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§7. Unitarity

$\mathcal{H}_{phys.}$ (a vector space over $\mathbb{C}$) has a positive definite norm (no negative norm states)

§7.0 What if $\mathcal{H}_{phys}$ does not have a Fock-space structure?

(a Fock-space structure

$$\Longleftrightarrow \mathcal{H}_{phys} \simeq \mathbb{C}|\Omega\rangle \oplus \left\{\begin{matrix}\text{1-particle}\\ \text{states}\end{matrix}\right\} \oplus \left\{\begin{matrix}\text{2-particle}\\ \text{states}\end{matrix}\right\} \oplus \cdots$$

)

e.g. physics at a strongly coupled quantum critical point.

Still, symmetry of the system has its unitary representat'n on $\mathcal{H}_{phys}$.

$\Updownarrow$ def. norm preserving.

translation rotation boost....

(also scale invariance at a critical point)

Just like the theory of finite-dim. unitary representation of $SU(2)$ yields a constraint $|\langle L_z\rangle| \leq l$

some constraints can be derived from the theory of infinite-dim unitary representation of certain symmetry algebra.

A combination of the unitary representation theory of the **conformal algebra** and the crossing symmetry. can be a very powerful tool in determing the operator dimensions in a conformal field-theory.

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§7.05 Negative norm states.

(case 1) a scalar field theory with the wrong-sign kinetic term.

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) + \frac{1}{2} m^2 \phi^2 \qquad \eta^{\mu\nu} = \mathrm{diag}(+,-,-,-)$$

$$\hookrightarrow -\frac{1}{2}(\dot{\phi})^2 + \frac{1}{2}(\vec{\partial}\phi)^2 + \frac{1}{2} m^2 \phi^2 \quad \text{instead of } +\frac{1}{2}(\dot{\phi})^2 - \frac{1}{2}(\vec{\partial}\phi)^2 - \frac{1}{2} m^2\phi^2$$

~~The equation of motion remains the same.~~

- mode expansion: $\phi(\vec{x},t) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(e^{-ip\cdot x} a_p + e^{ip\cdot x} a_{\vec{p}}^\dagger \right)$
- The canonical conjugate momentum of $\phi(\vec{x})$ is $-\dot{\phi}(\vec{x})$, not $\dot{\phi}(\vec{x})$.

$\Rightarrow$ The canonical quantization relation $[\phi(\vec{x}), \pi(\vec{y})] = i\delta^3(\vec{x}-\vec{y})$

results in $[a_{\vec{p}}, a_{\vec{q}}^\dagger] = (2\pi)^3 \delta^3(\vec{p}-\vec{q}) \times (-1)$.

$\Rightarrow \exists$ two possibilities

(a) $\exists\, |0\rangle \in \mathcal{H}$. $a_{\vec{p}}|0\rangle = 0 \Rightarrow (a_{\vec{p}}^\dagger)^n|0\rangle$: positive-energy excitations.

$$\| a_{\vec{p}}^\dagger |0\rangle \| = \langle 0| a_{\vec{p}} a_{\vec{p}}^\dagger |0\rangle = \langle 0| [a_{\vec{p}}, a_{\vec{p}}^\dagger] + a_{\vec{p}}^\dagger \underbrace{a_{\vec{p}} |0\rangle}_{\to 0}$$

$$= \langle 0| [a_{\vec{p}}, a_{\vec{p}}^\dagger] |0\rangle < 0.$$

negative norm states.

(b) $\exists\, |\infty\rangle \in \mathcal{H}$ $a_{\vec{p}}^\dagger |\infty\rangle = 0$

~~then "the excited state" $a_{\vec{p}}|\infty\rangle$ has~~

$$\| a_{\vec{p}} |\infty\rangle \| = \langle \infty | a_{\vec{p}}^\dagger a_{\vec{p}} |\infty\rangle = \langle \infty | [a_{\vec{p}}^\dagger, a_{\vec{p}}] |\infty\rangle > 0.$$

~~but the excitation energy is negative.~~

The Hamiltonian eigenvalue is not bounded from below.

