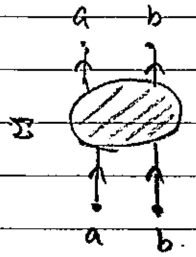
No.

Date

§ 6.3 BS wavefunction in a QFT process



$$=: (2\pi)^4 \delta^4(P'_{CM} - P_{CM})\, G(P_{CM}; p', p)$$

$$G(P_{CM}; p', p) = \sum_n \chi_n(p', \vec{P}'_{CM}) \frac{i}{(P_{CM}^2 - M_n^2 + i\varepsilon)} \chi_n^*(p, \vec{P}_{CM})$$

$$+ (\text{non-bound-state contribution})$$

$$\langle \Omega | T\{\psi_a(-p_3)\, \psi_b(-p_4)\} | n; \vec{P}_{CM}\rangle = (2\pi)^4 \delta^4(p_3 + p_4 - P_{CM})\, \chi_n(p', \vec{P}_{CM})$$

↑ BS wavefcn

$$\psi(p) := \int d^4x\, \psi(x)\, e^{-ip\cdot x}$$

Under the non-relativistic & weak coupling approximat'n

$$\chi_n(p', \vec{P}_{CM}) \cong \frac{i}{\left[\eta_a(\Delta E) + \omega' - \frac{(\eta_a \vec{P}_{CM} + \vec{p}')^2}{2m_a} + i\varepsilon\right]} \frac{i}{\left[\eta_b(\Delta E) - \omega' - \frac{(\eta_b \vec{P}_{CM} - \vec{p}')^2}{2m_b} + i\varepsilon\right]} \underline{\underline{\chi_n}}(\vec{p}', \vec{P}_{CM})$$

$$\psi_n(\vec{p}', \vec{P}_{CM}) := \int \frac{d\omega'}{2\pi} \chi_n(p', \vec{P}_{CM}) \cong \frac{i}{\left[\Delta E - \frac{(\vec{P}_{CM})^2}{2(m_a + m_b)} - \frac{(\vec{p}')^2}{2\mu_{ab}}\right]} \underline{\chi_n}(\vec{p}', \vec{P}_{CM})$$

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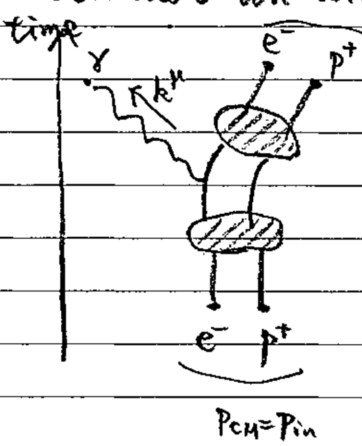
Consider an atomic transition process bnd state (i) → bnd state (f) + γ



$$i\mathcal{M} \times (2\pi)^4 \delta^4(p_{in} - p_{out} - k)$$

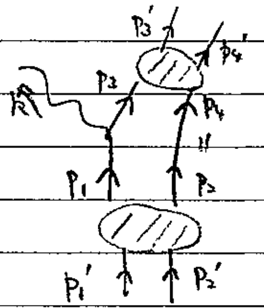
$$= \text{Residue}\left[\langle \Omega | T\{A_\mu(-k)\, \psi_{e^-}(-p_3)\, \psi_{p^+}(-p_4)\, \psi^\dagger_{p^+}(p_2)\, \psi^\dagger_{e^-}(p_2)\} | \Omega\rangle\right]$$

(LSZ formula)

residue of the poles $\left(\frac{1}{k^2}\right), \left(\frac{1}{P_{CM}^2 - M_i^2}\right), \left(\frac{1}{P_{CM}^2 - M_f^2}\right)$.

