§9. Path integral formulation of QFT's

§9.1 Repeating the derivation.

Example 1 a free scalar boson on $(d+1)$-dimensional space-time.

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 \iff \mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2$$

$$[\phi(\vec{x},t), \pi(\vec{y},t)] = i\,\delta^3(\vec{x}-\vec{y})$$

Fourier transformation $\Rightarrow$ collection of infinitely many harmonic oscillators w/ $\omega = \sqrt{k^2+m^2}$ labeled by $\vec{k} \in \mathbb{R}^d$.

$$\boxed{\begin{aligned}&\langle\Omega| T\{\phi(x_1^\mu)\phi(x_2^\mu)\cdots \pi(y_1^\mu)\pi(y_2^\mu)\cdots\}|\Omega\rangle \\ &= \frac{\int \mathcal{D}\phi\,\mathcal{D}\pi_{+i\varepsilon}\, e^{i\int d^dx\,dt\,\{-\mathcal{H}+\pi\, i(\partial_t\phi)\}}\,(\phi(x_1)\phi(x_2)\cdots\pi(y_1)\pi(y_2)\cdots)}{\int \mathcal{D}\phi\,\mathcal{D}\pi_{+i\varepsilon}\, e^{i\int d^dx\,dt\,\{-\mathcal{H}+\pi\, i(\partial_t\phi)\}}}\end{aligned}}$$

The same story when $\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4$.

Example 2 Dirac theory $\mathcal{L} = \bar{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi$

$$\boxed{\begin{aligned}&\langle\Omega| T\{\Psi(x_1)\Psi(x_2)\cdots \bar{\Psi}(y_1)\bar{\Psi}(y_2)\cdots\}|\Omega\rangle \\ &= \frac{\int \mathcal{D}\Psi\,\mathcal{D}\bar{\Psi}_{+i\varepsilon}\, e^{i\int d^4x\,\bar{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi}\,(\Psi(x_1)\Psi(x_2)\cdots\bar{\Psi}(y_1)\bar{\Psi}(y_2)\cdots)}{\int \mathcal{D}\Psi\,\mathcal{D}\bar{\Psi}_{+i\varepsilon}\, e^{i\int d^4x\,\bar{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi}}\end{aligned}}$$

Ψ & $\bar{\Psi}$: Grassmann coordinates (fields).

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Supplementary Notes.

scalar:

$$\phi(\vec{x}) = \Sigma_{\vec{k}}\; e^{i\vec{k}\cdot\vec{x}}\left(\phi^R_{\vec{k}} + i\phi^I_{\vec{k}}\right) \qquad \left(\phi^I_{-\vec{k}} = -\phi^I_{\vec{k}},\ \phi^R_{-\vec{k}} = \phi^R_{\vec{k}}\right)$$

$$H = \int d^3x \left(\frac{1}{2}(\dot{\phi})^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\right)$$

$$= \mathrm{vol(space)} \cdot \sum_{\vec{k}} \left(\frac{1}{2}\left\{(\dot{\phi}^R_{\vec{k}})^2 + E_k^2(\phi^R_{\vec{k}})^2\right\} + \frac{1}{2}\left\{(\dot{\phi}^I_{\vec{k}})^2 + E_k(\phi^I_{\vec{k}})^2\right\}\right)$$

In the operator formalism.

$$\phi^R_{\vec{k}} = \frac{1}{\sqrt{2E_{\vec{k}}}}\,\frac{1}{2}\left(a_{\vec{k}} + a^\dagger_{\vec{k}} + a^\dagger_{-\vec{k}} + a_{-\vec{k}}\right)$$

$$\phi^I_{\vec{k}} = \frac{1}{\sqrt{2E_{\vec{k}}}}\,\frac{1}{2i}\left(a_{\vec{k}} - a^\dagger_{\vec{k}} + a^\dagger_{-\vec{k}} - a_{-\vec{k}}\right) \qquad [a_{\vec{k}}, a^\dagger_{\vec{q}}] = \delta_{\vec{k},\vec{q}}$$

The path integral measure should be

$$\mathcal{D}\phi\,\mathcal{D}\pi := \prod_{\vec{k}/\{\pm1\}}\left[\mathcal{D}(\phi^R_{\vec{k}}(t))\,\mathcal{D}(\pi^R_{\vec{k}}(t))\,\mathcal{D}(\phi^I_{\vec{k}}(t))\,\mathcal{D}(\pi^I_{\vec{k}}(t))\right] \approx \prod_{k^\mu}\left[d\phi(k)\,d\pi(k)\right]$$

$$\left(\pi^{R/I}_{\vec{k}} = \mathrm{vol(space)}\ \dot{\phi}^{R/I}_{\vec{k}}\right)$$