Both (a) & (b) are bad situations.

a field theory with a higher spin field has this problem.

$$\mathcal{L} = \frac{(-1)^J}{2} (\partial_\nu A_{\mu_1\mu_2\cdots\mu_J})(\partial^\nu A^{\mu_1\mu_2\cdots\mu_J}) \longleftrightarrow \text{wrong sign for } A_{\mu_1\cdots\mu_J} \text{ with odd } \mu_i\text{'s set to } 0 \text{(time)}$$

OK. if such degree of freedom is absent in fact!

electro-magnetism / gravity: "gauge "symmetry" (redundance) is a "must" to avoid this problem.

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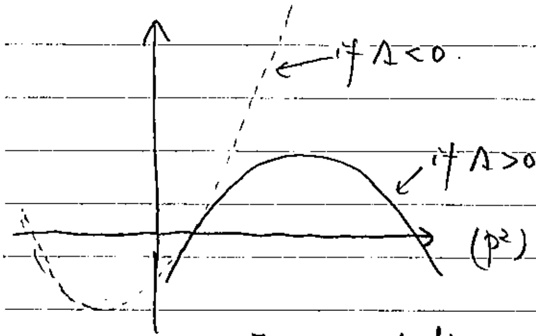
(case 2) a scalar field theory with the correct sign

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 - \frac{1}{2\Lambda}(\partial_\mu\partial_\nu\phi)(\partial^\mu\partial^\nu\phi) \qquad \begin{pmatrix}\text{assume} \\ m^2 \ll |\Lambda|\end{pmatrix}$$

$\Rightarrow$ In the momentum space $\int \frac{d^4p}{(2\pi)^4}\left(\tilde{\mathcal{L}} = \frac{1}{2}\phi(-p)\left(p^2 - m^2 - \frac{(p^2)^2}{\Lambda}\right)\phi(p)\right)$

$$\Rightarrow \int \langle 0|T\{\phi(x)\phi(0)\}|0\rangle e^{ip\cdot x} d^4x = \frac{i}{\left(p^2 - m^2 - \frac{(p^2)^2}{\Lambda} + i\varepsilon\right)}$$

The denominator always has 2 poles



$$\begin{cases} @\ p^2 \simeq m^2 & \dfrac{i}{p^2 - m^2} \quad \left(\dfrac{(m^2)^2}{|\Lambda|} \ll m^2 \text{ negligible}\right) \\ @\ p^2 \simeq \Lambda & \dfrac{-i}{p^2 - \Lambda} \quad \left(m^2 \ll |\Lambda| \text{ negligible}\right) \end{cases}$$

$\Downarrow$

Interpretation through the spectral representation.

$$\int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip\cdot x}\, i}{(p^2 - m^2 + i\varepsilon)} \Leftrightarrow \int \langle 0|\phi(x)|\vec{p}_{(m^2)}\rangle\langle\vec{p}_{(m^2)}|\phi(0)|0\rangle \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{2E_{\vec{p}}(m^2)}$$

~~so there must also be~~

$$\int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip\cdot x}(-i)}{(p^2 - \Lambda + i\varepsilon)} \Leftrightarrow \int \langle 0|\phi(x)|\vec{p}(\Lambda)\rangle\langle\vec{p}(\Lambda)|\phi(0)|0\rangle \frac{d^3\vec{p}}{(2\pi)^3}\frac{(-1)}{2E_{\vec{p}}(\Lambda)}$$

as a part of insertion of a complete system of states of the Hilbert space.

$$\int \frac{d^3\vec{p}}{(2\pi)^3}\frac{-1}{2E_{\vec{p}}} |\vec{p}(M^2)\rangle\langle\vec{p}(M^2)|$$

is for a negative norm state.

§7.1 Partial wave unitarity

Weakly coupled QFT's have two Fock-space structures:

the in-state basis & the out-state basis.

The Hermitian inner product on $\mathcal{H}_{phys}$ looks precisely the same in those two bases.

$$C_{\beta\alpha} = \delta_{\beta\alpha} \cdot \Pi_i \left((2\pi)^3 \delta^3(\vec{p}_i - \vec{q}_i)(2E_{\vec{q}_i}) \right) \quad \text{in both.}$$

$\downarrow$

$$C^{-1}_{\delta\gamma} = \delta_{\delta\gamma} \; \Pi_{i=1}^{n_i} \left((2\pi)^3 \delta^3(\vec{p}_i - \vec{q}_i)(2E_{\vec{q}_i}) \right)$$