PLUS

set notation of the kinematics



$$\left[k^\mu = (k, \vec{k})\right] + \left[p_3^\mu = \left(m_e + \eta_e(\Delta E_f(-\vec{k})) + \omega', -\eta_e\vec{k} + \vec{p}'\right)\right] \quad \left[p_4^\mu = \left(m_p + \eta_p(\Delta E_f(-\vec{k})) - \omega', -\eta_p\vec{k} - \vec{p}'\right)\right]$$

$$\| \qquad\qquad\qquad\qquad \|$$

$$\left[p_1^\mu = \left(m_e + \eta_e(\Delta E_i) + \omega, \vec{0} + \vec{p}\right)\right] \qquad \left[p_2^\mu = \left(m_p + \eta_p(\Delta E_i) - \omega, \vec{0} - \vec{p}\right)\right]$$

from $\vec{p}_2^{\,\mu} = \vec{p}_4^{\,\mu}$: $\quad \vec{p} = \eta_p\vec{k} + \vec{p}'$

from $(p_1+p_2)^0 = (k+p_3+p_4)^0$: $\quad \underset{\text{neg.}}{(\Delta E_i)} = \underset{\text{pos.}}{k} + (\Delta E_f(-\vec{k})) \quad \longrightarrow (\Delta E_i) - (\Delta E_f(-\vec{k})) > 0$

even more neg.

~~from $p_2^0 = p_4^0$: $\quad \omega = \omega' + \eta_p\{(\Delta E_i) - (\Delta E_f(-\vec{k}))\} = \omega' + \eta_p k$~~

Now write down the amplitude by picking up the residues.

set. $\vec{P}_{CM.in} = \vec{0}$ $\quad$ (so $\vec{P}_{CM.out} = -\vec{k}$)

$$\Delta(i\mathcal{M}) = \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{d\omega}{2\pi}\, \chi_f^*(\omega', \vec{p}'; -\vec{k}) \left(\frac{i}{\eta_p(\Delta E_i) - \omega - \frac{(\vec{p})^2}{2m_p} + i\varepsilon}\right)^{-1} \chi_i(\omega, \vec{p}; \vec{0})$$

γ on e^- line.

$$\mathcal{L}_{int} \supset \frac{eQ_e}{2m_e}\vec{A}\cdot(\vec{P}_{e.in} + \vec{P}_{out}) \longrightarrow \times \left(\frac{eQ_e}{2m_e}\, i(\vec{P}_{e.in} + \vec{P}_{e.out})\cdot\vec{\varepsilon}^{\,*}(-\vec{k})\right)$$

$$\vec{P}_{e.in} = \eta_e\cdot\vec{0} + \vec{P} \qquad \vec{P}_{e.out} = -\eta_e\vec{k} + \vec{p}' = \vec{p} - \vec{k}$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3}\left(\frac{eQ_e}{2m_e}\vec{\varepsilon}^{\,*}(-\vec{k})\cdot(\vec{p}2 - \vec{k})\right) \underline{\chi_f^*}(\vec{p} - \eta_p\vec{k}; -\vec{k})\ \underline{\chi_i}(\vec{p}; \vec{0})$$

$$\times \int \frac{d\omega}{2\pi} \frac{1}{\left(\eta_e(\Delta E_i) + \omega - \frac{(\vec{p})^2}{2m_e} + i\varepsilon\right)\left(\eta_e(\Delta E_f(-\vec{k})) + (\underset{\omega'}{\underbrace{\omega - \eta_p k}}) - \frac{(\vec{p}-\vec{k})^2}{2m_e} + i\varepsilon\right)\left(\eta_p(\Delta E_i) - \omega - \frac{(\vec{p})^2}{2m_p} + i\varepsilon\right)}$$

~~$(-\eta_e\vec{k}) + \vec{p}' = \vec{p} - \vec{k}$~~

The ω-integration is straightforward by using the following straightforward calculation

$$\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\frac{1}{(\omega+A+i\epsilon)(\omega+B+i\epsilon)(-\omega+C+i\epsilon)}$$

$$=\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\frac{-1}{(A-B)(A+C)(B+C)}\left\{\frac{(B+C)}{(\omega+A+i\epsilon)}-\frac{(A+C)}{(\omega+B+i\epsilon)}+\frac{(A-B)}{(\omega-C-i\epsilon)}\right\}$$

$$=(\text{log divergence})\times\frac{\{-(B+C)+(A+C)-(A-B)=0\}}{(A-B)(A+C)(B+C)}+\frac{(-\pi i)}{2\pi}\frac{-(B+C)+(A+C)+(A-B)}{(A-B)(A+C)(B+C)}$$