Dirac theory:

In the operator formalism $\;\Psi(\vec{x}) = \sum_{\vec{k},s} \frac{1}{\sqrt{2E_{\vec{k}}}}\left(u_{\vec{k},s}\, a_{\vec{k},s}\, e^{i\vec{k}\cdot\vec{x}} + v_{\vec{k},s}\, b^\dagger_{\vec{k},s}\, e^{-i\vec{k}\cdot\vec{x}}\right)$

$$\mathcal{H}_{\text{Hilb.sp.}} \cong \bigotimes_{\substack{\vec{k},s=\uparrow\downarrow\\ f=e^-,e^+}}\left(\underset{\substack{\Uparrow\\ \text{2-state system in Week 2.}}}{\mathbb{C}^2_{(\vec{k},s,f)}}\right) \otimes \underbrace{\mathcal{H}_{\text{others}}}_{\gamma\text{, etc. if there is any.}}$$

$\Rightarrow$ Introduce Grassmann-odd variables and make replacements

$$a_{\vec{k},s} \to \theta_{\vec{k},s,e^-} \qquad a^\dagger_{\vec{k},s} \to \bar{\theta}_{\vec{k},s,e^-}$$

$$b_{\vec{k},s} \to \theta_{\vec{k},s,e^+} \qquad b^\dagger_{\vec{k},s} \to \bar{\theta}_{\vec{k},s,e^+}$$

In the path integral formulation,

$$\Psi(\vec{x},t) = \Sigma_{\vec{k},s}\,\frac{1}{\sqrt{2E_{\vec{k}}}}\left(u_{\vec{k},s}\,\theta_{\vec{k},s,e^-}(t)\,e^{i\vec{k}\cdot\vec{x}} + v_{\vec{k},s}\,\bar{\theta}_{\vec{k},s,e^+}(t)\,e^{-i\vec{k}\cdot\vec{x}}\right)$$

and the measure is

$$\mathcal{D}\Psi\,\mathcal{D}\bar{\Psi} := \prod_{\substack{\vec{k},s=\uparrow\downarrow\\ f=e^-,e^+}}\left[\mathcal{D}\theta_{\vec{k},s,f}(t)\,\mathcal{D}\bar{\theta}_{\vec{k},s,f}(t)\right] \approx \prod_{\substack{k^\mu,s=\uparrow\downarrow\\ f=e^-,e^+}}\left[d\theta_{k^\mu,s,f}\,d\bar{\theta}_{k^\mu,s,f}\right]$$

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§9.6 Effective theory and coarse graining

When a system (theory, model) has both heavy partiles and light particles (ϕ and ψ), it is possible to integrate out ϕ at the very beginning, if you are interested only in correlation functions involving ψ's, (not ϕ's).

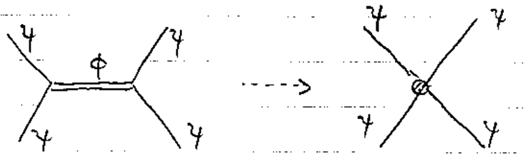
$$Z[\chi_H, \chi_L] := \int \mathcal{D}\phi \mathcal{D}\psi \; e^{i\int d^{d+1}x \left[\left(\frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}M^2\phi^2\right) + \frac{1}{2}(\partial_\mu\psi)^2 \overbrace{- g\phi\psi\psi}^{-V(\psi)} + \phi\chi_H + \psi\chi_L\right]}$$

external fields (so Z is a generating functional) of all correlation fcn's

- ☆ complete a square in the exponent
- ☆ carry out the Gaussian integral.

$$\Longrightarrow Z[\chi_H, \chi_L] \propto \int \mathcal{D}\psi \; e^{i\int d^{d+1}x \overbrace{\left[\frac{1}{2}(\partial_\mu\psi)^2 + \psi\chi_L + (g\psi\psi - \chi_H)\frac{1}{\partial^2 + m^2}(g\psi\psi - \chi_H)\right]}^{-V(\psi)}}$$

to get the action of the low-energy effective theory.



by "integrating out" the heavy field ϕ.

Also possible (and useful) to integrate out

- only two components [(-1)-eg space of γ^0] in the four component Dirac fermion
- only negative frequency modes of a complex scalar

so we obtain a non-relativistic effective theory of $\left\{ \begin{array}{l} e^- \\ \text{cold atom} \end{array} \right.$

The idea dates back to Born–Oppenheimer approximation.

