§7.2 Optical theorem (applications)

The unitarity of the S-matrix also implies

$$\mathbb{1} = S^\dagger S = \left(\mathbb{1} + (2\pi)^4\delta^4(p_\gamma - p_\beta)(-i\mathcal{M}^*_{\beta\gamma})\right)\left(\mathbb{1} + (2\pi)^4\delta^4(p_\beta - p_\alpha)\, i\mathcal{M}_{\beta\alpha}\right)$$

$$= \mathbb{1} + (2\pi)^4\delta^4(p_\gamma - p_\alpha)\left\{ i\left[\mathcal{M}_{\gamma\alpha} - (\mathcal{M}_{\alpha\gamma})^*\right] + \sum_\beta (\mathcal{M}_{\beta\gamma})^*(\mathcal{M}_{\beta\alpha})(2\pi)^4\delta^4(p_\beta - p_\alpha)\right\}$$

so

$$\boxed{\frac{\mathcal{M}_{\gamma\alpha} - (\mathcal{M}_{\alpha\gamma})^*}{(i)} = \int \prod_{j=1}^{N_\beta}\left[\frac{d\vec{p}_j^{\,3}}{(2\pi)^3}\frac{1}{(2E_j)}\right](2\pi)^4\delta^4(p_\beta - p_\alpha)(\mathcal{M}_{\beta\gamma})^*(\mathcal{M}_{\beta\alpha})}$$

As a particular case $\gamma = \alpha$,

$$2\,\mathrm{Im}(\mathcal{M}_{\alpha\alpha}) = \int \prod_{j=1}^{N_\beta}\left[\frac{d\vec{p}_j^{\,3}}{(2\pi)^3}\frac{1}{(2E_j)}\right](2\pi)^4\delta^4(p_\beta - p_\alpha)\left|\mathcal{M}_{\beta\alpha}\right|^2 = \begin{cases}\sigma_{tot}\cdot(4E_1E_2v_{r}) \\ \text{or} \\ \Gamma_{tot}\cdot(2E)\end{cases}$$

optical theorem

Useful because ....

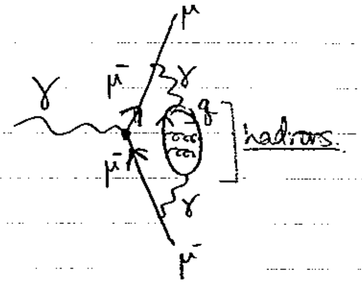
☆ $\boxed{\mathrm{Im}(\mathcal{M}_{\alpha\alpha}) \Longrightarrow \Gamma_{tot},\ \sigma_{tot}}$

$e^+ + e^- \to \gamma$ or $Z \to$ hadrons $\left(\sigma_{tot}\cdot 2s\right) \cong 2\,\mathrm{Im}\left[\mathcal{M}\left(\right.\right.$  $\left.\left.\right)\right]$
at $E_{CM} \gg$ GeV

[Perturbative calculations are available for "inclusive enough" observables.]
such as σ_{tot}.

☆ $\boxed{\Gamma_{tot},\ \sigma_{tot} \Longrightarrow \mathrm{Im}(\mathcal{M}_{\alpha\alpha})}$

to estimate contributions to anomalous magnetic moment of μ^-,



we can use $\left[\sigma_{tot}(e^+ + e^- \to \text{hadron})\cdot s\right]$ to determine $\mathrm{Im}(\mathcal{M}_{\gamma^*\gamma^*})$.
unitarity

and $\mathrm{Re}(\mathcal{M}_{\gamma^*\gamma^*})(s + i\varepsilon) = \int \frac{ds'}{\pi}\frac{\mathrm{Im}(\mathcal{M}_{\gamma^*\gamma^*}(s'))}{s' - s}$.

dispersion integral $\left(\mathcal{M}(s + i\varepsilon) : \text{holomorphic in } s\right)$
[Kramers–Kronig relation]

separate "principle"

★ $\boxed{\sigma_{tot} \longrightarrow \mathrm{Im}(\mathcal{M}_{\alpha\alpha})}$ for perturbative calculation.