$$=\frac{(-i)}{(A+C)(B+C)}$$

$$\Delta(i\mathcal{M})=\int\frac{d^3\vec{p}}{(2\pi)^3}\left(\frac{-ieQ_e}{2m_e}\vec{\epsilon}^*(-\vec{k})\cdot(2\vec{p}-\vec{k})\right)\frac{\chi_i(\vec{p};\vec{0})}{\left[(\Delta E_i)-\frac{(\vec{p})^2}{2\mu_{ep}}\right]}\frac{\chi_f^*(\vec{p}',-\vec{k})}{\left[(\Delta E_f(-\vec{k}))-\frac{(\vec{k})^2}{2(m_e+m_p)}-\frac{(\vec{p}')^2}{2\mu_{ep}}\right]}$$

$$=\int\frac{d^3\vec{p}}{(2\pi)^3}\left(\frac{ieQ_e}{2m_e}\vec{\epsilon}^*(-\vec{k})\cdot(2\vec{p}-\vec{k})\right)\psi_f^*(\underset{(\vec{p}-\eta_p\vec{k})}{\vec{p}'};-\vec{k})\,\psi_i(\vec{p};\vec{0})$$

$$=ieQ_e\;2(m_e+m_p)\int\frac{d^3\vec{p}}{(2\pi)^3}\psi^*_{NR.f}(\vec{p}-\eta_p\vec{k};-\vec{k})\frac{\vec{\epsilon}^*(-\vec{k})\cdot(2\vec{p}-\vec{k})}{2m_e}\psi_{NR.i}(\vec{p};\vec{0})$$

$$\left([i\mathcal{M}]=\quad +1\quad +3\quad -3/2\quad\quad 0\quad -3/2\;=+1\right)$$

as expected in $1\to 2$ body decay

Roughly speaking, $i\mathcal{M}\sim m_H\times 2ieQ_e\,\vec{\epsilon}^*\cdot\langle\vec{v}\rangle$. $\quad\langle\vec{v}\rangle\sim\langle f|\frac{\vec{p}}{m_e}|i\rangle$

Further evaluation

☆ Because $k\sim\Delta E_i-\Delta E_f\sim O(m_e\alpha^2)$

$\vec{p}\sim 1/r_B\sim m_e\alpha$ so $k\ll|\vec{p}|$.

the $\vec{p}$-integral may be approximated

$$\left(\begin{array}{l}\cdot\ (2\vec{p}-\vec{k})\Rightarrow 2\vec{p}\\ \cdot\ \chi_f^*(\vec{p}-\eta_p\vec{k};-\vec{k})\Rightarrow\chi_f^*(\vec{p};\vec{0})\end{array}\right)$$

to be. $\displaystyle\int\frac{d^3\vec{p}}{(2\pi)^3}\psi^*_{NR.f}(\vec{p})\frac{\vec{\epsilon}^*\cdot\vec{p}}{m_e}\psi_{NR.i}(\vec{p})=\int d^3\vec{x}\,\psi^*_{NR.f}(\vec{x})\frac{\vec{\epsilon}^*\cdot(-i\vec{\partial})}{m_e}\psi_{NR.i}(\vec{x})$.

☆ $\displaystyle\frac{\vec{p}}{m_e}=-i[\vec{x}_e,H]$ so $\langle f|\frac{\vec{p}}{m_e}|i\rangle=-i\langle f|[\vec{x},H]|i\rangle=-i(\Delta E_i-\Delta E_f)\langle f|\vec{x}|i\rangle$.

$\left(\vec{x}=i\frac{\partial}{\partial\vec{p}}\right)$

so,

$$\Delta(i\mathcal{M}) \cong 2m_H \big(\overset{k}{\overset{\parallel}{(\Delta E_i) - (\Delta E_f)}}\big) \int d^3\vec{x}\ \psi^*_{NR,f}(\vec{x})\, (eQ_e \vec{x})\, \psi_{NR,i}(\vec{x}) \cdot \big(\vec{\varepsilon}^{\,*}(-\vec{k})\big)$$

$$= 2m_H\ k\ \langle f | \vec{d} | i \rangle \cdot \vec{\varepsilon}^{\,*}(-\vec{k})$$

↑ electric dipole $eQ_e \langle \vec{x} \rangle$ matrix element (off-diagonal)

called dipole approximation.