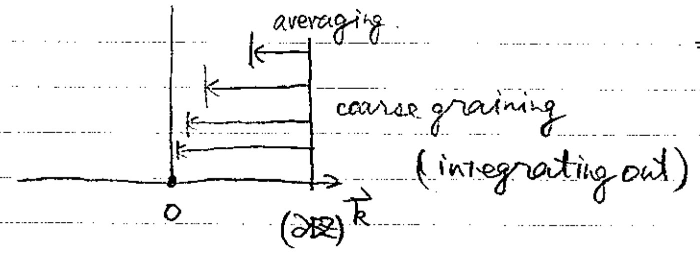
Consider a spin system turned into a field theory (eg. ϕ^4-theory),

- spacial momentum : $|\vec{k}| \leq$ bdry of the Brillouin zone. $\sim \pi/a$
- scalar field(s) : local average of magnetization

(already a length scale of smearing is in place)
$>a$. (averaging)

- coarse graining (粗視化) / block spin transformation.

$\Leftrightarrow$ integrating out $\phi(\vec{k})$'s with larger $\vec{k}$'s



$\Rightarrow$ In the effective action,

$\boxed{\text{new terms may show up, and coupling constants may be modified accordingly.}}$

- coupling constants : fns of $|\vec{k}|_{max}$ $g(|\vec{k}|_{max})$
- free energy / effective action (GL potential) : depend on $g(|\vec{k}|_{max})$ & $|\vec{k}|_{max}$ apparently

but should not dep. on $|\vec{k}|_{max}$ in the end

$\frac{\partial g(|\vec{k}|_{max})}{\partial \ln(|\vec{k}|_{max})}$ is related to the $|\vec{k}|_{max}$ dependence of the computation of the effective action.

$\boxed{\text{renormalization (group)}}$

§9.2 Partition function, free energy, effective action (formal)

We call the followings as the partion function:

$$\boxed{Z(\beta;\text{theory}) := \mathrm{tr}_{\text{Hilb.sp.}}\left(e^{-\beta H}\right) = \sum_{\text{all states}} e^{-\beta E_i}}$$

$$\boxed{= \int \mathcal{D}\Pi\mathcal{D}\phi\; e^{\left[-\int d^dx\int_0^\beta d\tau\,\mathcal{H} + i\int_0^\beta d\tau\int d^dx\,\Pi(\partial_\tau\phi)\right]}}$$

$\beta\to\infty$ $(T\to 0)$ limit.

$$\boxed{Z(\text{theory}) := \int_{+i\varepsilon\text{ prescr.}} \mathcal{D}\Pi\mathcal{D}\phi\; e^{\int d^dx\,dt\left\{-i\mathcal{H} + i\Pi(\partial_t\phi)\right\}}}$$

The following generalizations are useful and are also called partition functions.

$$\boxed{Z(\beta,\text{theory}, J_I(\vec{x})) := \int \mathcal{D}\Pi\mathcal{D}\phi\; e^{\int d^dx\int_0^\beta d\tau\left\{-\mathcal{H} + i\Pi(\partial_\tau\phi) + \Sigma_I\mathcal{O}_I(\vec{x})J_I(\vec{x})\right\}}}$$

$$\boxed{Z(\text{theory}, J_I(\vec{x},t)) := \int_{+i\varepsilon} \mathcal{D}\Pi\mathcal{D}\phi\; e^{i\int d^dx\int dt\left\{-\mathcal{H} + \Pi(\partial_t\phi) + \mathcal{O}_I(\vec{x},t)J_I(\vec{x},t)\right\}}}$$

They are the generating functions of various τ-/t- ordered correlation functions.

$$\boxed{\left\langle \mathcal{O}_I(\vec{x})\right\rangle_{\text{th.eq}}^{J_I=0} = \frac{1}{\beta}\frac{\partial}{\partial J_I(\vec{x})}\ln\left(Z(\beta,\text{theory}, J_I(\vec{x}))\right)\Big|_{J=0}}$$

If $\left\langle \mathcal{O}_I(\vec{x})\right\rangle_{\text{th.eq}}^{J\neq 0} \sim$ linear in $J_I(\vec{x}) =: \chi\cdot J_I(\vec{x})$

then

$$\boxed{\begin{aligned}\chi &= \left[\frac{\partial}{\partial J_J(\vec{y})}\left(\left\langle \mathcal{O}_I(\vec{x})\right\rangle_{\text{th.eq}}^{J\neq0}\right)\right]\Big|_{J=0} = \left[\frac{1}{\beta}\frac{\partial^2}{\partial J_J(\vec{y})\,\partial J_I(\vec{x})}\ln\left(Z(\beta, J_K)\right)\right]\Big|_{J=0}\\ &= \beta\left(\left\langle \mathcal{O}_J(\vec{y})\mathcal{O}_I(\vec{x})\right\rangle_{\text{th.eq}}^{J=0} - \left\langle \mathcal{O}_J(\vec{y})\right\rangle_{\text{th.eq}}^{J=0}\left\langle \mathcal{O}_I(\vec{x})\right\rangle_{\text{th}}^{J=0}\right)\end{aligned}}$$

The linear respose relation.