Think of a theory with $\mathcal{L} = \mathcal{L}_{kin.} + g\,\phi\,\bar{\Psi}\Psi$. ϕ: scalar

$$\Rightarrow \int \frac{d^3\vec{p}_\Psi}{(2\pi)^3}\frac{1}{2E_{p_\Psi}} \int \frac{d^3\vec{p}_{\bar{\Psi}}}{(2\pi)^3}\frac{1}{2E_{p_{\bar{\Psi}}}} (2\pi)^4\delta^4(p_\Psi + p_{\bar{\Psi}} - P_\phi) \underbrace{|\mathcal{M}|^2}_{\text{spin-sum}} = \frac{g^2}{4\pi}\left\{E^2 - (2m_\Psi)^2\right\}.$$

$$\left[\begin{array}{l} i\mathcal{M}(\phi \longrightarrow \Psi + \bar{\Psi}) \\ \quad\Downarrow \\ \quad (E,\vec{0})\ \text{CM frame} \end{array}\right]$$

straightforward calculation. tree level (= 0-loop)

E: allowed to be free ($\neq M_\phi$)

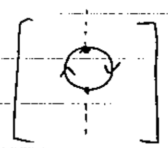
So...

$$\boxed{2\,\mathrm{Im}\left[\mathcal{M}(\phi\to\phi)\right] \Leftarrow \frac{g^2}{4\pi}\left\{s - (2m_\Psi)^2\right\}\Theta\left(s - (2m_\Psi)^2\right)} \Leftarrow \text{application of the optical thm.}$$

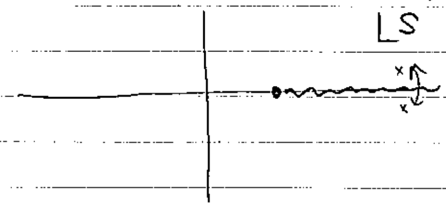
$$\to \mathcal{M}(\phi\to\phi) \text{ at } s = \frac{+g^2}{8\pi^2}\left\{s - (2m_\Psi)^2\right\}\ln\left(\frac{(2m_\Psi)^2 - (s + i\varepsilon)}{(2m_\Psi)^2}\right) + \left(\text{rational, real (for real } s)\right)$$

$(+\varepsilon \to 0)$

$\|$



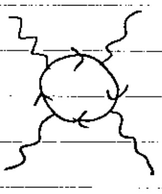
(i)



$[$ branch cut at $(2m_\Psi)^2 \le s$ $]$

We have managed to obtain an expression for a 1-loop graph without doing 1-loop computation.

More generally...



$= i\mathcal{M}(s,t)$ should be a holomorphic function of (s,t) except poles and branch cuts that satisfy all of $\left\{\begin{array}{l} s\text{-channel} \\ t\text{-channel} \\ u\text{-channel}\end{array}\right\}$ unitarity relation.

§ 8 Low-energy effective theory

Here, "theory" is in the sense of <u>model</u>.

Example 1 QED with $\gamma, e^\pm, \mu^\pm \longrightarrow$ QED with $\gamma, e^\pm$

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \bar{\Psi}_{(e)}(i\gamma^\mu D_\mu - m_{(e)})\Psi_{(e)} + \bar{\Psi}_{(\mu)}(i\gamma^\mu D_\mu - m_{(\mu)})\Psi_{(\mu)} \quad \text{—} (*)$$

Start from a theory (=model) above.

If we are interested in physics with energy below $m_{(\mu)} \sim 106$ MeV,

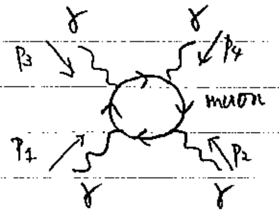
we do not have to maintain $\Psi_{(\mu)}$ in the Lagrangian.

But we have to use

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \bar{\Psi}_{(e)}(i\gamma^\mu D_\mu - m_{(e)})\Psi_{(e)} + \frac{e^4 C^{(1)}_{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}}{16\pi^2 m_{(\mu)}^4} F^{\mu_1\nu_1}F^{\mu_2\nu_2}F^{\mu_3\nu_3}F^{\mu_4\nu_4}$$

$$+ \cdots \quad \text{—} (**)$$

in order to account for the $\gamma+\gamma \to \gamma+\gamma$ scattering amplitude



$$= i\mathcal{M}(m_{(\mu)}, p_i^\mu, \varepsilon_i^\nu)$$ expand in power series of $p/m_{(\mu)}$

$$= i \frac{e^4 C^{(1)}_{\mu_1\nu_1\cdots\mu_4\nu_4}}{16\pi^2 m_{(\mu)}^4} (p_1^{\mu_1}\varepsilon_1^{\nu_1} - p_1^{\nu_1}\varepsilon_1^{\mu_1})(p_2^{\mu_2}\varepsilon_2^{\nu_2} - p_2^{\nu_2}\varepsilon_2^{\mu_2})(p_3^{\mu_3}\varepsilon_3^{\nu_3} - p_3^{\nu_3}\varepsilon_3^{\mu_3})(p_4^{\mu_4}\varepsilon_4^{\nu_4} - p_4^{\nu_4}\varepsilon_4^{\mu_4})$$

$$+ \mathcal{O}\left(\frac{p^6}{m_{(\mu)}^6}\right)$$

(Weinberg vol I. p.523)