To bring all together

$$\Delta(d\Gamma) \cong \frac{1}{2m_H} \underbrace{\frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{2k} \frac{1}{(2E_f)} (2\pi)\delta(k - ((\Delta E_i) - (\Delta E_f)))}\ |\mathcal{M}|^2$$

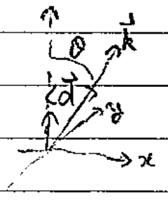
bnd state (i) at rest ↑ initial state; final state phase space & 4-momentum conservat'n

$$\cong \frac{(2m_H)^2}{(2m_H)^2} \frac{d^2\Omega(\hat{k})}{(2\pi)^3}\, dk\, \delta(k - \cdots)\, \frac{k^2}{k}\, \pi \times k^2 \left|\vec{\varepsilon}^{\,*} \langle \vec{d} \rangle\right|^2$$

$$= \frac{d^2\Omega}{8\pi^2} k^3 \left|\vec{\varepsilon}^{\,*} \cdot \langle \vec{d} \rangle\right|^2 \sim \alpha k^3 r_B^2 \sim \alpha \times (m_e\alpha^2)^3 \times \frac{1}{(m_e\alpha)^2} \sim m_e \alpha^5.$$

↑ dipole formula.

The decay width $\Gamma \sim m_e\alpha^5$ is smaller than the fine structure splitting $O(m_e\alpha^4)$ but is larger than the hyperfine splitting.



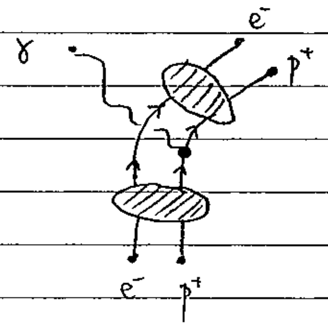
memo: $\Delta(d\Gamma) = \int \frac{d^2\Omega}{8\pi^2} k^3 |\langle \vec{d} \rangle|^2 \left\{ (\vec{\varepsilon}_x \cdot \hat{e}_z)^2 + (\vec{\varepsilon}_y \cdot \hat{e}_z)^2 \right\}$ with $\begin{pmatrix} \vec{\varepsilon}_x = \cos\theta\, \hat{e}_x - \sin\theta\, \hat{e}_z \\ \vec{\varepsilon}_y = \hat{e}_y \\ \vec{k} \cdot \vec{\varepsilon} = 0 \end{pmatrix}$

$$= \int \frac{d\phi\, d\theta \sin\theta}{8\pi^2} k^3 |\langle \vec{d} \rangle|^2 \{\sin^2\theta + 0\}$$

$$= \int_{-1}^{+1} \frac{d\cos\theta}{4\pi} (1 - \cos^2\theta)\, k^3 |\langle \vec{d} \rangle|^2 = \frac{|\langle \vec{d} \rangle|^2}{3\pi} k^3 = \frac{4\alpha}{3} |\langle \vec{r} \rangle|^2 k^3$$

polarized photon emitted for a fixed $\langle d \rangle$

— Another contribution



is evaluated similarly.

It turs out that this contribution is for

$$\frac{ieQ_e}{2m_e}\vec{\varepsilon}^*(-\vec{k})\cdot(2\vec{p}-\vec{k}) \Longrightarrow \frac{ieQ_p}{2m_p}\vec{\varepsilon}^*(-\vec{k})\cdot(2\vec{p}+\vec{k}).$$

So we can ignore this contribution due to $\left(m_e/m_p\right) \ll 1$.

— The selection rule in the electric dipole transition processes:

$\langle f|\vec{d}|i\rangle \neq 0$ only when $L_f \otimes (L=\text{spin-}1) \otimes L_i$ irreducible decomposition of $SU(2) \simeq SO(3)$ space rotation contains the $(L=0)$ component.

$$\Rightarrow L_i \otimes (L=1) = \begin{cases} (L_i+1)\oplus(L_i)\oplus(L_i-1) & \text{if } L_i \geq 1 \ \left(\text{p-wave or higher}\right) \\ (L=1) & \text{if } L_i=0 \ (\text{s-wave.}) \end{cases}$$

↑ on spherical harmonics (not about earthquakes!)