(The susceptibility is given by a 2-pt function (fluctuation))

Ex: In a free theory $\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2$

$$\boxed{Z(m, J) :\cong \int_{+i\varepsilon} \mathcal{D}\phi \; e^{i\int d^{d+1}x \left(\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 + J(x)\phi(x)\right)}}$$

$$= \int \prod_{p\in\mathbb{R}^{1,d}} \left[d\tilde{\phi}(p)\right] \; e^{i\int \frac{d^{d+1}p}{(2\pi)^{d+1}} \frac{1}{2}\tilde{\phi}(p)^*(p^2-m^2)\tilde{\phi}(p) + \left(\tilde{J}(p)^*\tilde{\phi}(p) + \tilde{J}(p)\tilde{\phi}(p)^*\right)\frac{1}{2}}$$

$$= \int \prod_{p\in\mathbb{R}^{1,d}} \left[d\tilde{\phi}(p)\right] \; e^{i\int \frac{d^{d+1}p}{(2\pi)^{d+1}} \cdot \frac{1}{2}\left(\tilde{\phi}(p) + \frac{1}{p^2-m^2}\tilde{J}(p)\right)^*(p^2-m^2)\left(\tilde{\phi}(p) + \frac{1}{p^2-m^2}\tilde{J}(p)\right)}$$

$$e^{-i\int \frac{d^{d+1}p}{(2\pi)^{d+1}} \frac{1}{2}\tilde{J}^*(p)\frac{1}{p^2-m^2}\tilde{J}(p)}$$

$$\propto \; e^{-i\int \frac{d^{d+1}p}{(2\pi)^{d+1}} \frac{1}{2}\tilde{J}^*(p)\frac{1}{p^2-m^2}\tilde{J}(p)} \times \text{const}\cdot(m, \cancel{J})$$

So

$$\frac{\langle\Omega| T\{\tilde{\phi}(p)^* \tilde{\phi}(p')\}|\Omega\rangle}{\langle\Omega|\Omega\rangle_{(\text{often } \langle\Omega|\Omega\rangle = 1)}} = -\frac{\partial^2}{\partial\tilde{J}(p)\,\partial\tilde{J}(p')^*} \ln\left(Z(m,J)\right)\Bigg|_{J=0}$$

$$= -\frac{\partial^2}{\partial\tilde{J}(p)\,\partial\tilde{J}^*(p')}\left((-i)\int \frac{d^{d+1}p}{(2\pi)^{d+1}} \frac{1}{2}\tilde{J}^*(p)\frac{1}{p^2-m^2}\tilde{J}(p)\right)\Bigg|_{J=0}$$

$$= (2\pi)^{d+1}\delta^{d+1}(p-p')\frac{i}{(p^2-m^2)}$$

as expected.

$$Z(\beta, \text{theory}, J_I(\vec{x})) =: e^{-\beta F(\beta, \text{theory}, J_I(\vec{x}))}$$

$$Z(\text{theory}, J_I(x^\mu)) =: e^{-i\int d^d\vec{x}\, dt\, \mathcal{F}(\text{theory}, J_I(\vec{y}, t'))}$$

F: called the free energy.

Two questions will come to our minds then.

◎ How is the Landau-Ginzburg theory related to $F(\beta, \text{thry}, J_I(\vec{x}))$?

◎ In thermodynamics / statistical mechanics, we say that the "phase 1" is realized when "$F(\beta, \text{thry}, \text{phase}1) < F(\beta, \text{thry}, \text{phase}2)$".

How do we compute "$F(\beta, \text{thry}, \text{phase})$" ?

Here is a QFT version of such stories.

$$\boxed{\text{a phase} \Leftrightarrow \text{an ansatz on which set of operators have non-zero expectation values.}}$$

(order parameters $\Leftrightarrow$ hypothetically chosen operator expectation values)

Suppose that there is a choice $J_I(\vec{x})$ that reproduces an order parameter $\langle \mathcal{O}_J(\vec{y})\rangle^J_{\text{th.eq}} = -\frac{\partial}{\partial J_J(\vec{y})}\left(F(J_K(\vec{z}))\right)$.

