



At $\vec{P}_{CM} = \vec{0}$

$$(\underline{\underline{\Delta E}}_n)\tilde{\psi}_n(\vec{r}; \vec{P}_{CM} = \vec{0}) \cong \left\{\frac{-\vec{\partial}_r^2}{2\mu_{ab}} - \frac{\vec{\partial}_r^4}{8m_e^3} + \left(\frac{e^2(Q_{e^-}Q_{p^+})}{4\pi r}\right) - \frac{(eQ_{e^-})(\vec{\partial}\cdot\vec{E})}{8m_e^2} - \frac{(\vec{E}\times(-i\vec{\partial}))\cdot\frac{\vec{\tau}}{2}}{2m_e^2}\right\}\tilde{\psi}_n(\vec{r}; \vec{P}_{CM} = \vec{0})$$

At the leading order (Schrödinger eq.) $r \sim 1/m_e\alpha$ $\quad p \sim m_e\alpha$.

$\Rightarrow$ correction terms to $(\underline{\underline{\Delta E}}_n)$ are of order $(m_e\cdot\alpha^4)$.

$\vec{L}\cdot\vec{S}$ coupling $\longrightarrow$ $\underline{(L^2, S^2, J^2, J_z)\text{ eigenstates}}$

§ 6.2.2 Hyperfine structure

Let us now include $(\varphi, -\vec{A}) = A_\mu$ exchange not just φ.

$$\mathcal{L} = \psi_a^\dagger \left\{ i\partial t - m_a - eQ_a\varphi + \frac{(\vec{\partial}_i - ieQ_a\vec{A}_i)^2}{2m_a} - \left(\frac{g_a}{2}\right)\frac{\left(\frac{\tau^i}{2}\right)(\epsilon^{ijk}\partial_j A_k)e}{m_a} + \cdots \right\} \psi_a$$

$\left(g_a/2\right) = \underset{=+1}{-Q_e}$ for e^-. $\quad (g_a/2)$ for p^+ : just keep it as a parameter.

In addition to

$$K_{irr.}^{LO} \cong (-ieQ_{e^-})(-ieQ_{p^+})\frac{(-i)(+1)}{(-|\vec{q}|^2)}, \quad \text{now. we have}$$

$$\Delta K_{irr} \cong \frac{(-i)\times(-1)}{-|\vec{q}|^2} \times \left((ieQ_{e^-})\frac{(\vec{P}_{e.in} + \vec{P}_{e.out})}{m_e} + \frac{eg_e}{2m_e}(\vec{S}_e \times \vec{q})\right) \cdot \left((ieQ_{p^+})\frac{(\vec{P}_{p.in} + \vec{P}_{p.out})}{m_p} + \frac{eg_p}{2m_p}(\vec{S}_p \times (-\vec{q}))\right).$$

Two terms out of the four terms : don't do much (just correction (spherical or $\vec{S}_e \cdot \vec{L}$).

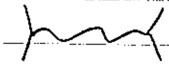
Two other terms : in terms of "potential"

$$\frac{\alpha}{r} \cdot \frac{(\vec{q} \times \vec{P}_e)}{m_e m_p} \cdot \vec{S}_p \sim \frac{\alpha}{r^3}\frac{\vec{L} \cdot \vec{S}_p}{m_e m_p} \quad \text{and} \quad e^2 \delta^3(\vec{r})\frac{\vec{S}_e \cdot \vec{S}_p}{m_e m_p}$$

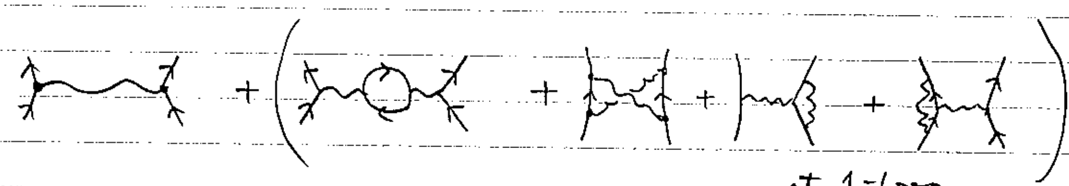
$$\text{extra}(\Delta E) \sim \frac{\alpha}{m_e m_p} \times (m_e\alpha)^3 \sim \frac{m_e^2}{m_p}\alpha^4$$

Hydrogen atom in the 1s state : 21 cm line $\left(\begin{array}{l} = 1.4 \text{ GHz} \\ = 5.9 \times 10^{-6} \text{eV} \end{array}\right)$.

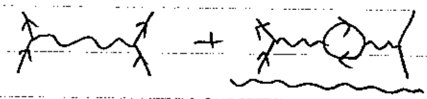
§ 6.2.3 Lamb shift

So far just one 2PI graph  has been taken into account.

But...

K_{irr} =  at 1-loop.

+ higher loop contributions

☆  modifies

$$K_{irr} \simeq (-ieQ_{e^-})(-ieQ_{p^+})\frac{i}{|\vec{q}|^2} \xrightarrow{} (-ieQ_{e^-})(-ieQ_{p^+})\left(\frac{i}{|\vec{q}|^2} + i\frac{\alpha}{15\pi}\frac{}{m_e^2}\right)$$

(approximation at $|\vec{q}| \ll m_e$
(say $|\vec{q}| \sim O(m_e\alpha)$))

need 1-loop computation, which has not been covered in this course yet.

$\Rightarrow$ modifies the Schrödinger eq. by

$$\frac{\alpha}{r} \longrightarrow \left(\frac{\alpha}{r} + \frac{4}{15}\frac{\alpha^2}{m_e^2}\delta^3(\vec{r})\right) \qquad (\Delta E) \text{ changes by } O\left(m_e\alpha^5/\pi\right)$$

☆ three other graphs : subtle treatment required.

[see Landau-Lifshitz vol.4 (QED) §123.]

still (ΔE) change by $O\left(m_e\frac{\alpha^5}{\pi} \times \ln(1/\alpha)\right)$

Summary

$\boxed{(\Delta E) \sim (m_e\alpha^2)}$ non-rela Schrödinger

$\to$ e^- relativistic correction (φ-exchange) $\to$ (fine structure) $\boxed{\text{extr } m_e\alpha^2 \times (\beta_{(e)}^2 \sim \alpha^2) \sim m_e\alpha^4}$

$\to$ 1-loop correction to K_{irr} $\to$ extra $\boxed{m_e\alpha^4 \times \frac{\alpha}{\pi}\ln(1/\alpha)}$ (Lamb shift)

$\to$ ($\vec{A}$-exchange) relativistic corr. both e^- & p^+ $\to$ (hyperfine structure) $\boxed{\text{extra } m_e\alpha^2 \times (\beta_{(e)} \cdot \beta_{(p)}) \sim m_e\alpha^2 \cdot \left(\alpha \cdot \frac{m_e}{m_p}\alpha\right)}$ $\sim \boxed{\frac{m_e^2}{m_p}\alpha^4}$