$\Rightarrow$ The basis change matrix $(P^{o/i})_{\gamma\alpha}$

$$|\alpha\rangle^{in} = |\gamma\rangle^{out} (P^{o/i})_{\gamma\alpha}$$

is unitary with respect to the norm above (norm-preserving)

$$(C^{-1})_{\delta\gamma} (P^{o/i})_{\gamma\alpha} \left((P^{o/i})_{\delta\beta} \right)^{c.c} = C_{\beta\alpha}$$

$\Updownarrow$

$$S_{\beta\alpha} := C_{\beta\gamma}(P^{o/i})_{\gamma\alpha} = {}^{out}\langle\beta|\alpha\rangle^{in}$$

$$(C^{-1})_{\delta\gamma} S_{\gamma\alpha} (S_{\delta\beta})^{c.c.} = C_{\beta\alpha} \quad \text{: called the S-matrix unitarity}$$

S-matrix unitarity $\Rightarrow$ partial wave unitarity.

e.g. a finite-dim unitary matrix U $\left(U^\dagger U = \mathbb{1}\right)$

- $\sum_{\text{all } i} |U_{ij}|^2 = 1$
- $\sum_{\text{some } i} |U_{ij}|^2 \le 1$ — ($*$)

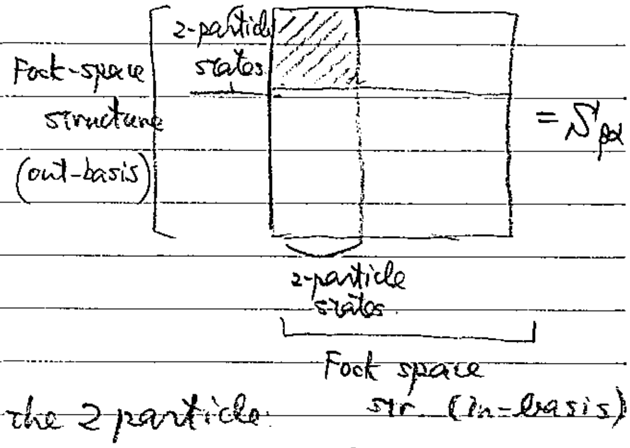
for any j. (for any column vector of the matrix U)

Apply ($*$) to the S'-matrix

$$\boxed{\sum_{\substack{\delta,\gamma:\\ \text{2-particle}\\ \text{states}}} C^{-1}_{\delta\gamma}\, S_{\gamma\alpha}\, (S_{\delta\alpha})^{c.c.} \le C_{\alpha\alpha}}$$

(for any 2 particle state α.)

(partial wave unitarity)



Let us rewrite the sum over the 2 particle states to make the most of the partial wave unitarity.

- set the state $|\alpha\rangle^{in}$ (2 particle) to be in its center of mass frame.
- get rid of common factors

$$\Rightarrow \boxed{\frac{1}{(4\pi)^4}\,\frac{p_1}{M}\,\frac{p_3}{M}\int d^2\Omega(\vec p_3)\; \underline{S^{cc.}_{34,78}}\; \underline{S_{34,12}} \le \delta^2(\Omega_{\vec p_1} - \Omega_{\vec p_7})}$$

$$S^{2\to2}_{\beta\alpha} =: (2\pi)^4 \delta^4(p_\beta - p_\alpha)\, \underline{S_{\beta\alpha}}$$

$\alpha \Rightarrow (E_1, \vec p_1) + (E_2, -\vec p_1)$
$\beta \Rightarrow (E_3, \vec p_3) + (E_4, -\vec p_3)$
$\alpha' \Rightarrow (E_7, \vec p_7) + (E_8, -\vec p_7)$
$E_1 = E_7,\ E_2 = E_8$

(derivation in the next page)

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$$(S'_{\delta\alpha'})^{cc}\, C^{-1}_{\delta\gamma}\, S_{\gamma\alpha} \qquad \alpha \Rightarrow p_1^\mu + p_2^\mu$$

$$\gamma \Rightarrow p_3^\mu + p_4^\mu \qquad \delta \Rightarrow p_5^\mu + p_6^\mu \qquad \alpha' \Rightarrow p_7^\mu + p_8^\mu$$