(we have a $J_I(\vec{x}) \Leftrightarrow \langle \mathcal{O}_J(\vec{y})\rangle_{\text{th.eq}}$ dictionary then)

Define

$$\boxed{-\Gamma_{\text{th.eq}}\left[\langle \mathcal{O}_J(\vec{y})\rangle_{\text{th.eq}}\right] := \left(F\left[J_I(\vec{x})\right] + \int d^d\vec{x}\, J_I(\vec{x}) \langle \mathcal{O}_I(\vec{x})\rangle^J_{\text{th.eq}}\right)}$$

all J's translate to $\langle \mathcal{O}\rangle_{\text{th.eq}}$'s

Legendre transformation w.r.t. $J_I(\vec{x})$

Then

$$\boxed{-\frac{\partial}{\partial \langle \mathcal{O}_J(\vec{y})\rangle_{\text{th.eq}}}\left(\Gamma_{\text{th.eq}}\left[\langle \mathcal{O}_K(\vec{y})\rangle_{\text{th.eq}}\right]\right) = J_J(\vec{y})}$$

so the order parameters that may be realized in the original theory ($J_I(\vec{x}) = 0$) should satisfy

$$\boxed{\frac{\partial}{\partial \langle \mathcal{O}_J(\vec{y})\rangle_{\text{th.eq}}} \Gamma_{\text{th.eq}}\left[\langle \mathcal{O}\rangle_{\text{th.eq}}\right] = 0}$$

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The order parameter $\langle O_I(\vec{x})\rangle_{th.eq}$ may have $\vec{x}$-dependence
(e.g. ~~Pierls transition~~ $\langle$phonon$(k=\pi/a)\rangle$)

In a phase with $\langle O_I(\cancel{\vec{x}})\rangle_{th.eq}$.

$$\Gamma_{th.eq} =: -\int d^d\vec{x}\ \underbrace{V_{th.eq}(\langle O_I\rangle_{th.eq})}$$

↳ the Landau-Ginzburg potential
$(\beta,\ \text{theory},\ \langle O_I\rangle_{th.eq})$
↑ order parameter.

In each phase

$$\boxed{F(\beta,\text{theory},\text{phase}) := -\Gamma_{th.eq}(\beta,\text{theory},\langle O_I(\vec{x})\rangle)\Big|_{\frac{\partial\Gamma_{th.eq}}{\partial\langle O_I(\vec{x})\rangle}=0}}\quad J=0,\ F=-\Gamma$$

~~and we can apply the thermodynamical principle~~

"phase 1" realized when $F(\text{phase }1) < F(\text{phase }2)$.

Example: in a condensed matter system with

$$\mathcal{H} = \sum_{s=\uparrow\downarrow}\left(\psi_s^\dagger(\vec{k})\ \varepsilon(\vec{k})\ \psi_s(\vec{k})\right) + (\psi_\uparrow^\dagger\psi_\uparrow)(\psi_\downarrow^\dagger\psi_\downarrow)\ c$$

↑ induced from (photon / phonon) exchange

magnetic phase $\langle(\psi_\uparrow^\dagger\psi_\uparrow)(\vec{k})\rangle \neq 0,\ \langle(\psi_\downarrow^\dagger\psi_\downarrow)(\vec{k})\rangle = 0$ for some $\vec{k}$.

super conducting phase $\langle(\psi_\uparrow\psi_\downarrow)\rangle \neq 0$.

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For $T \to 0$ ($\beta \to \infty$) the vacuum case

$$\boxed{\langle \mathcal{O}_I(x^\mu) \rangle_{vac.} = -\frac{\partial}{\partial J_I(x^\mu)} \mathcal{F} = -\frac{\delta}{\delta J_I(x^\mu)} \left(\int d^{d+1}x \ \mathcal{F} \right)}$$

$$-\Gamma_{vac}[\langle \mathcal{O}_I(x^\mu) \rangle_{vac}] := \int d^{d+1}x \left\{ \mathcal{F} + J_I(x^\mu) \langle \mathcal{O}_I(x^\mu) \rangle_{vac} \right\}$$

(Legendre transformation)

The operator vacuum expectation value should satisfy $\boxed{\frac{\delta}{\delta \langle \mathcal{O}_I(x) \rangle_{vac}} \Gamma[\langle \mathcal{O}_K(x^\mu) \rangle_{vac}] = 0.}$

$\Gamma_{vac}[\langle \mathcal{O}_I(x) \rangle_{vac}]$: called the effective action.

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