⇒ If the initial bound state is in an s-wave state (e.g. 2s, 3s, etc.)

then the dipole transition is possible only to a p-wave state. (2s: nowhere to go)

If the initial bound state is not in an s-wave state (e.g. 2p, 3p, 3d, ...)

then the dipole transition is possible when $|L_f - L_i| \leq 1$.

$$\Delta(i\mathcal{M}) = \int \frac{d\vec{p}}{(2\pi)^3}\frac{d\omega}{2\pi}\, \chi_f^*(\omega', \vec{p}\,'; -\vec{k}) \left(\frac{i}{\eta_e(\Delta E_i) + \omega - \frac{(\vec{p})^2}{2m_e} + i\varepsilon} \right)^{-1} \chi_i(\omega, \vec{p}\,; \vec{0})$$

↑ γ to p^+ line

$$\times \left(\frac{eQ_p}{2m_p}\, i(\vec{p}_{p.in} + \vec{p}_{p.out}) \cdot \vec{\varepsilon}^{\,*}(-\vec{k}) \right)$$

$$\vec{p} = -\eta_e \vec{k} + \vec{p}\,' \qquad \omega = \omega' - \eta_e\{(\Delta E_i) - \Delta E_f(-\vec{k})\} = \omega' - \eta_e k.$$

$$\vec{p}_{p.in} = \vec{0} - \vec{p} \qquad \vec{p}_{p.out} = -\eta_p \vec{k} - \vec{p}\,' = -\eta_p \vec{k} - \eta_e \vec{k} - \vec{p} = -\vec{k} - \vec{p}$$

$$(\vec{p}_{p.in} + \vec{p}_{p.out}) = (-2\vec{p} - \vec{k})$$

$$= \int \frac{d^3 p}{(2\pi)^3} \left(\frac{eQ_p}{2m_p} \vec{\varepsilon}^{\,*}(-\vec{k}) \cdot (-2\vec{p} - \vec{k}) \right) \chi_f^*(\vec{p} + \eta_e \vec{k}\,; -\vec{k})\ \chi_i(\vec{p}\,; \vec{0})$$

$$\int \frac{d\omega}{2\pi} \frac{1}{\left(\eta_p(\Delta E_i) - \omega - \frac{(\vec{p})^2}{2m_p} + i\varepsilon\right)\left(\eta_p(\Delta E_f(-\vec{k})) - \underset{\hookrightarrow (\omega + \eta_e k)}{\omega'} - \frac{(-\vec{k}-\vec{p})^2}{2m_p} + i\varepsilon\right)\left(\eta_e(\Delta E_i) + \omega - \frac{(\vec{p})^2}{2m_e} + i\varepsilon\right)}$$

$$\int \frac{d\omega}{2\pi} \frac{1}{(-\omega + A)(-\omega + B)(\omega + C)}$$

$$= \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \left\{ \frac{(B+C)}{(-\omega+A)} - \frac{(A+C)}{(-\omega+B)} + \frac{(B-A)}{(\omega+C)} \right\} \frac{1}{(B+C)(B-A)(C+A)}$$

$$= \frac{(\log\ div)}{2\pi} \cdot \frac{\{-(B+C) + (A+C) + (B-A) = 0\}}{(B+C)(B-A)(C+A)} + \frac{(-\pi i)}{2\pi} \frac{\{-(B+C) + (A+C) - (B-A) = 2(A-B)\}}{(B+C)(B-A)(C+A)}$$

$$= \frac{i}{(B+C)(C+A)}$$

$$= \int \frac{d\vec{p}}{(2\pi)^3} \left(\frac{ieQ_p}{2m_p} \vec{\varepsilon}^{\,*}(-\vec{k}) \cdot (-2\vec{p} - \vec{k}) \right) \frac{\chi_f^*(\vec{p} + \eta_e \vec{k}\,; -\vec{k})}{\left[\Delta E_f(-\vec{k}) - \frac{(\vec{k})^2}{2(m_e+m_p)} - \frac{(\vec{p}\,')^2}{2\mu_{ep}} \right]} \frac{\chi_i(\vec{p}\,; \vec{0})}{\left[(\Delta E_i) - \frac{(\vec{p})^2}{2\mu_{ep}} \right]}$$

$$= \int \frac{d\vec{p}}{(2\pi)^3} \left(\frac{+ieQ_p}{2m_p} \vec{\varepsilon}^{\,*}(-\vec{k}) \cdot (2\vec{p} + \vec{k}) \right) \psi_f^*(\vec{p} + \eta_e \vec{k}\,; -\vec{k})\ \psi_i(\vec{p}\,; \vec{0})$$