The right hand side

$$C_{\alpha'\alpha} \Rightarrow (2\pi)^3\,\delta^3(\vec{p}_1-\vec{p}_7)\,(2\pi)^3\,\delta^3(\vec{p}_{CM}-\vec{p}\,'''_{CM})(2E_1)(2E_2)$$

rewrite $\delta^3(\vec{p}_1-\vec{p}_7) = \delta(p_1-p_7)\,\dfrac{1}{p_1^2}\,\delta^2(\theta\phi_1-\theta\phi_7)$

(here, $\theta\phi_1$ and $\theta\phi_7$ specify the direction of $\vec{p}_1$ & $\vec{p}_7$

$\int d^2\Omega_{(\vec{p}_7)}\,\delta^2(\theta\phi_1-\theta\phi_7)\,f(\theta_7,\phi_7) = f(\theta_1,\phi_1)$

is the definition of $\delta^2(\theta\phi_1-\theta\phi_7)$)

So, the RHS is $\delta^2(\theta\phi_1-\theta\phi_7)\times(2\pi)^6\,\dfrac{(2E_1)(2E_2)}{p_1^2}\,\delta^3(\vec{p}_{CM}-\vec{p}\,'''_{CM})\,\delta(p_1-p_7)$

$\vec{p}_{CM}=\vec{p}_1+\vec{p}_2,\quad \vec{p}\,'''_{CM}=\vec{p}_7+\vec{p}_8$

The left hand side

$$(S'_{\delta\alpha'})^{cc}\, C^{-1}_{\delta\gamma}\,(S_{\gamma\alpha})$$

C^{-1}

$$= \int \frac{d^3p_5}{(2\pi)^3}\frac{1}{2E_5}\frac{d^3p_6}{(2\pi)^3}\frac{1}{2E_6}\int\frac{d^3p_3}{(2\pi)^3}\frac{1}{2E_3}\frac{d^3p_4}{(2\pi)^3}\frac{1}{2E_4}\left(\begin{array}{c}(2\pi)^3\delta^3(\vec{p}_3-\vec{p}_5)(2\pi)^3\delta^3(\vec{p}\,'_{CM}-\vec{p}\,''_{CM})\\ \times(2E_3)(2E_4)\end{array}\right)$$

$$\times(2\pi)^4\delta(E_7+E_8-E_1-E_2)\,\delta^3(\vec{p}\,'''_{CM}-\vec{p}_{CM})\,\left(S_{34,78}\right)^{cc}$$

$$\times(2\pi)^4\delta(E_3+E_4-E_1-E_2)\,\delta^3(\vec{p}\,'_{CM}-\vec{p}_{CM})\,\left(S_{34,12}\right)$$

$$=\int\frac{d^3p_3}{(2\pi)^3}\frac{1}{2E_3\,2E_4}(2\pi)^2\delta(E_3+E_4-E_1-E_2)\,\delta(E_7+E_8-E_1-E_2)\,S^{cc}_{34,78}\,S_{34,12}$$

$$\times(2\pi)^3\delta^3(\vec{p}\,'''_{CM}-\vec{p}_{CM})$$

$$=\int dp_3\int d^2\Omega(\vec{p}_3)\,\frac{(p_3)^2}{2E_3\,2E_4}\,\frac{E_3E_4}{p_3\,E_{CM}}\,\frac{1}{(2\pi)}\,\delta(p_3-p_{3,\text{right value}})\,\frac{E_1E_2}{p_1\,E_{CM}}\,\delta(p_7-p_1)$$

$$\times S^{cc}_{34,78}\,S_{34,12}$$

$$\times(2\pi)^3\delta^3(\vec{p}_{CM}-\vec{p}\,'''_{CM})$$

$$=\int d^2\Omega(\vec{p}_3)\,\frac{1}{2^4(2\pi)^4}\,\frac{p_3}{E_{CM}}\,\frac{p_1}{E_{CM}}\,S^{cc}_{34,78}\,S_{34,12}\times\boxed{(2\pi)^6\delta^3(\vec{p}_{CM}-\vec{p}\,'''_{CM})\,\delta(p_1-p_7)\,\frac{(2E_1)(2E_2)}{p_1^2}}$$

$$\Rightarrow\ \delta^2(\theta\phi_1-\theta\phi_7) = \int d^2\Omega(\vec{p}_3)\,\frac{1}{(4\pi)^4}\,\frac{p_3}{E_{CM}}\,\frac{p_1}{E_{CM}}\,S^{cc}_{34,78}\,S_{34,12}$$