<u>The latter (**) is the low-energy effective theory (=model) of the former (*).</u>

The latter theory with just the $\mathcal{O}(1/m_{(\mu)}^4)$ term

will violate partial wave unitarity at $E \sim m_{(\mu)}$.

But all the terms in the $p/m_{(\mu)}$ expansion are equally important

in the partial wave unitarity at $E \sim m_{(\mu)}$

We should use the high-energy theory (*) at $E \gtrsim m_{(\mu)}$

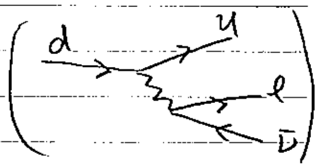
$\boxed{\text{Exampl 2}}$ The Standard Model $\longrightarrow$ QCD + QED + 4-fermi term

$$\mathcal{L}_i \supset \left[\bar{u}_i\, i\gamma^\mu \left(\frac{1-\gamma_5}{2}\right) i g_w \left(\frac{A^1_\mu + iA^2_\mu}{2}\right) d_j\right] V_{ij} \quad - \frac{1}{4 g_w^2}\left(\underbrace{F^1_{\mu\nu}F^{1\mu\nu} + F^2_{\mu\nu}F^{2\mu\nu}}_{\text{W-boson Kinetic Term}} + F^3_{\mu\nu}F^{3\mu\nu}\right)$$

$$+ \left[\bar{\ell}\, i\gamma^\nu \left(\frac{1-\gamma_5}{2}\right) i g_w \left(\frac{A^1_\nu - iA^2_\nu}{2}\right) \nu\right] \quad + \cdots \qquad \text{——} (*)$$

- i: subscripts in u_i, d_i $\Rightarrow$ generation ($i = 1, 2, 3$)
- V_{ij}: a 3×3 unitary matrix (called Cabbibo Kobayashi Maskawa (CKM) matrix)
- In the Peskin - Schroeder convention,

$$\gamma^0 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \tau^i \\ -\tau^i & 0 \end{pmatrix} \qquad \gamma_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$



This high-energy theory yields.

$$i\mathcal{M}(d_j \to u_i + \ell + \bar{\nu}) = \left[\bar{u}(\vec{p}_{u_i})\gamma^\mu(1-\gamma_5)u(\vec{p}_{d_j})\right] V_{ij} \times \frac{\left(\frac{ig_w}{4}\right)\left(-\frac{ig_w}{4}\right)\left(-i\times\eta_{\mu\nu}\times 2\right)}{(p_d - p_u)^2 - m_W^2 + i\varepsilon} \left[\bar{u}(\vec{p}_e)\gamma^\nu(1-\gamma_5)v(\vec{p}_\nu)\right].$$

Since $m_W \cong 80\,\text{GeV}$, it makes sense (when applied to low-energy physics) to expand in P/m_W

$$\frac{1}{(p_d - p_u)^2 - m_W^2} \Rightarrow \frac{1}{-m_W^2} - \frac{(p_d - p_u)^2}{(m_W^2)^2} - \frac{(p_d - p_u)^4}{(m_W)^6} - \cdots$$

and retain just a few terms.

The LO term in the amplitude is reproduced by

$$\mathcal{L}_{int} \mathrel{+}\cong -\left(\frac{g_w^2}{8m_W^2}\right)\left[\bar{u}_i\gamma^\mu(1-\gamma_5)d_j\right]\left[\bar{\ell}\gamma_\mu(1-\gamma_5)\nu\right] V_{ij} \qquad \text{——} (**)$$

to $\mathcal{L}_{QED\times QCD}$.. (without $W^\pm$ bosons)