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So, it is convenient to introduce

$$\underline{S^{2\to 2}_{34,12}} =: (4\pi)^2 \sqrt{\frac{v_{12}}{v_1 v_2}} \sqrt{\frac{v_{34}}{v_3 v_4}} \; \underline{S^{2\to 2}_{34,12}}_{\,\mathrm{red}}.$$

$v_1 = P_1/E_1$, etc. $\quad v_{12} E_1 E_2 = |E_1 P_{z,2} - E_2 P_{z,1}|$
(when $\vec{p}_1 \parallel e_z$ and $\vec{p}_2 \parallel e_z$)

$\rightarrow$ in the center of mass frame

$$\frac{v_{12}}{v_1 v_2} = \frac{|E_1 P_{z,1} - E_2 P_{z,2}|}{|P_{z,1}|\,|P_{z,2}|} = \frac{(E_1+E_2)}{P_1} = \frac{E_{CM}}{P_1}.$$

$(\; P_{z,1} = P_1 . \quad P_{z,2} = -P_1 \;)$

because the partial wave unitarity condition is rewritten as

$$\boxed{\int d^2\Omega(\vec{p}_3) \;\; \underline{S_{34,78}}^{cc}_{\,\mathrm{red}} \;\; \underline{S_{34,12}}_{\,\mathrm{red}} \le \delta^2(\theta\phi_1 - \theta\phi_7)}$$

- $\delta^2(\theta\phi_1 - \theta\phi_7) = \sum_{l,m} Y_{l,m}(\theta,\phi_7)\,(Y_{l,m}(\theta,\phi_1))^{cc}$

- here. $\int d^2\Omega \; Y_{l,m}(\theta,\phi)\,(Y_{l',m'}(\theta,\phi))^{cc} = \delta_{l,l'}\,\delta_{m,m'}$

$$Y_{l,0}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}}\, P_l(\cos\theta)$$

- parametrize $\underline{S^{2\to 2}_{34,12}}_{\,\mathrm{red}} =: \sum_{l,m} Y_{l,m}(\theta,\phi_3)\; e^{2i\delta_l(E_{CM})}\,(Y_{l,m}(\theta,\phi_1))^{c.c.}$

Then. $\sum_{l,m} Y_{l,m}(\theta,\phi_7)\; |e^{2i\delta_l(E_{CM})}|^2\,(Y_{l,m}(\theta,\phi_1))^{cc} \le \sum_{l,m} Y_{l,m}(\theta,\phi_7)\,(Y_{l,m}(\theta,\phi_1))^{c.c}$

$$\boxed{|e^{2i\delta_l(E_{CM})}| \le 1}$$

If $|e^{2i\delta_l(E)}| < 1$ for some l. (not a pure phase shift)

$\boxed{\text{then } \Sigma_\gamma |S^{2\to n\geq 3}_{\gamma\alpha}|^2 > 0}$

When the particles in the Fock space have non-0 spins.

$\underline{S^{2\to 2}_{\beta\alpha}}_{\text{red}}$ should be parametrized by $e^{2i\delta_J(E_{cm})}$

summed over the total angular momentum $\vec{J}$

(see Weinberg I. §3.7)

<Let us keep the story simple in this lecture note>

A relation between the partial wave unitarity constraint and perturbative calculation.

We may compute $i\mathcal{M}^{2\to 2}_{\beta\alpha}$ using Feynman rule.

$$S^{2\to 2}_{\beta.\alpha} = \mathbb{1}_{\beta.\alpha} + (2\pi)^4\delta^4(p_\beta - p_\alpha)\, i\mathcal{M}_{\beta\alpha}$$

$$\| $$

$$(2\pi)^4\delta^4(p_\beta - p_\alpha)\cdot(4\pi)^2\sqrt{\frac{v_{12}}{v_1 v_2}}\sqrt{\frac{v_{34}}{v_3 v_4}}\,\underline{S}^{2\to 2}_{\text{red}}$$

Similarly we introduce $i\mathcal{M}^{2\to 2}_{\beta.\alpha} = (4\pi)^2\sqrt{\frac{v_{12}}{v_1 v_2}}\sqrt{\frac{v_{34}}{v_3 v_4}}\left(\underline{i\mathcal{M}}_{\beta.\alpha\,\text{red}}\right).$

$$\text{Then } \boxed{\underline{i\mathcal{M}^{2\to 2}_{34,12}}_{\text{red}} = \sum_{l.m} Y_{l.m}(\theta.\phi_3)\left(e^{2i\delta_l} - 1\right)\left(Y_{l.m}(\theta.\phi_1)\right)^*}$$

It follows that.

$$d\sigma^{2\to 2} = \frac{1}{(2E_1)(2E_2)\,v_{12}}\int\frac{d^3p_3}{(2\pi)^3}\frac{1}{(2E_3)}\frac{d^3p_4}{(2\pi)^3}\frac{1}{(2E_4)}(2\pi)^4\delta^4(p_3+p_4-p_1-p_2)\,|\mathcal{M}|^2$$

$$= \cdots = \frac{\pi}{p_2^2}\sum_{l=0}^{\infty}(2l+1)\,\left|e^{2i\delta_l(E_{cm})} - 1\right|^2$$

(derivation : see p(x) upper half.)

☆ In a QFT model with small coupling constants

Feynman rule calculations often yield

$i\mathcal{M}^{2\to2}_{\beta\alpha}$ ~ dimensionless $\underbrace{\text{imaginary}}_{}$ (almost pure) and small.

$$e^{2i\delta_\ell} = 1 + i(\mathcal{M})_{\ell\text{ component}} \simeq 1.$$

δ_ℓ ~ small phase shift.

☆ In a QFT model where known contributions to $i\mathcal{M}^{2\to2}$

do not satisfy the partial wave unitarity,

(for some ℓ and at certain energy scale)

$$\left| 1 + \frac{R}{2(4\pi)^2} \int d^2\Omega(\vec{p}_3)\, (\Delta\, i\mathcal{M}^{2\to2})(\theta,\phi_3)\, (Y_{\ell,m}(\theta,\phi_3))^* \left(Y_{\ell,0}(\theta=0) = \sqrt{\frac{2\ell+1}{4\pi}}\right)^{-1} \right| > 1,$$

$$\left(R := \sqrt{\frac{v_1 v_2}{(v_{12}/2)}} \sqrt{\frac{v_3 v_4}{(v_{34}/2)}} \right)$$

(see p.(X) to extract "ℓ"-component)

then that is an indication that there must be

additional contributions (operators)

(interactions terms)

in the model. (higher loop)

$v \simeq 246\,\text{GeV}$.

$\left(\frac{2}{v^2} = 2\sqrt{2}G_F\right)$

example

$$\mathcal{L} \simeq \left(\frac{2}{v^2}\right)\left(\bar{\Psi}\gamma^\mu\left(\frac{1-\gamma_5}{2}\right)\Psi\right)_q \left(\bar{\Psi}\gamma_\mu\left(\frac{1-\gamma_5}{2}\right)\Psi\right)_\ell \Rightarrow \mathcal{M} \sim G_F E_{CM}^2.$$

$$\mathcal{L} \sim \frac{1}{M}(\ell h)(\ell h) \Longrightarrow \mathcal{M} \sim \left(\frac{E_{CM}}{M}\right)$$

something has to happen @ $\begin{cases} E \sim 1/\sqrt{G_F} \\ E \sim M \end{cases}$

☆ The black disc limit : observed in hadron-hadron scattering

$\overset{\text{def}}{\Leftrightarrow}$ $|e^{2i\delta_\ell(E)}| \ll 1$. for some (range of) ℓ's at $\underbrace{E = E_{CM}}_{\text{energy}}$

That is when $e^{2i\delta_\ell} \simeq (1 + i \times \underset{\approx}{i})$

$$\mathcal{M} \sim \sum_{\ell,m} Y_{\ell,m}(\theta,\phi_3)\, \underset{\approx}{i}\, (Y_{\ell,m}(\theta,\phi_3))^{cc}$$

$$\Rightarrow \left\{ \begin{array}{l} \sum_\beta^{2\text{-body}} |\mathcal{M}_{\beta\alpha}|^2 = 1 \\ \sum_\beta^{n\geq3\text{ body}} |S'_{\beta\alpha}|^2 = 1 \end{array} \right\} \Rightarrow \left[\sigma_\ell^{\text{elast}} \simeq \sigma_\ell^{2\to(n\geq3)} \right]$$