called 4-fermi interaction

$$\frac{g^2}{2m_W^2} = \frac{2}{v^2} = 2\sqrt{2}\,G_F$$

g, m_W: microscopic-theory parameters. G_F: effective theory parameters

$\boxed{\text{(**) should be modified @ 1-loop.}}$

$\boxed{\text{Example 3 = Question (homework)}}$ The seesaw mechanism simplified

In a theory with one scalar ϕ and two Dirac fermions Ψ and Ψ', suppose that

$$\mathcal{L} = (\partial_\mu \phi)(\partial^\mu \phi) + \bar{\Psi} i\gamma^\mu (\partial_\mu \Psi) + \bar{\Psi}'(i\gamma^\mu \partial_\mu - M)\Psi' + \lambda \phi \bar{\Psi}\Psi' + \lambda^* \phi \bar{\Psi}'\Psi . \quad (*)$$

So, ϕ and Ψ are massless, but Ψ' is massive.

Now, verify that the low-energy effective theory of $(*)$ at $E \ll M$ is given by

$$\mathcal{L} = (\partial_\mu \phi)(\partial^\mu \phi) + \bar{\Psi} i\gamma^\mu (\partial_\mu \Psi) + \frac{|\lambda|^2}{M} \phi \bar{\Psi}\Psi \phi . \quad (**)$$

analogy: ϕ: Higgs doublet

Ψ: left-handed neutrino

Ψ': right-handed neutrino

$$\boxed{\underbrace{\left(\text{The SM} + \text{RH}\nu + \nu\,\text{Yukawa int.}\right)}_{(*)} \longrightarrow \underbrace{\left(\text{The SM} + \frac{|\lambda|^2}{M}(\ell h)(\ell h)\right)}_{(**)}}$$

In this case (the seesaw mechanism), we should deal with

$\begin{cases} \phi \text{ as a complex boson} \\ \Psi, \Psi' \text{ as Weyl spinors (2-component spinors)} \end{cases}$ in fact.

- low-energy approximation
- derivative/mass expansion (and truncation)
- Born – Oppenheimer approximation

$\boxed{\text{Example 4}}$ QCD $\longrightarrow$ hadrons.

quark + gluon $\rightarrow$ $\mathcal{L} = (\partial_\mu \pi^a)(\partial^\mu \pi^a) + \cdots$ (**)

$$- (\partial_\mu \rho_\nu^a - \partial_\nu \rho_\mu^a)(\partial^\mu \rho^{a\nu} - \partial^\nu \rho^{a\mu}) - \frac{1}{2} m^2 \rho_\mu^a \rho^{a\mu}$$

$$+ \bar{N} \gamma^\mu \gamma_5 (\partial_\mu \pi) N (g_A / F_\pi) + \cdots$$

(perturbative calculation cannot determine all the information of (**).)

$\boxed{\text{Example 5}}$ quantum hall system (e^-'s in $\langle \vec{B} \rangle \neq 0$ in 2+1 dim) $\longrightarrow$ Chern-Simons theory + scalar field

(hw E-4)

$\boxed{\text{Example 6}}$ QED $(\gamma, e^\pm)$ $\longrightarrow$ $E \ll m_e$ (cf. hw in week 1? 2?)

pair creation cannot take place anymore.

- Initial states with just photons $\Rightarrow$ an effective theory of γ.
- Initial states with just one e^- (+ γ's ?)

$$\Rightarrow \mathcal{L} \cong \psi^\dagger \left(i\partial_t - m - eQ_e\varphi - \frac{(i\vec{\partial} + eQ_e\vec{A})^2}{2m_e} - \cdots \right) \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad (**)$$

This two component "ψ" is a part of the 4-component as

$$\underset{4\text{ componet}}{\Psi(\vec{p}, t)} \cong \begin{pmatrix} \psi(\vec{p}, t) \\ \frac{\vec{p}\cdot\vec{\sigma}}{2m_e} \psi(\vec{p}, t) \end{pmatrix} + (\text{positron}) .$$

- Even furtherer in low-energy (so there is no ionization)

$\Rightarrow$ an effective theory of bound states —— (***)

$$\mathcal{L} = \phi_{1s}^\dagger \left(i\partial_t - \frac{(i\vec{\partial})^2}{2m_{1s}} + \cdots \right) \phi_{1s} + \vec{\Phi}_{2p}^\dagger \left(i\partial_t - \frac{(i\vec{\partial})^2}{2m_{2p}} + \cdots \right) \vec{\Phi}_{2p} + \cdots$$

$$+ \mathcal{L}_{int.} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

Sometimes the low-energy effective theory can be just an empty system.