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Supplementary Notes

◎ $$d\sigma^{2\to 2} = \frac{1}{(2E_1)(2E_2)\,v_{12}}\int\frac{d^3p_3}{(2\pi)^3}\frac{1}{2E_3}\int\frac{d^3p_4}{(2\pi)^3}\frac{1}{2E_4}(2\pi)^4\delta^4(p_3+p_4-p_1-p_2)\,|\mathcal{M}|^2$$

$$= \frac{1}{(2E_1)(2E_2)\,v_{12}}\int\frac{d^3p_3}{(2\pi)^3}\frac{1}{2E_3\,2E_4}(2\pi)\,\delta(E_3+E_4-E_1-E_2)\left(|\mathcal{M}|^2=(4\pi)^4\frac{v_{34}}{v_3v_4}\frac{v_{12}}{v_1v_2}|\mathcal{M}_{red}|^2\right)$$

$$= \frac{1}{2E_1\,2E_2}\int d p_3\ p_3^2\ d^2\Omega(\vec{p}_3)\ \frac{2^4(2\pi)^2}{2E_3\,2E_4}|\mathcal{M}_{red}|^2\,\delta(p_3-p_{3,kin.})\left(\frac{p_3}{E_3}+\frac{p_4=p_3}{E_4}\right)^{-1}\frac{v_{34}}{v_3v_4}\ \ v_1v_2$$

use the CM-frame $$= \frac{2^2}{2E_1\,2E_2\,v_1v_2}\left(\frac{p_3^2\ 2^2\ E_3E_4}{2E_3\,2E_4\ p_3(E_3+E_4)}\underbrace{\frac{v_{34}}{v_3\,v_4}}_{=1}\right)(2\pi)^2\int d^2\Omega\ |\mathcal{M}_{red}|^2$$

$$= \frac{(2\pi)^2}{(p_1)^2}\int d^2\Omega(\vec{p}_3)\ |\mathcal{M}_{red}|^2$$

$$= \frac{(2\pi)^2}{(p_2)^2}\int d^2\Omega(\vec{p}_3)\ \left|\Sigma_{l,m}Y_{l,m}(\theta,\phi_3)\left(e^{2i\delta_l(E_1+E_2)}-1\right)Y_{l,m}(\theta,\phi_1)\right|^2$$

set $\hat{p}_2=(\theta,\phi_1)$ s.t. $\theta_1=0 \Rightarrow Y_{l,m}(\theta,\phi_1)=\delta_{m,0}\sqrt{\frac{2l+1}{4\pi}}$

$$= \frac{(2\pi)^2}{(p_1)^2}\int d^2\Omega(\hat{p}_3)\left(\sum_l\sum_{l'}Y_{l,0}(\theta,\phi_3)\left(Y_{l',0}(\theta,\phi_3)\right)^{c.c}\right)\left(e^{2i\delta_l}-1\right)\left(e^{2i\delta_{l'}}-1\right)^{c.c}\times\frac{\sqrt{(2l+1)(2l'+1)}}{4\pi}$$

$$= \frac{(2\pi)^2}{(p_1)^2}\sum_l\frac{(2l+1)}{4\pi}\left|e^{2i\delta_l}-1\right|^2 = \frac{\pi}{(p_2)^2}\sum_l(2l+1)\left|e^{2i\delta_l(E_{CM})}-1\right|^2$$

◎ given $i\mathcal{M}(\theta\phi_3;(\theta\phi_1/\!/\hat{e}_z))$, to extract $(e^{2i\delta_l}-1)$ ...

$$i\mathcal{M}_{red}(\theta,\phi_3;(\theta,\phi_1/\!/\hat{e}_z)) = \sum_{l,m}Y_{l,m}(\theta,\phi_3)\left(e^{2i\delta_l}-1\right)\left(Y_{l,m}(\theta,\phi_1)=\delta_{m,0}\sqrt{\frac{2l+1}{4\pi}}\right)^{c.c}$$

$$\Rightarrow \sqrt{\frac{4\pi}{2l+1}}\times\int d^2\Omega(\hat{p}_3)\ i\mathcal{M}_{red}(\theta\phi_3;\theta\phi_1=\hat{e}_z)\cdot\left(Y_{l,0}(\theta,\phi_3)\right)^{c.c} = \left(e^{2i\delta_l}-1\right)$$